From SAT to Maximum Independent Set: A New Approach to Characterize Tractable Classes

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Motivations & Objectives

- Characterizing new tractable classes remains important for both SAT and AI in general
- Cornerstone step towards understanding the practical effectiveness of SAT solvers

- Exploit the polynomial reducibility, one of the fundamental concepts in complexity theory
- to characterize new tractable classes in SAT thanks to tractability results obtained for other NP-Compete problems (e.g. maximum independent set problem)

Outline

Plan

Propositional Satisfiability (SAT) - Maximum Independent Set (MIS)

Tractability Results: from MIS to SAT

A Connection with Minimal Models

Conclusion and perspectives

Conjunctive Normal Form (CNF) and SAT

- A conjunction of clauses:

$$\overbrace{(x_1 \vee \cdots \vee x_l)}^{clause} \wedge (y_1 \vee \cdots \vee y_m) \wedge (z_1 \vee \cdots \vee z_n) \cdots$$

- Clause: a disjunction of literals (x, $\neg x$)
- Example :

$$\Phi = \overbrace{(p \vee \neg q \vee \neg r)}^{1} \wedge \overbrace{(p \vee \neg q \vee s)}^{1} \wedge \overbrace{p}^{1} \wedge \overbrace{(r \vee \neg s)}^{1}$$

$\mathcal{I}(p) = 1$ and $\mathcal{I}(r) = 1$ (Partial interpretation)

Satisfiability: $\exists \mathcal{I}, \mathcal{I}(\Phi) = 1$ (NP-complete [Cook 71])

- Tractable classes : 2-SAT, (Renamable) Horn, etc.

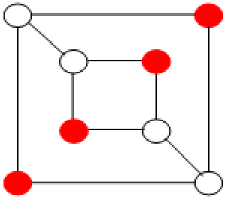
Boolean Satisfiability Problem (SAT)

- Spectacular progress $\rightarrow$ Modern SAT solvers
  - Application instances with millions of variables and clauses
- Many applications
  - Formal Verification
  - Planning
  - Bioinformatics
  - Cryptography
  - ...
- Around SAT
  - Max-SAT, (Weighted) Partial Max-SAT, QBF, ...
- CRIL Projects
  - Microsoft Research (UK): 2007-2012 (with Youssef Hamadi)
  - RATP (France): 2015-2017
  - ANR Project TUPLES "Tractability for Understanding and Pushing forward the Limits of Efficient Solvers", 2010-2014

Maximum Independent set problem

- Given an undirected graph $G = (V, E)$,
- An *independent set* of G is a set of non-adjacent vertices.

- A *Maximum independent* is a sub-set $V' \subset V$ of maximum cardinality such that for all $u, v \in V', (u, v) \notin E$. We note $\alpha(G)$ this maximum size.

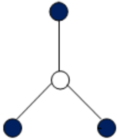


- *Maximum Independent Set Problem (MIS)* : Given a graph G, find a maximum independent set of G (NP-Hard)

- MIS tractable classes: *claw-free graphs*, perfect graphs

Tractable classes: claw-free graph

- A *claw-free graph* is a graph that does not have a *claw* as an induced subgraph.



- Finding a MIS in this class of graphs is tractable [Minty 80, Sbihi 80]

Plan

PBN Formulæ

Definition (PBN formula)

A PBN formula ϕ is a CNF formula where each clause is either a positive clause or a binary negative clause.

- Notations: $Pos(\phi)$ (resp. $Neg(\phi)$) to denote the set of its positive (resp. negative) clauses.
- Example:

$$\phi_{PBN} = \begin{cases} (p \vee q \vee r \vee s) \ \wedge \\ (r \vee s \vee t) \ \wedge \\ (\neg p \vee \neg v) \quad \wedge \\ (\neg q \vee \neg s) \end{cases}$$

- Checking the satisfiability of an PBN formula is NP-complete.

Transformation of CNF formula to PBN

- Associate to each negative literal l, a fresh variable r_l
- Replace each clause of the form:
  $p_1 \vee \cdots \vee p_m \vee \neg q_1 \vee \cdots \vee \neg q_n$ with
  $\{p_1 \vee \cdots \vee p_m \vee r_{\neg q_1} \vee \cdots \vee r_{\neg q_n}, \neg q_1 \vee \neg r_{\neg q_1}, \ldots, \neg q_n \vee \neg r_{\neg q_n}\}$
- Remark: the first clause can be obtained from several applications of the resolution rule on the last set of clauses

$$\phi_{CNF} = \begin{cases} (p \vee q \vee \neg r) \wedge \\ (p \vee \neg s) \end{cases}$$

$$\phi_{PBN} = \begin{cases} (p \vee q \vee t_{\neg r}) \wedge \\ (p \vee t_{\neg s}) \wedge \\ (\neg r \vee \neg t_{\neg r}) \wedge \\ (\neg s \vee \neg t_{\neg s}) \end{cases}$$

SPBN And IPBN

Definition (SPBN formula)

An SPBN formula ϕ is a PBN formula where, for all $c \in Pos(\phi)$, $|\mathcal{P}(Pos(\phi) \setminus \{c\}) \cap c| \leqslant 1$.

Example

$$\phi_{SPBN} = (p \vee q) \wedge (p \vee r) \wedge (t \vee u) \wedge (\neg q \vee \neg r) \wedge (\neg p \vee \neg u)$$

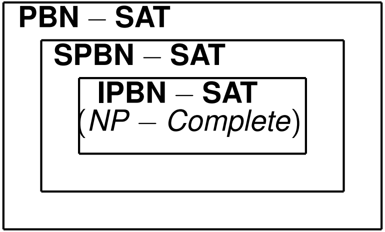
Definition (IPBN formula)

An IPBN formula ϕ is a PBN formula where, for all $c, c' \in Pos(\phi)$ with $c \neq c'$, $c \cap c' = \emptyset$.

Example

$$\phi_{IPBN} = (p \vee q \vee v) \wedge (r \vee s \vee w) \wedge (t \vee u) \wedge (\neg p \vee \neg t) \wedge (\neg q \vee \neg r)$$

A hierarchy of syntactic fragments of SAT



Example of a IPBN formula

- The pigeon-hole principle PHP_n^b: mapping b pigeons to n holes, with $b > n$
- The pigeon-hole principle php_n^b is expressed using the following IPBN formula:

$$\bigvee_{j=1}^{n} p_{ij}, 1 \leqslant i \leqslant b \tag{1}$$

$$\neg p_{ij} \vee \neg p_{kj}, 1 \leqslant i < k \leqslant b \text{ and } 1 \leqslant j \leqslant n \tag{2}$$

where $p_{ij} = 1$ iff the pigeon i is put in the hole j.

- Each positive clause of $\phi_{php_n^b}$ does not share any literals with the other positive clauses.

PBN formula

- Given a PBN formula ϕ, $\mathcal{R}(\phi)$ to denote the PBN formula $\phi \cup \{\neg p \vee \neg q \mid p \neq q \textit{ and } \exists c \in Pos(\phi) \ s.t. \ \{p, q\} \subseteq c\}$.

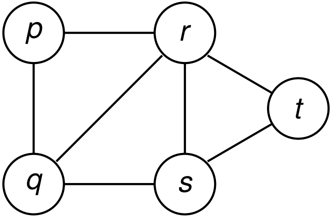
Proposition

Given an PBN formula ϕ, ϕ is satisfiable iff $\mathcal{R}(\phi)$ is satisfiable.

- The previous proposition shows that we only need to satisfy one literal in each positive clause.

Tractable Classes in IPBN-SAT: transformation

$$\phi_{IPBN} = \begin{cases} (p \vee q \vee r) \;\; \wedge \\ (s \vee t) \;\; \wedge \\ (\neg q \vee \neg s) \;\; \wedge \\ (\neg r \vee \neg s) \wedge \\ (\neg r \vee \neg t) \end{cases}$$



Tractable Classes in IPBN-SAT: SAT and MIS

Let ϕ be an IPBN formula.
$G_\phi = (V, E)$ denotes the undirected graph of ϕ. Where $V = Var(\phi)$ and $\{p, q\} \in E$ iff p and q are in the same clause.

Theorem

Given an IPBN formula ϕ, ϕ is satisfiable iff $\alpha(G_\phi) \geqslant n$ where $n = |Pos(\phi)|$.

- It suffices to satisfy one literal by positive clause.

Tractable Classes in IPBN-SAT: recognition

- If G_ϕ is claw-free so the corresponding IPBN formula ϕ is claw-free.

Theorem

(recognition) Checking whether a CNF formula is claw-free is tractable.

- Checking whether the corresponding graph is claw-free is tractable

Tractable Classes in IPBN-SAT: claw-free IPBN

Theorem

(satisfiability) Checking the satisfiability of a claw-free IPBN formula is tractable.

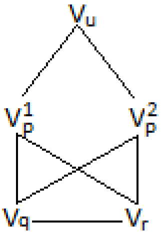
- A direct consequence of the first theorem.
- MIS in the class of claw-free graphs is tractable.

Proposition

The pigeon-hole principle is an instance of claw-free IPBN-SAT.

Tractable Classes in SPBN-SAT: transformation

- Example: $\phi = (p \vee q) \wedge (p \vee r) \wedge (\neg p \vee \neg u) \wedge (\neg q \vee \neg r)$
- $S(p, \phi) = \{V_p^1, V_p^2\}$, $S(q, \phi) = \{V_q\}$,...
- Edges are added between all occurrences of p ($\{v_p^1, v_p^2\}$) and q ($\{v_q\}$).



Tractable Classes in SPBN-SAT

Theorem

Given an SPBN formula ϕ, ϕ is satisfiable iff $\alpha(G_\phi \geqslant n$ where $n = |Pos(\phi)|$.

- Generalization of the first theorem.
- The occurrences of the positive literals are taken into account, since positive clauses may share literals.

Tractable Classes in PBN-SAT: CC-PBN

- Let ϕ be a PBN formula
- We note $C(p, \phi) = \{c \in \phi | p \in c\}$ the "*cover*" of p in ϕ

Definition

A CC-PBN formula ϕ is a PBN formula where, for all $p, q \in Var(\phi)$ with $p, q \in c$ for a clause $c \in Pos(\phi)$, we have $C(p, Pos(\phi)) \subseteq C(q, Pos(\phi))$ or $C(q, Pos(\phi)) \subseteq C(p, Pos(\phi))$.

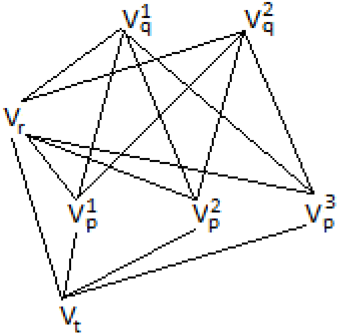
Tractable Classes in PBN-SAT: CC-PBN example

Example

ϕ is a CC-PBN formula:

$$\phi = ([p] \vee [q] \vee r) \wedge \\ ([p] \vee [q]) \wedge \\ ([p] \vee t) \wedge \\ (\neg r \vee \neg t)$$

Note: $C(q, Pos(\phi)) \subset C(p, Pos(\phi))$



Tractable Classes in PBN-SAT: CC-PBN

Proposition

Given an CC-PBN formula ϕ, ϕ is satisfiable iff $\mathcal{R}(\phi)$ is satisfiable.

Theorem

Given a CC-PBN formula ϕ, ϕ is satisfiable iff $\alpha(G_{\phi}^{\mathcal{O}}) \geqslant n$ where $n = |Pos(\phi)|$.

Plan

Propositional Satisfiability (SAT) - Maximum Independent Set (MIS)

Tractability Results: from MIS to SAT

A Connection with Minimal Models

Conclusion and perspectives

A Connection with Minimal Models

Characterization of the minimal models in CC-PBN formulæ using the MIS.

Definition

Let ϕ be a propositional formula and $\mathcal{I}$ a model of ϕ. Then $\mathcal{I}$ is said to be a *minimal model* of ϕ if there is no model strictly smaller than *I* w.r.t. $\preceq$.

$\rightarrow \mathcal{I} \preceq \mathcal{I}'$ if $\{p \in \mathcal{P}(\phi) | \mathcal{I}(p) = 1\} \subseteq \{p \in \mathcal{P}(\phi) | \mathcal{I}'(p) = 1\}$

Proposition

Let ϕ be a CC-PBN formula and $\mathcal{I}$ a model of ϕ. Then, $\mathcal{I}$ is a minimal model of ϕ iff $\mathcal{I}$ satisfies exactly one literal in each positive clause.

Theorem

Let ϕ be a CC-PBN formula and $\mathcal{I}$ a Boolean interpretation of ϕ. $\mathcal{I}$ is a minimal model of ϕ iff $\alpha(G_\phi) = |Pos(\phi)|$ and $\bigcup_{p \in E} S(p, \phi)$ is a maximum independent set of G_ϕ where $E = \{p \in \mathcal{P}(\phi) \mid \mathcal{I}(p) = 1\}$.

Plan

Conclusion and perspectives

- Cross-fertilization between MIS and SAT
- New tractable classes.
- Pigeonhole problem belongs to one of these classes: IPBN.

- Characterization of new tractable classes modulo occurrences of the variables in the positives clauses, e.g: the Tovey classes.
- PBN is a suitable form for connecting tractables classes between CSP and SAT

Thank you.