Deep Proof Search in MELL

Ozan Kahramanoğulları

Department of Mathematics, University of Trento

Microsoft Research – University of Trento
Centre for Computational and Systems Biology

Deep Inference

Proofs are sequences of inference rule instances.

$$[a \parr (\bar{a} \otimes [b \parr \bar{b}])]$$

MLS - Multiplicative Linear Logic

$$\mathsf{ai}{\downarrow}\,\frac{S\{1\}}{S[a \parr \bar{a}]} \qquad \mathsf{s}\,\frac{S([U \parr R] \otimes T)}{S[U \parr (R \otimes T)]}$$

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Shallow inference
as in sequent calculus

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Deep inference,
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$$[(\bar{a} \otimes \bar{b} \otimes \bar{c}) \parr a \parr b \parr c]$$

42 deep instances – 6 provide proofs, 3 redundant

Reducing Non-determinism in Deep Inference

Non-determinism is mainly due to context management.

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$$\mathsf{s}\,\frac{(\bar{a} \otimes [\bar{b} \parr a \parr b])}{[(\bar{a} \otimes \bar{b}) \parr a \parr b]}\ ✗ \qquad \mathsf{s}\,\frac{[([a \parr \bar{b}] \otimes \bar{a}) \parr b]}{[(\bar{a} \otimes \bar{b}) \parr a \parr b]}\ ✗ \qquad \mathsf{s}\,\frac{[([b \parr \bar{b}] \otimes \bar{a}) \parr a]}{[(\bar{a} \otimes \bar{b}) \parr a \parr b]}\ ✓$$

Reducing Non-determinism in Deep Inference

Theorem: **MLSdlis** is complete for Multiplicative Linear Logic.

MLSdlis - Multiplicative Linear Logic with deep lazy interaction switch

$$\mathsf{ai}\!\downarrow\frac{S\{1\}}{S[a \parr \bar{a}]} \qquad \mathsf{dlis}\,\frac{S([U \parr R] \otimes T)}{S[U \parr (R \otimes T)]}$$

$$\mathsf{u}_1\!\downarrow\frac{S\{R\}}{S[R \parr \bot]} \qquad \mathsf{u}_2\!\downarrow\frac{S\{R\}}{S(R \otimes 1)}$$

Reducing Non-determinism in Deep Inference

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Reducing Non-determinism in Deep Inference

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INTERACTION AND DEPTH
AGAINST NONDETERMINISM IN PROOF SEARCH

OZAN KAHRAMANOĞULLARI

Also, for 6 other systems intermediate between MLS and MLSdli.

MLSdlis - Multiplicative Linear Logic with deep lazy interaction switch

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Multiplicative Exponential Linear Logic

extends MLS with the rules for exponentials.

ELSu - Multiplicative Exponential Linear Logic

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$$\mathsf{u}_1\!\downarrow \frac{S\{R\}}{S[R \parr \bot]} \qquad \mathsf{u}_2\!\downarrow \frac{S\{R\}}{S(R \otimes 1)}$$

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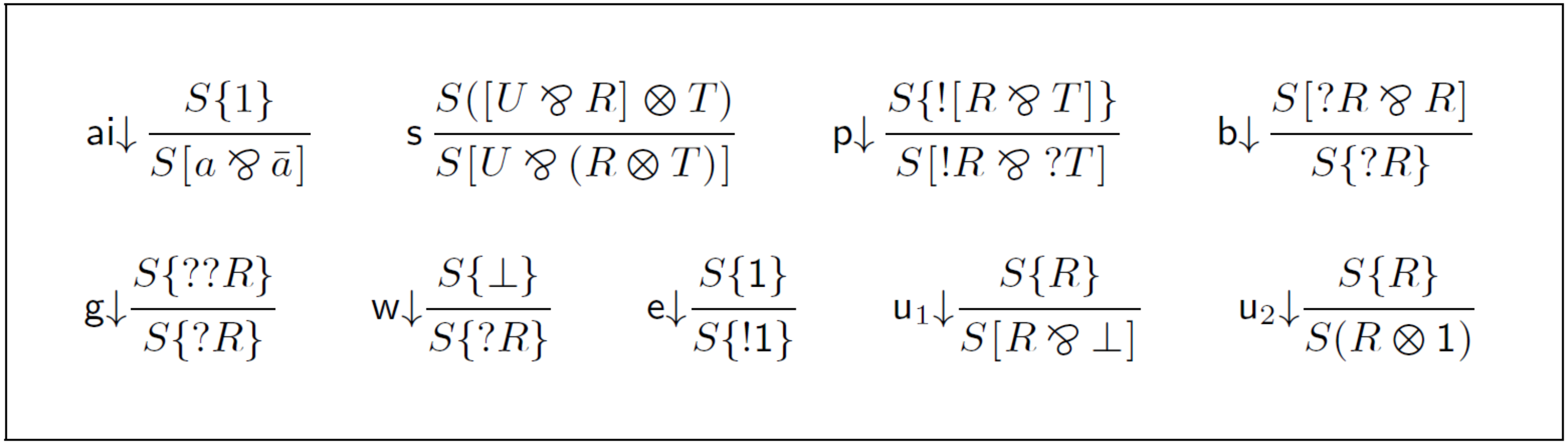

$$\mathsf{ai}\!\downarrow \frac{S\{1\}}{S[a \parr \bar{a}]} \qquad \mathsf{s}\, \frac{S([U \parr R] \otimes T)}{S[U \parr (R \otimes T)]} \qquad \mathsf{p}\!\downarrow \frac{S\{![R \parr T]\}}{S[!R \parr ?T]} \qquad \mathsf{b}\!\downarrow \frac{S[?R \parr R]}{S\{?R\}}$$

$$\mathsf{g}\!\downarrow \frac{S\{??R\}}{S\{?R\}} \qquad \mathsf{w}\!\downarrow \frac{S\{\bot\}}{S\{?R\}} \qquad \mathsf{e}\!\downarrow \frac{S\{1\}}{S\{!1\}} \qquad \mathsf{u}_1\!\downarrow \frac{S\{R\}}{S[R \parr \bot]} \qquad \mathsf{u}_2\!\downarrow \frac{S\{R\}}{S(R \otimes 1)}$$

Multiplicative Exponential Linear Logic

extends MLS with the rules for exponentials.

Can we reduce nondeterminism in proof search in MELL?

ELSu - Multiplicative Exponential Linear Logic

$$\mathsf{ai}\downarrow \frac{S\{1\}}{S[a \parr \bar{a}]} \qquad \mathsf{s}\,\frac{S([U \parr R] \otimes T)}{S[U \parr (R \otimes T)]} \qquad \mathsf{p}\downarrow \frac{S\{![R \parr T]\}}{S[!R \parr ?T]} \qquad \mathsf{b}\downarrow \frac{S[?R \parr R]}{S\{?R\}}$$

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Multiplicative Exponential Linear Logic

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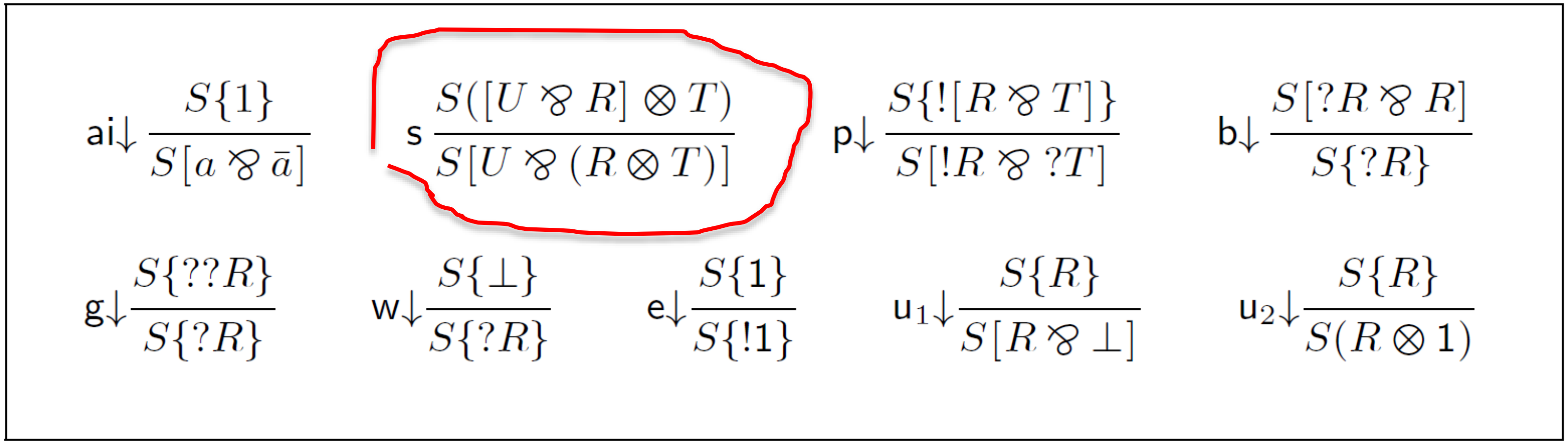
ELSu



$$\mathsf{ai}\!\downarrow \frac{S\{1\}}{S[a \parr \bar{a}]} \qquad \mathsf{s} \frac{S([U \parr R] \otimes T)}{S[U \parr (R \otimes T)]} \qquad \mathsf{p}\!\downarrow \frac{S\{![R \parr T]\}}{S[!R \parr ?T]} \qquad \mathsf{b}\!\downarrow \frac{S[?R \parr R]}{S\{?R\}}$$

$$\mathsf{g}\!\downarrow \frac{S\{??R\}}{S\{?R\}} \qquad \mathsf{w}\!\downarrow \frac{S\{\bot\}}{S\{?R\}} \qquad \mathsf{e}\!\downarrow \frac{S\{1\}}{S\{!1\}} \qquad \mathsf{u}_1\!\downarrow \frac{S\{R\}}{S[R \parr \bot]} \qquad \mathsf{u}_2\!\downarrow \frac{S\{R\}}{S(R \otimes 1)}$$

Multiplicative Exponential Linear Logic

extends MLS with the rules for exponentials.

U and R have common atoms -
they can interact

$$\text{dlis}\;\frac{S([U \parr R] \otimes T)}{S[U \parr (R \otimes T)]} \qquad \text{if at}\,\overline{R} \cap \text{at}\,U \neq \emptyset$$

Lazy: Not a disjunction

Deep: Not a conjunction

ELSu

$$\text{ai}\!\downarrow\frac{S\{1\}}{S[a \parr \bar{a}]} \qquad \text{s}\,\frac{S([U \parr R] \otimes T)}{S[U \parr (R \otimes T)]} \qquad \text{p}\!\downarrow\frac{S\{![R \parr T]\}}{S[!R \parr ?T]} \qquad \text{b}\!\downarrow\frac{S[?R \parr R]}{S\{?R\}}$$

$$\text{g}\!\downarrow\frac{S\{??R\}}{S\{?R\}} \qquad \text{w}\!\downarrow\frac{S\{\bot\}}{S\{?R\}} \qquad \text{e}\!\downarrow\frac{S\{1\}}{S\{!1\}} \qquad \text{u}_1\!\downarrow\frac{S\{R\}}{S[R \parr \bot]} \qquad \text{u}_2\!\downarrow\frac{S\{R\}}{S(R \otimes 1)}$$

Multiplicative Exponential Linear Logic

extends MLS with the rules for exponentials.

ELS dli

$$\mathsf{ai}{\downarrow}\,\frac{S\{1\}}{S[a \parr \bar{a}]} \qquad \mathsf{dlis}\,\frac{S([U \parr R] \otimes T)}{S[U \parr (R \otimes T)]} \qquad \mathsf{p}{\downarrow}\,\frac{S\{![R \parr T]\}}{S[!R \parr ?T]} \qquad \mathsf{b}{\downarrow}\,\frac{S[?R \parr R]}{S\{?R\}}$$

$$\mathsf{g}{\downarrow}\,\frac{S\{??R\}}{S\{?R\}} \qquad \mathsf{w}{\downarrow}\,\frac{S\{\bot\}}{S\{?R\}} \qquad \mathsf{e}{\downarrow}\,\frac{S\{1\}}{S\{!1\}} \qquad \mathsf{u}_1{\downarrow}\,\frac{S\{R\}}{S[R \parr \bot]} \qquad \mathsf{u}_2{\downarrow}\,\frac{S\{R\}}{S(R \otimes 1)}$$

Multiplicative Exponential Linear Logic

extends MLS with the rules for exponentials.

$$\mathsf{p}\!\downarrow \frac{S\{![R \parr T]\}}{S[!R \parr ?T]}$$

ELS dli

$$\mathsf{ai}\!\downarrow \frac{S\{1\}}{S[a \parr \bar{a}]} \qquad \mathsf{dlis} \frac{S([U \parr R] \otimes T)}{S[U \parr (R \otimes T)]} \qquad \mathsf{p}\!\downarrow \frac{S\{![R \parr T]\}}{S[!R \parr ?T]} \qquad \mathsf{b}\!\downarrow \frac{S[?R \parr R]}{S\{?R\}}$$

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Multiplicative Exponential Linear Logic

extends MLS with the rules for exponentials.

R and T have common atoms -
they can interact

$$\mathsf{ip}{\downarrow}\ \frac{S\{![R \parr T]\}}{S[!R \parr ?T]} \quad \text{if at}\,\overline{R} \cap \text{at}\,T \neq \emptyset$$

ELS dli

$$\mathsf{ai}{\downarrow}\ \frac{S\{1\}}{S[a \parr \bar{a}]} \qquad \mathsf{dlis}\ \frac{S([U \parr R] \otimes T)}{S[U \parr (R \otimes T)]} \qquad \mathsf{p}{\downarrow}\ \frac{S\{![R \parr T]\}}{S[!R \parr ?T]} \qquad \mathsf{b}{\downarrow}\ \frac{S[?R \parr R]}{S\{?R\}}$$

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Multiplicative Exponential Linear Logic

extends MLS with the rules for exponentials.

ELS dli

$$\mathsf{ai}\!\downarrow \frac{S\{1\}}{S[a \parr \bar{a}]} \qquad \mathsf{dlis} \frac{S([U \parr R] \otimes T)}{S[U \parr (R \otimes T)]} \qquad \mathsf{ip}\!\downarrow \frac{S\{![R \parr T]\}}{S[!R \parr ?T]} \qquad \mathsf{b}\!\downarrow \frac{S[?R \parr R]}{S\{?R\}}$$

$$\mathsf{g}\!\downarrow \frac{S\{??R\}}{S\{?R\}} \qquad \mathsf{w}\!\downarrow \frac{S\{\bot\}}{S\{?R\}} \qquad \mathsf{e}\!\downarrow \frac{S\{1\}}{S\{!1\}} \qquad \mathsf{u}_1\!\downarrow \frac{S\{R\}}{S[R \parr \bot]} \qquad \mathsf{u}_2\!\downarrow \frac{S\{R\}}{S(R \otimes 1)}$$

Multiplicative Exponential Linear Logic

extends MLS with the rules for exponentials.

$$\mathsf{r}\!\downarrow \frac{S\{?[R \parr T]\}}{S[?R \parr ?T]}$$

ELS dli

$$\mathsf{ai}\!\downarrow \frac{S\{1\}}{S[a \parr \bar{a}]} \qquad \mathsf{dlis}\, \frac{S([U \parr R] \otimes T)}{S[U \parr (R \otimes T)]} \qquad \mathsf{i\,p}\!\downarrow \frac{S\{![R \parr T]\}}{S[!R \parr ?T]} \qquad \mathsf{b}\!\downarrow \frac{S[?R \parr R]}{S\{?R\}}$$

$$\mathsf{g}\!\downarrow \frac{S\{??R\}}{S\{?R\}} \qquad \mathsf{w}\!\downarrow \frac{S\{\bot\}}{S\{?R\}} \qquad \mathsf{e}\!\downarrow \frac{S\{1\}}{S\{!1\}} \qquad \mathsf{u}_1\!\downarrow \frac{S\{R\}}{S[R \parr \bot]} \qquad \mathsf{u}_2\!\downarrow \frac{S\{R\}}{S(R \otimes 1)}$$

Multiplicative Exponential Linear Logic

extends MLS with the rules for exponentials.

R and T have common atoms -
they can interact

$$\text{ir}\downarrow\ \frac{S\{?[R \parr T]\}}{S[?R \parr ?T]} \quad \text{if at } \overline{R} \cap \text{at } T \neq \emptyset$$

ELS dli

$$\text{ai}\downarrow\ \frac{S\{1\}}{S[a \parr \bar{a}]} \qquad \text{dlis}\ \frac{S([U \parr R] \otimes T)}{S[U \parr (R \otimes T)]} \qquad \text{ip}\downarrow\ \frac{S\{![R \parr T]\}}{S[!R \parr ?T]} \qquad \text{b}\downarrow\ \frac{S[?R \parr R]}{S\{?R\}}$$

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R and T have common atoms -
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$$\mathsf{ir}{\downarrow}\,\frac{S\{?[R \parr T]\}}{S[?R \parr ?T]} \quad \text{if } \mathsf{at}\,\overline{R} \cap \mathsf{at}\,T \neq \emptyset$$

ELS dlip

ir↓

$$\mathsf{ai}{\downarrow}\,\frac{S\{1\}}{S[a \parr \bar{a}]} \qquad \mathsf{dlis}\,\frac{S([U \parr R] \otimes T)}{S[U \parr (R \otimes T)]} \qquad \mathsf{ip}{\downarrow}\,\frac{S\{![R \parr T]\}}{S[!R \parr ?T]} \qquad \mathsf{b}{\downarrow}\,\frac{S[?R \parr R]}{S\{?R\}}$$

$$\mathsf{g}{\downarrow}\,\frac{S\{??R\}}{S\{?R\}} \qquad \mathsf{w}{\downarrow}\,\frac{S\{\bot\}}{S\{?R\}} \qquad \mathsf{e}{\downarrow}\,\frac{S\{1\}}{S\{!1\}} \qquad \mathsf{u}_1{\downarrow}\,\frac{S\{R\}}{S[R \parr \bot]} \qquad \mathsf{u}_2{\downarrow}\,\frac{S\{R\}}{S(R \otimes 1)}$$

Multiplicative Exponential Linear Logic

The interaction condition preserves completeness for MELL.

Theorem: **ELSdlip** is complete for MELL.

ELS dlip

ir↓

$$\mathsf{ai}{\downarrow}\,\frac{S\{1\}}{S[a \parr \bar{a}]} \qquad \mathsf{dlis}\,\frac{S([U \parr R] \otimes T)}{S[U \parr (R \otimes T)]} \qquad \mathsf{ip}{\downarrow}\,\frac{S\{![R \parr T]\}}{S[!R \parr ?T]} \qquad \mathsf{b}{\downarrow}\,\frac{S[?R \parr R]}{S\{?R\}}$$

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Multiplicative Exponential Linear Logic

The interaction condition preserves completeness for MELL.

Theorem: **ELSdlip** is complete for MELL.

that is, ELSdlip and ELS prove the same formulae,

and so do 14 other systems intermediate between ELS and ELSdlip.

ELS dlip

ir↓

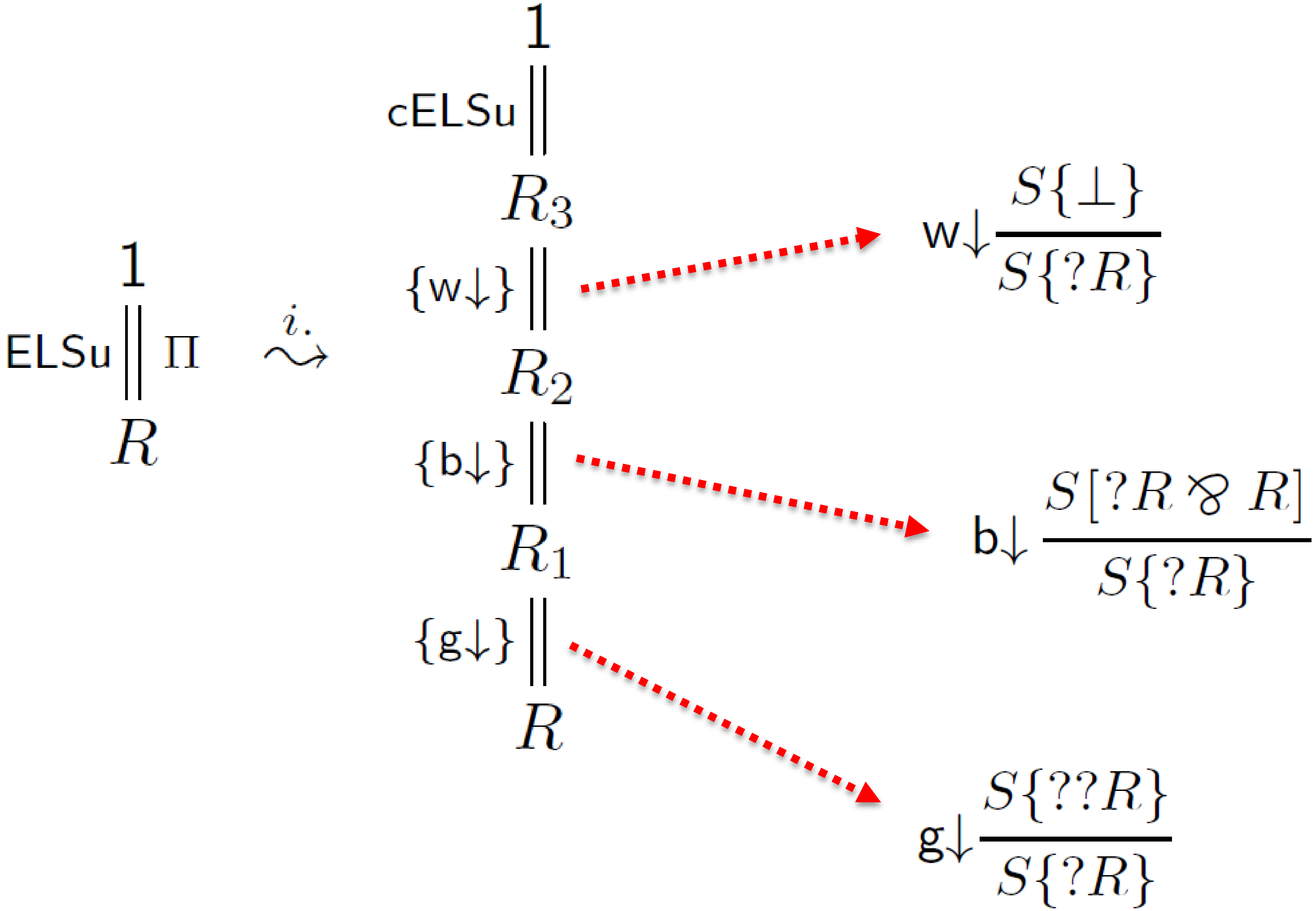
$$\mathsf{ai}\!\downarrow \frac{S\{1\}}{S[a \parr \bar{a}]} \qquad \mathsf{dlis} \frac{S([U \parr R] \otimes T)}{S[U \parr (R \otimes T)]} \qquad \mathsf{ip}\!\downarrow \frac{S\{![R \parr T]\}}{S[!R \parr ?T]} \qquad \mathsf{b}\!\downarrow \frac{S[?R \parr R]}{S\{?R\}}$$

$$\mathsf{g}\!\downarrow \frac{S\{??R\}}{S\{?R\}} \qquad \mathsf{w}\!\downarrow \frac{S\{\bot\}}{S\{?R\}} \qquad \mathsf{e}\!\downarrow \frac{S\{1\}}{S\{!1\}} \qquad \mathsf{u}_1\!\downarrow \frac{S\{R\}}{S[R \parr \bot]} \qquad \mathsf{u}_2\!\downarrow \frac{S\{R\}}{S(R \otimes 1)}$$

Decomposition of ELS Proofs

$$\begin{array}{c} 1 \\ \mathsf{ELSu} \left\Vert \vphantom{\int_0^1} \right. \Pi \\ R \end{array}$$

Decomposition of ELS Proofs



$$\begin{array}{c}1\\ \text{ELSu}\,\Big\|\,\Pi\\ R\end{array} \quad \overset{i.}{\leadsto} \quad \begin{array}{c}1\\ \text{cELSu}\,\Big\|\\ R_3\\ \{\mathsf{w}\!\downarrow\}\,\Big\|\\ R_2\\ \{\mathsf{b}\!\downarrow\}\,\Big\|\\ R_1\\ \{\mathsf{g}\!\downarrow\}\,\Big\|\\ R\end{array}$$

$$\mathsf{w}\!\downarrow\frac{S\{\bot\}}{S\{?R\}}$$

$$\mathsf{b}\!\downarrow\frac{S[?R \parr R]}{S\{?R\}}$$

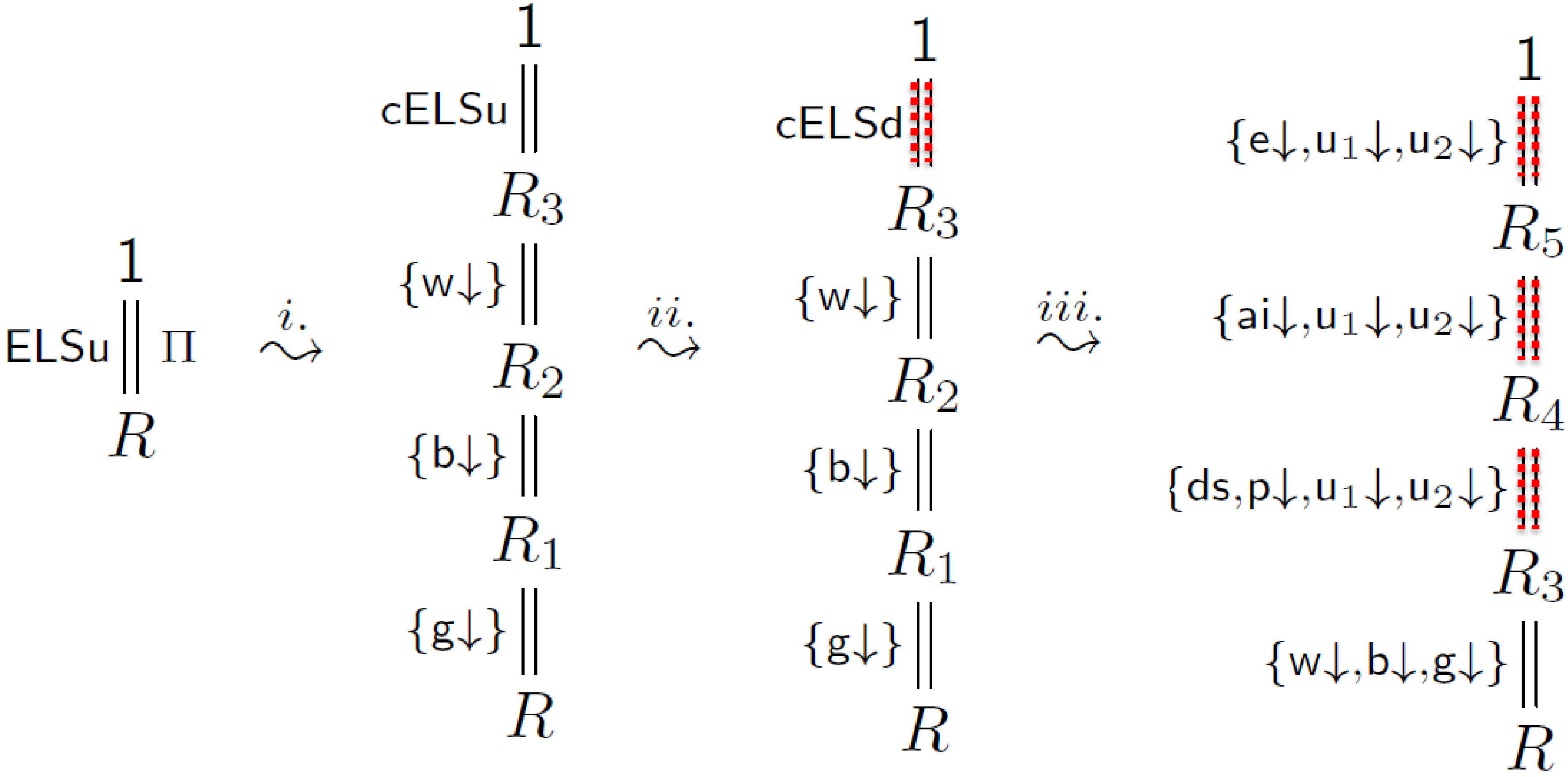
$$\mathsf{g}\!\downarrow\frac{S\{??R\}}{S\{?R\}}$$

Decomposition of ELS Proofs

$$
\begin{array}{c} 1 \\ \mathsf{ELSu}\,\Big\Vert\;\Pi \\ R \end{array}
\quad\overset{i.}{\rightsquigarrow}\quad
\begin{array}{c} 1 \\ \mathsf{cELSu}\,\Big\Vert \\ R_3 \\ \{\mathsf{w}\downarrow\}\,\Big\Vert \\ R_2 \\ \{\mathsf{b}\downarrow\}\,\Big\Vert \\ R_1 \\ \{\mathsf{g}\downarrow\}\,\Big\Vert \\ R \end{array}
\quad\overset{ii.}{\rightsquigarrow}\quad
\begin{array}{c} 1 \\ \mathsf{cELSd}\,\Big\Vert \\ R_3 \\ \{\mathsf{w}\downarrow\}\,\Big\Vert \\ R_2 \\ \{\mathsf{b}\downarrow\}\,\Big\Vert \\ R_1 \\ \{\mathsf{g}\downarrow\}\,\Big\Vert \\ R \end{array}
$$

the instances of switch have the "deep" condition.

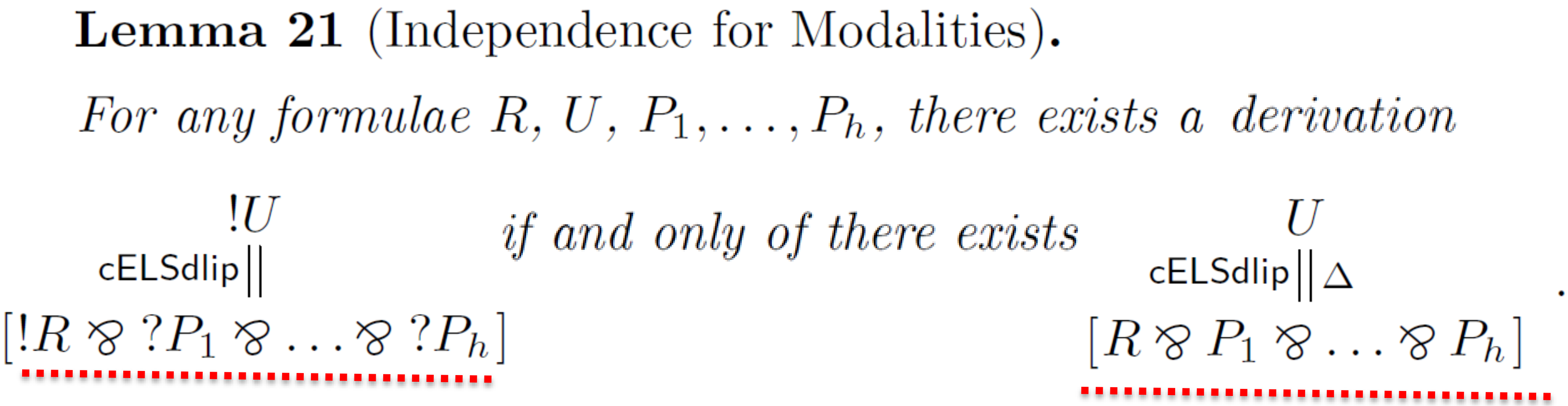
Decomposition of ELS Proofs



$$\begin{array}{r} 1 \\ \mathsf{ELSu}\,\Big\Vert\;\Pi \\ R \end{array} \;\overset{i.}{\leadsto}\; \begin{array}{r} 1 \\ \mathsf{cELSu}\,\Big\Vert \\ R_3 \\ \{\mathsf{w}\downarrow\}\,\Big\Vert \\ R_2 \\ \{\mathsf{b}\downarrow\}\,\Big\Vert \\ R_1 \\ \{\mathsf{g}\downarrow\}\,\Big\Vert \\ R \end{array} \;\overset{ii.}{\leadsto}\; \begin{array}{r} 1 \\ \mathsf{cELSd}\,\Big\Vert \\ R_3 \\ \{\mathsf{w}\downarrow\}\,\Big\Vert \\ R_2 \\ \{\mathsf{b}\downarrow\}\,\Big\Vert \\ R_1 \\ \{\mathsf{g}\downarrow\}\,\Big\Vert \\ R \end{array} \;\overset{iii.}{\leadsto}\; \begin{array}{r} 1 \\ \{\mathsf{e}\downarrow,\mathsf{u}_1\downarrow,\mathsf{u}_2\downarrow\}\,\Big\Vert \\ R_5 \\ \{\mathsf{ai}\downarrow,\mathsf{u}_1\downarrow,\mathsf{u}_2\downarrow\}\,\Big\Vert \\ R_4 \\ \{\mathsf{ds},\mathsf{p}\downarrow,\mathsf{u}_1\downarrow,\mathsf{u}_2\downarrow\}\,\Big\Vert \\ R_3 \\ \{\mathsf{w}\downarrow,\mathsf{b}\downarrow,\mathsf{g}\downarrow\}\,\Big\Vert \\ R \end{array}$$

Independence Lemma for ELSdlip

The non-interacting formulae do not interfere with each other.



Lemma 21 (Independence for Modalities).

For any formulae R, U, $P_1, \ldots, P_h$, *there exists a derivation*

$$\begin{array}{c} !U \\ \text{cELSdlip}\,\Vert \\ \underline{[\,!R \parr ?P_1 \parr \ldots \parr ?P_h\,]} \end{array} \quad \textit{if and only of there exists} \quad \begin{array}{c} U \\ \text{cELSdlip}\,\Vert\,\Delta \\ \underline{[\,R \parr P_1 \parr \ldots \parr P_h\,]} \end{array} .$$

Independence Lemma for ELSdlip

The non-interacting formulae do not interfere with each other.

Lemma 21 (Independence for Modalities).

For any formulae R, U, $P_1, \ldots, P_h$, there exists a derivation

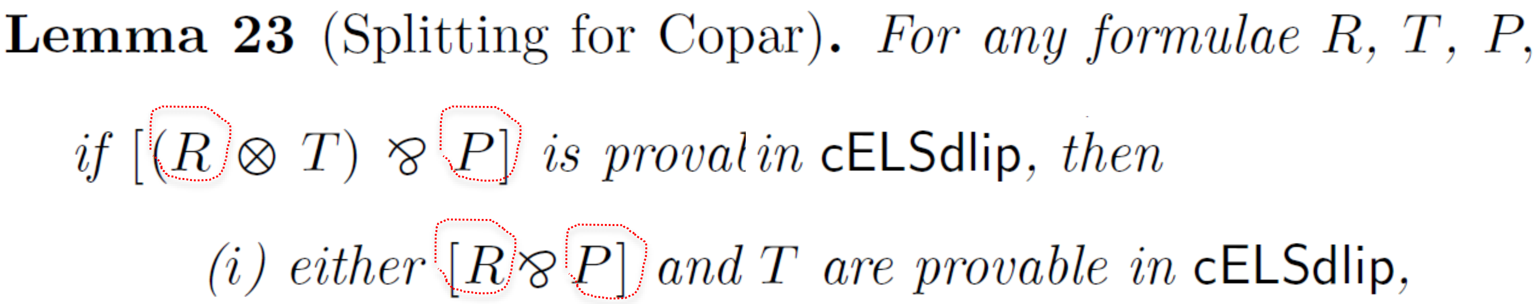
$$\begin{array}{c} !U \\ \text{cELSdlip}\,\Vert \\ \underline{[!R \parr ?P_1 \parr \ldots \parr ?P_h]} \end{array} \quad \textit{if and only of there exists} \quad \begin{array}{c} U \\ \text{cELSdlip}\,\Vert\,\Delta \\ \underline{[R \parr P_1 \parr \ldots \parr P_h]} \end{array} .$$

Lemma 22 (Independence for Copar).

For any formulae P, U and R, if $[\boxed{P} \parr \boxed{U}]$ has a proof in cELSdlip,

$$\textit{then there is a derivation} \quad \begin{array}{c} R \\ \text{cELSdlip}\,\Vert \\ [(R \otimes \boxed{P}) \parr \boxed{U}] \end{array} .$$

Splitting Lemma for ELSdlip - conjunction

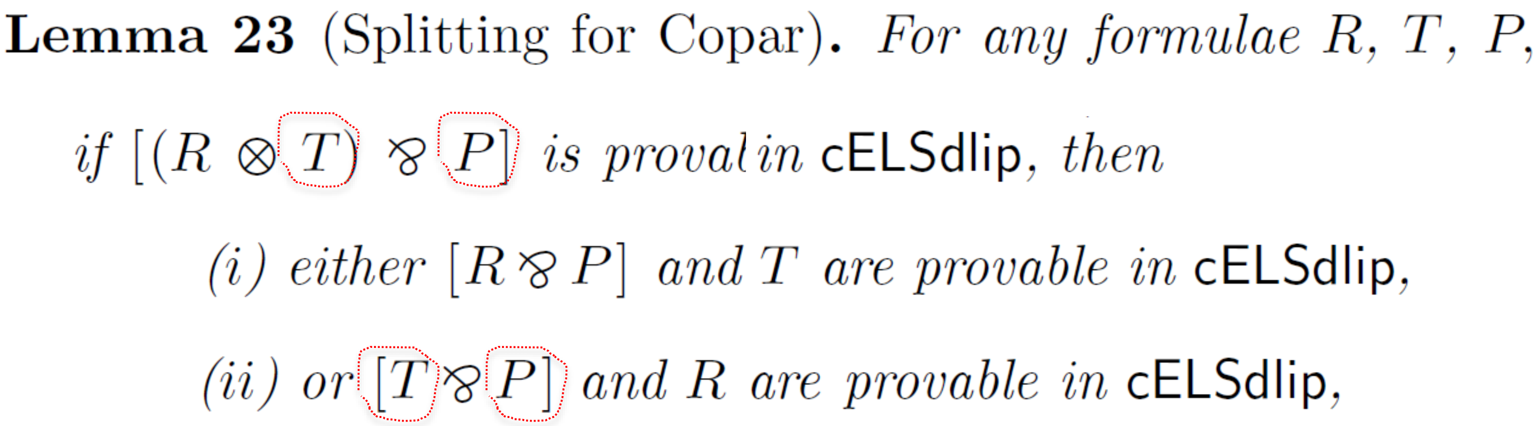


Lemma 23 (Splitting for Copar). *For any formulae* R, T, P,

if $[(R \otimes T) \parr P]$ *is proval in* cELSdlip, *then*

(i) either $[R \parr P]$ *and* T *are provable in* cELSdlip,

Splitting Lemma for ELSdlip - conjunction



Lemma 23 (Splitting for Copar). *For any formulae R, T, P,*

if $[(R \otimes T) \parr P]$ *is proval in* cELSdlip, *then*

(i) either $[R \parr P]$ *and* T *are provable in* cELSdlip,

(ii) or $[T \parr P]$ *and* R *are provable in* cELSdlip,

Splitting Lemma for ELSdlip - conjunction

Lemma 23 (Splitting for Copar). *For any formulae R, T, P,*

if $[(\underline{R} \otimes \underline{T}) \parr \boxed{P}]$ *is proval in* cELSdlip, *then*

(i) either $[R \parr P]$ *and* T *are provable in* cELSdlip,

(ii) or $[T \parr P]$ *and* R *are provable in* cELSdlip,

(iii) or there are formulae P_R *and* P_T, *such that*

$$\begin{array}{c}[P_R \parr P_T]\\ \text{cELSdlip}\,\Big\Vert\\ \boxed{P}\end{array}\quad,\qquad \begin{array}{c}1\\ \text{cELSdlip}\,\Big\Vert\\ [\underline{R} \parr P_R]\end{array}\qquad \textit{and}\qquad \begin{array}{c}1\\ \text{cELSdlip}\,\Big\Vert\\ [\underline{T} \parr P_T]\end{array}\quad.$$

Splitting Lemma for ELSdlip - !

Lemma 24 (Splitting for Modality !). *For any formulae R and P,*

if $[\underline{!R} \parr \boxed{P}]$ is provable i cELSdlip, *then*

(i) either there are formulae $P_1, \dots, P_h$ for some $h > 1$, such that

$$\begin{array}{c} [?P_1 \parr \dots \parr ?P_h] \\ \mathsf{cELSdlip} \, \Big\Vert \\ \boxed{P} \end{array} \qquad \textit{and} \qquad \begin{array}{c} 1 \\ \mathsf{cELSdlip} \, \Big\Vert \\ [\underline{R} \parr P_1 \parr \dots \parr P_h] \end{array}$$

Splitting Lemma for ELSdlip - !

Lemma 24 (Splitting for Modality !). *For any formulae R and P,*

if $[!R \parr P]$ *is provable* cELSdlip, *then*

(i) either there are formulae $P_1, \dots, P_h$ *for some* $h > 1$, *such that*

$$\begin{array}{c} [?P_1 \parr \dots \parr ?P_h] \\ \text{cELSdlip} \Big\Vert \\ P \end{array} \qquad \textit{and} \qquad \begin{array}{c} 1 \\ \text{cELSdlip} \Big\Vert \\ [R \parr P_1 \parr \dots \parr P_h] \end{array}$$

$$\textit{(ii)} \quad \textit{or} \quad \begin{array}{c} \bot \\ \text{cELSdlip} \Big\Vert \\ P \end{array} \qquad \textit{and} \qquad \begin{array}{c} 1 \\ \text{cELSdlip} \Big\Vert \\ R \end{array} \quad .$$

Splitting Lemma for ELSdlip - ?

Lemma 25 (Splitting for Modality ?). *For any formulae R and P,*

if $[?R \parr P]$ *is prova* cELSdlip,

$$\begin{array}{c} [?R \parr !T \parr ?Q_1 \parr \ldots \parr ?Q_h] \\ \text{cELSdlip}\,\Big\Vert \\ [?R \parr P] \end{array} \quad .$$

Splitting Lemma for ELSdlip - ?

Lemma 25 (Splitting for Modality ?). *For any formulae R and P,*

if $[\underline{?R} \parr P]$ *is prova* cELSdlip,

$$\begin{array}{c} 1 \\ \text{cELSdlip} \Vert \\ [\underline{?R} \parr !T \parr ?Q_1 \parr \ldots \parr ?Q_h] \qquad . \\ \text{cELSdlip} \Vert \\ [?R \parr P] \end{array}$$

Context Reduction for ELSdlip

Lemma 26 (Context reduction).

For any formula R and context $S\{\ \}$, if $S\{R\}$ is provable in cELSdlip, *then either there is a formula P_R such that*

$$(i) \qquad \begin{array}{c} !\ldots ![R \parr P_R] \\ \mathsf{cELSdlip} \Big\Vert \\ S\{R\} \end{array} \qquad \text{and} \qquad \begin{array}{c} 1 \\ \mathsf{cELSdlip} \Big\Vert \\ [R \parr P_R] \end{array}$$

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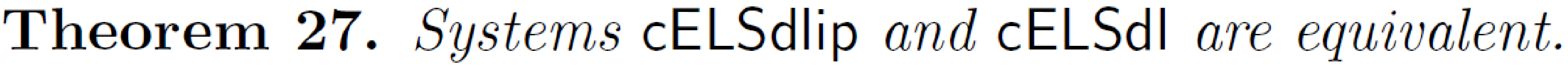
$$(i) \qquad \begin{array}{c} !\ldots![R \parr P_R] \\ \text{cELSdlip}\,\Big\Vert \\ S\{R\} \end{array} \qquad \textit{and} \qquad \begin{array}{c} 1 \\ \text{cELSdlip}\,\Big\Vert \\ [R \parr P_R] \end{array}$$

$$\textit{or} \qquad (ii) \qquad \begin{array}{c} !\ldots!R \\ \text{cELSdlip}\,\Big\Vert \\ \underline{S\{R\}} \end{array} \qquad \textit{and} \qquad \begin{array}{c} 1 \\ \text{cELSdlip}\,\Big\Vert \\ R \end{array} \quad .$$

Completeness of ELSdlip



Theorem 27. *Systems* cELSdlip *and* cELSdl *are equivalent.*

Completeness of ELSdlip

Theorem 27. *Systems* cELSdlip *and* cELSdl *are equivalent.*

Proof.

$$\Pi = \left\{ \begin{array}{c} 1 \\ \mathsf{cELSdlip} \Big\Vert \, \Pi' \\ \rho \, \dfrac{S\{Q\}}{S\{T\}} \\ \mathsf{cELSdl} \Big\Vert \\ R \end{array} \right.$$

Completeness of ELSdlip

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Completeness of ELSdlip

Theorem 27. *Systems* cELSdlip *and* cELSdl *are equivalent.*

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Theorem 28 (Cut elimination).

The cut rule is admissible for system ELSdlip.

Reduction in Nondeterminism: Example

$$[?a \parr ?b \parr ?c \parr (!\bar{a} \otimes !\bar{b} \otimes !\bar{c})]$$

proof step	1.
cELSu	42
cELSu*	45
cELSdlip	3

Reduction in Nondeterminism: Example

$$\mathsf{dlis};\mathsf{ip}{\downarrow}\,\frac{[?b \parr ?c \parr (![a \parr \bar{a}] \otimes !b \otimes !\bar{c})]}{[?a \parr ?b \parr ?c \parr (!\bar{a} \otimes !\bar{b} \otimes !\bar{c})]}$$

proof step	1.	2.	3.
`cELSu`	42	19	19
`cELSu*`	45	20	20
`cELSdlip`	3	3	3

Reduction in Nondeterminism: Example

$$
\mathsf{dlis};\,\mathsf{ip}\!\downarrow\;\dfrac{\mathsf{ai}\!\downarrow;\,\mathsf{e}\!\downarrow;\,\mathsf{u}_2\!\downarrow\;\dfrac{\mathsf{dlis};\,\mathsf{ip}\!\downarrow\;\dfrac{\mathsf{ai}\!\downarrow;\,\mathsf{e}\!\downarrow;\,\mathsf{u}_2\!\downarrow\;\dfrac{\mathsf{ip}\!\downarrow\;\dfrac{\mathsf{ai}\!\downarrow;\,\mathsf{e}\!\downarrow\;\dfrac{1}{![c \parr \bar{c}]}}{[?c \parr !\bar{c}]}}{[?c \parr (![b \parr \bar{b}] \otimes !\bar{c})]}}{[?b \parr ?c \parr (!\bar{b} \otimes !\bar{c})]}}{[?b \parr ?c \parr (![a \parr \bar{a}] \otimes !\bar{b} \otimes !\bar{c})]}}{[?a \parr ?b \parr ?c \parr (!\bar{a} \otimes !\bar{b} \otimes !\bar{c})]}
$$

proof step	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.	13.
cELSu	42	19	19	19	19	6	3	3	3	3	1	1	1
cELSu*	45	20	20	20	20	7	3	3	3	3	1	1	1
cELSdlip	3	3	3	3	3	2	2	2	2	2	1	1	1

Reduction in Nondeterminism: Example

$$\mathsf{b}{\downarrow};\mathsf{b}{\downarrow};\mathsf{w}{\downarrow};\mathsf{u}_1{\downarrow}\,\dfrac{\mathsf{dlis}\,\dfrac{\mathsf{dlis}\,\dfrac{\mathsf{ai}{\downarrow};\mathsf{u}_2{\downarrow}\,\dfrac{\mathsf{ai}{\downarrow};\mathsf{u}_2{\downarrow}\,\dfrac{\mathsf{ai}{\downarrow};\mathsf{u}_2{\downarrow}\,\dfrac{\mathsf{dlis}\,\dfrac{\mathsf{dlis}\,\dfrac{([\bar a \parr \bar b \parr (a \otimes b)] \otimes [\bar a \parr \bar b \parr (a \otimes b)])}{[(a \otimes b) \parr ([\bar a \parr \bar b] \otimes [\bar a \parr \bar b \parr (a \otimes b)])]}}{[(a \otimes b) \parr (a \otimes b) \parr ([\bar a \parr \bar b] \otimes [\bar a \parr \bar b])]}}{[(a \otimes b) \parr (a \otimes b) \parr ([\bar a \parr \bar b] \otimes [\bar a \parr \bar b] \otimes [e \parr \bar e])]}}{[(a \otimes b) \parr (a \otimes b) \parr ([\bar a \parr \bar b] \otimes [\bar a \parr \bar b] \otimes [d \parr \bar d] \otimes [e \parr \bar e])]}}{[(a \otimes b) \parr (a \otimes b) \parr ([\bar a \parr \bar b] \otimes [c \parr \bar c] \otimes [\bar a \parr \bar b] \otimes [d \parr \bar d] \otimes [e \parr \bar e])]}}{[(a \otimes b) \parr (a \otimes b) \parr ([(([\bar a \parr \bar b] \otimes [c \parr \bar c] \otimes [\bar a \parr \bar b] \otimes d) \parr \bar d] \otimes [e \parr \bar e])]}}{[(a \otimes b) \parr (a \otimes b) \parr ([(([\bar a \parr \bar b] \otimes c \otimes [\bar a \parr \bar b] \otimes d) \parr \bar c \parr \bar d] \otimes [e \parr \bar e])]}}{[?(a \otimes b) \parr ([(([\bar a \parr \bar b] \otimes c \otimes [\bar a \parr \bar b] \otimes d) \parr \bar c \parr \bar d] \otimes [e \parr \bar e])]}$$

proof step	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.
`cELSu`	41	22	53	53	28	28	15	15	8	10
`cELSdlip`	6	6	6	6	5	5	4	4	3	6

Conclusions

Systems with deep inference bring about **shorter proofs**
but also greater **nondeterminism** in proof search.

With a purely proof theoretical technique, this nondeterminism
can be reduced, also for MELL, without sacrificing **theoretical hygiene**.

The systems can implemented as term rewriting systems, and
orthogonal techniques, e.g., **focusing**, **concurrency**, **heuristics**,
can be used together: **theorem provers** are under development.

Modal Logics with deep inference should benefit from this techniques.

Next **milestone:** prioritizing the rule instances in a clean manner.

The results on MELL are of independent **theoretical interest**.