Bunched hypersequent calculi for distributive substructural logics

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Preliminaries: proof calculi for intuitionistic logic up to relevant logic

- Sequent calculus for propositional intuitionistic logic Ip (Gentzen 1935)

$$p \Rightarrow p \qquad \bot; \Gamma \Rightarrow D \qquad \Gamma \Rightarrow \top \qquad \frac{\Gamma \Rightarrow D}{\Gamma; \Delta \Rightarrow D}\,\text{weak}$$

$$\frac{\Gamma; \Delta; \Delta \Rightarrow D}{\Gamma; \Delta \Rightarrow D}\,\text{ctr} \qquad \frac{\Gamma \Rightarrow A \qquad B; \Delta \Rightarrow D}{A \to B; \Gamma; \Delta \Rightarrow D}\,{\to}\text{l} \qquad \frac{A; \Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B}\,{\to}r$$

$$\frac{A_1; A_2; \Gamma \Rightarrow D}{A_1 \wedge A_2; \Gamma \Rightarrow D}\,{\wedge}l \qquad \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\,{\wedge}r \qquad \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2}\,{\vee}r$$

$$\frac{\Gamma; A; B; \Delta \Rightarrow D}{\Delta; B; A; \Gamma \Rightarrow D}\,\text{e} \qquad \frac{A; \Gamma \Rightarrow D \qquad B; \Gamma \Rightarrow D}{A \vee B; \Gamma \Rightarrow D}\,{\vee}l$$

- $\Rightarrow A$ is derivable in the calculus iff $A \in$ Ip
- The antecedent is a semicolon-separated list of formulae
- The calculus has the subformula property: every formula in the premise is a subformula in the conclusion
- Aside. Permit multiple formula in succedent to get classical logic

Calculus for propositional intuitionistic logic Ip (Gentzen 1935)

$$p \Rightarrow p \qquad \bot; \Gamma \Rightarrow D \qquad \Gamma \Rightarrow \top \qquad \frac{\Gamma \Rightarrow D}{\Gamma; \Delta \Rightarrow D}\,\text{weak}$$

$$\frac{\Gamma; \Delta; \Delta \Rightarrow D}{\Gamma; \Delta \Rightarrow D}\,\text{ctr} \qquad \frac{\Gamma \Rightarrow A \qquad B; \Delta \Rightarrow D}{A \to B; \Gamma; \Delta \Rightarrow D}\,\to\text{l} \qquad \frac{A; \Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B}\,\to r$$

$$\frac{A_1; A_2; \Gamma \Rightarrow D}{A_1 \wedge A_2; \Gamma \Rightarrow D}\,\wedge l \qquad \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\,\wedge r \qquad \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2}\,\vee r$$

$$\frac{\Gamma; A; B; \Delta \Rightarrow D}{\Delta; B; A; \Gamma \Rightarrow D}\,\text{e} \qquad \frac{A; \Gamma \Rightarrow D \qquad B; \Gamma \Rightarrow D}{A \vee B; \Gamma \Rightarrow D}\,\vee l \qquad \frac{\Gamma \Rightarrow A \qquad A, \Delta \Rightarrow D}{\Gamma, \Delta \Rightarrow D}\,\text{cut}$$

- Contrast with the Hilbert calculus which contains the rule of

modus ponens: if A and $A \to B$, then B

- This rule violates subformula property
- Subformula property is critical for making arguments e.g.

Consistency ($\Rightarrow \bot$ not derivable) and PSPACE-complexity

- Gentzen generalises *modus ponens* to cut rule and then proves his cut-elimination theorem

Substructural logics. The Lambek calculus with exchange $\mathbf{FL}_e$

- Remove some of the properties of the structural connective to obtain substructural logics
- For FL_e, the rules of contraction and weakening have been deleted. This allows us to define distinct connectives $\otimes$ and $\wedge$.

$$p \Rightarrow p \qquad \bot, \Gamma \Rightarrow D \qquad \Rightarrow 1 \qquad \Gamma \Rightarrow \top \qquad \frac{A_1, A_2, \Gamma \Rightarrow D}{A_1 \otimes A_2, \Gamma \Rightarrow D}\ \otimes\mathrm{l}$$

$$\frac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A \otimes B}\ \otimes\mathrm{r} \qquad \frac{\Gamma \Rightarrow A \qquad B, \Delta \Rightarrow D}{A \multimap B, \Gamma, \Delta \Rightarrow D}\ \multimap\mathrm{l} \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \multimap B}\ \multimap r$$

$$\frac{A_i, \Gamma \Rightarrow D}{A_1 \wedge A_2, \Gamma \Rightarrow D}\ \wedge l \qquad \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\ \wedge r \qquad \frac{\Gamma, A, B, \Delta \Rightarrow D}{\Delta, B, A, \Gamma \Rightarrow D}\ \mathrm{e}$$

$$\frac{A, \Gamma \Rightarrow D \qquad B, \Gamma \Rightarrow D}{A \vee B, \Gamma \Rightarrow D}\ \vee l \qquad \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2}\ \vee r$$

- The antecedent is a comma-separated list of formulae
- (We could even delete the exchange rule and/or consider non-associativity)
- Once again, cut-elimination holds so we omit the cut rule

An observation: $\mathbf{FL}_e$ is not distributive

$$A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C) \text{ is not derivable}$$

Proof: what rule can be applied to obtain this sequent? (4 possibilities)

$$\frac{A \Rightarrow (A \wedge B) \vee (A \wedge C)}{A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C)} \wedge\mathrm{l}$$

$$\frac{B \vee C \Rightarrow (A \wedge B) \vee (A \wedge C)}{A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C)} \wedge\mathrm{l}$$

$$\frac{A \wedge (B \vee C) \Rightarrow A \wedge B}{A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C)} \vee\mathrm{r}$$

$$\frac{A \wedge (B \vee C) \Rightarrow A \wedge C}{A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C)} \vee\mathrm{r}$$

Observe that no premise is valid (or continue backward proof search)

- The need for distributivity arises e.g. in relevant logics.

A bunched calculus sDFL$_e$ for DFL$_e$ (Dunn 1974, Mints 1976)

$$p \Rightarrow p \qquad \bot, \Gamma \Rightarrow D \qquad \Rightarrow 1 \qquad \Gamma \Rightarrow \top \qquad \frac{\Gamma[A_1, A_2] \Rightarrow D}{\Gamma[A_1 \otimes A_2] \Rightarrow D}\ \otimes\mathrm{l}$$

$$\frac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A \otimes B}\ \otimes\mathrm{r} \qquad \frac{\Gamma \Rightarrow A \qquad \Sigma[B] \Rightarrow D}{\Sigma[\Gamma, A \to B] \Rightarrow D}\ \multimap\mathrm{l} \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B}\ \multimap r$$

$$\frac{\Gamma[A_1; A_2] \Rightarrow D}{\Gamma[A_1 \wedge A_2] \Rightarrow D}\ \wedge l \qquad \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\ \wedge r \qquad \frac{\Sigma[\Gamma, \Delta] \Rightarrow A}{\Sigma[\Delta, \Gamma] \Rightarrow A}\ \text{(m-e)}$$

$$\frac{\Gamma[A] \Rightarrow D \qquad \Gamma[B] \Rightarrow D}{\Gamma[A \vee B] \Rightarrow D}\ \vee l \qquad \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2}\ \vee r \qquad \frac{\Sigma[(X, Y), Z] \Rightarrow A}{\Sigma[X, (Y, Z)] \Rightarrow A}\ \text{(m-as)}$$

$$\frac{\Sigma[(X; Y); Z] \Rightarrow A}{\Sigma[X; (Y; Z)] \Rightarrow A}\ \text{(a-as)} \qquad \frac{\Sigma[X; Y] \Rightarrow A}{\Sigma[Y; X] \Rightarrow A}\ \text{(a-ex)} \qquad \frac{\Sigma[X] \Rightarrow A}{\Sigma[X; Y] \Rightarrow A}\ \text{(a-w)}$$

$$\frac{\Sigma[X; X] \Rightarrow A}{\Sigma[X] \Rightarrow A}\ \text{(a-ctr)}$$

- The antecedent has two structure connectives: comma and semicolon
- Comma $\rightsquigarrow$ multiplicative connectives. Semicolon $\rightsquigarrow$ additive connectives

Derivation of $A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C)$ in sDFL$_e$

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{A \Rightarrow A}{A; B \Rightarrow A}
\qquad
\dfrac{B \Rightarrow B}{A; B \Rightarrow B}
}{A; B \Rightarrow A \wedge B}
}{A; B \Rightarrow (A \wedge B) \vee (A \wedge C)}
\qquad
\dfrac{
\dfrac{
\dfrac{A \Rightarrow A}{A; C \Rightarrow A}
\qquad
\dfrac{C \Rightarrow C}{A; C \Rightarrow C}
}{A; C \Rightarrow A \wedge C}
}{A; C \Rightarrow (A \wedge B) \vee (A \wedge C)}
}{
\dfrac{A; B \vee C \Rightarrow (A \wedge B) \vee (A \wedge C)}{A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C)}
}
$$

Bunched (hyper)sequent calculi for distributive substructural logics

- How can we construct calculi with the subformula property for axiomatic extensions of DFL_e?
- (Ciabattoni, Galatos, Terui 2008) develop a general method for (hyper)sequent calculi
- To extend these methods to bunched (hyper)sequent calculi we
  - (i) Interpret the additional structure and prove a cut-elimination theorem on this extended structure.
  - (ii) (This yields an algorithm for transforming an axiom into a structural rule)
  - (iii) Characterise those axiom extensions that can be presented
  - (iv) We also consider the special case of the logic of bunched implication BI (DFL_e with two implications defined on $\Rightarrow$) where the above interpretation does not hold.
- Underlying aim: present logics in a simple extension of the sequent calculus, to permit applications of the calculus

e.g. decidability, complexity, proof search, interpolation, standard completeness arguments

Example: A calculus for $\mathrm{DFL}_e + (1 \wedge (p \otimes q)) \multimap p$

- $(1 \wedge (p \otimes q)) \multimap p \rightsquigarrow$ restricted weakening. Using invertible rules backwards:

$$\dfrac{\dfrac{1, (1; (p, q)) \Rightarrow p}{1, (1 \wedge (p \otimes q)) \Rightarrow p}}{1 \Rightarrow (1 \wedge (p \otimes q)) \multimap p}$$

- So it suffices to derive $1, (1; (p, q)) \Rightarrow p$. In the presence of cut the following equivalences hold ('Ackermann's lemma')

$$1, (1; (p, q)) \Rightarrow p \qquad \dfrac{X \Rightarrow p}{\varnothing_a, (\varnothing_a; (X, q)) \Rightarrow p}$$

$$\dfrac{X \Rightarrow p \qquad Y \Rightarrow q}{\varnothing_a, (\varnothing_a; (X, Y)) \Rightarrow p} \qquad \dfrac{X \Rightarrow p \quad Y \Rightarrow q \quad \Gamma[p] \Rightarrow B}{\varnothing_a, (\varnothing_a; (X, Y)) \Rightarrow B}$$

- Apply all possible cuts to the premises (assuming termination) to get the equivalent rules

$$\dfrac{\Gamma[X] \Rightarrow B \quad Y \Rightarrow q}{\varnothing_a, (\varnothing_a; (X, Y)) \Rightarrow B} \qquad \dfrac{\Gamma[X] \Rightarrow B}{\varnothing_a, (\varnothing_a; (X, Y)) \Rightarrow B}\,\mathrm{r}$$

- $s\mathrm{DFL}_e + r + cut$ is sound and complete for $\mathrm{DFL}_e + (1 \wedge (p \otimes q)) \multimap p$. By our cut-elimination theorem, so is $s\mathrm{DFL}_e + r$. This has the subformula property.

An example where the argument fails

$$\mathrm{DFL}_e + (p \multimap 0) \vee ((p \multimap 0) \multimap 0)$$

- Applying invertible rules to $1 \Rightarrow (p \multimap 0) \vee ((p \multimap 0) \multimap 0)$ we get

$$\varnothing_m \Rightarrow (p \multimap 0) \vee ((p \multimap 0) \multimap 0)$$

- Applying Ackermann lemma (below left), then invertible rule ($\vee$l):

$$\frac{(p \multimap 0) \vee ((p \multimap 0) \multimap 0) \Rightarrow X}{\varnothing_m \Rightarrow X} \qquad \frac{(p \multimap 0) \Rightarrow X \qquad ((p \multimap 0) \multimap 0) \Rightarrow X}{\varnothing_m \Rightarrow X}$$

- The rule above right violates the subformula property...
- ...and yet there is no way to proceed. There are no invertible rules to apply
- ...and Ackermann's lemma does not simplify premises
- It seems that structural rules extensions of $s\mathrm{DFL}_e$ are not expressive enough to present $\mathrm{DFL}_e + (p \multimap 0) \vee ((p \multimap 0) \multimap 0)$
- We need to extend the sequent formalism further...

Bunched hypersequent calculus for $\mathbf{DFL}_e + (p \multimap 0) \vee ((p \multimap 0) \multimap 0)$ **(I)**

- A natural extension of a sequent $\Gamma \Rightarrow A$ is to a non-empty set of sequents (Avron 1996, Pottingern 1983)

$$\Gamma_1 \Rightarrow A_1 \mid \Gamma_2 \Rightarrow A_2 \mid \ldots \mid \Gamma_{n+1} \Rightarrow A_{n+1}$$

- Here we take the analogous extension of $s\mathrm{DFL}_e$ with hypersquent structure
- The hypersequent calculus $h\mathrm{DFL}_e$ is obtained from $s\mathrm{DFL}_e$ as follows:

Add a hypersequent context "$g \mid$" to each rule. Also add rules manipulating the components

$$\frac{g \mid \Gamma, A \Rightarrow B}{g \mid \Gamma \Rightarrow A \multimap B} \multimap\mathrm{r} \qquad \frac{h \mid h \mid g}{h \mid g} \mathrm{EC} \qquad \frac{g}{h \mid g} \mathrm{EC}$$

Bunched hypersequent calculus for $\mathbf{DFL}_e + (p \multimap 0) \vee ((p \multimap 0) \multimap 0)$ (II)

- Prove soundness of $h\mathrm{DFL}_e$ wrt DFL_e interpreting | as disjunction
- (Contrast with FL_e: $\Gamma_1 \Rightarrow A_1 \mid \Gamma_2 \Rightarrow A_2 \rightsquigarrow ((\Gamma_1^l \multimap A_1) \wedge 1) \vee ((\Gamma_2^l \multimap A_2) \wedge 1))$
- Therefore the following is an equivalent calculus.

$$h\mathrm{DFL}_e + g \mid 1 \Rightarrow p \multimap 0 \mid 1 \Rightarrow (p \multimap 0) \multimap 0$$

- Applying invertible rules:

$$g \mid 1 \Rightarrow p \multimap 0 \mid 1 \Rightarrow (p \multimap 0) \multimap 0 \qquad g \mid \varnothing_m, p \Rightarrow \mathcal{O}_m \mid \varnothing_m, p \multimap 0 \Rightarrow \mathcal{O}_m$$

- Now repeatedly apply Ackermann's lemma to above right to get:

$$\frac{g \mid X \Rightarrow p \qquad g \mid Y \Rightarrow p \multimap 0}{g \mid \varnothing_m, X \Rightarrow \mathcal{O}_m \mid \varnothing_m, Y \Rightarrow \mathcal{O}_m}$$

- Applying invertible rules and all possible cuts we obtain a structural rule

$$\frac{g \mid X \Rightarrow p \qquad g \mid p, Y \Rightarrow \mathcal{O}_m}{g \mid \varnothing_m, X \Rightarrow \mathcal{O}_m \mid \varnothing_m, Y \Rightarrow \mathcal{O}_m} \qquad \frac{g \mid X, Y \Rightarrow \mathcal{O}_m}{g \mid \varnothing_m, X \Rightarrow \mathcal{O}_m \mid \varnothing_m, Y \Rightarrow \mathcal{O}_m}\,r$$

- $h\mathrm{DFL}_e + r$ (via cut-elimination) is a calculus for $\mathrm{DFL}_e + (p \multimap 0) \vee ((p \multimap 0) \multimap 0)$

The substructural hierarchy over DFL_e

- We can characterise the extensions of DFL_e that can be presented
- Following (Ciabattoni, Galatos, Terui 2008), set $\mathcal{N}_0^d$ and $\mathcal{P}_0^d$ as the set of propositional variables, and define

$$\mathcal{P}_{n+1}^d ::= 1 \mid \mathcal{N}_n^d \mid \mathcal{P}_{n+1}^d \otimes \mathcal{P}_{n+1}^d \mid \mathcal{P}_{n+1}^d \wedge \mathcal{P}_{n+1}^d \mid \mathcal{P}_{n+1}^d \vee \mathcal{P}_{n+1}^d$$
$$\mathcal{N}_{n+1}^d ::= \mathcal{O}_m \mid \mathcal{P}_n^d \mid \mathcal{N}_{n+1}^d \wedge \mathcal{N}_{n+1}^d \mid \mathcal{P}_{n+1}^d \multimap \mathcal{N}_{n+1}^d$$

- The positive classes $\mathcal{P}_i$ contain formulae whose most external connective is invertible on the left
- The negative classes $\mathcal{N}_i$) contain formulae whose most external connective is invertible on the right

Theorem

Every extension of DFL_e by a disjunction of $\mathcal{N}_2^d$ axioms computes a structural rule extension of $hDFL_e$ when the cuts on the premises terminate.

The logic of bunched implications BI (O'Hearn and Pym, 1999)

- BI can be used for resource composition and systems modelling and as a propositional fragment of separation logic
- Bunched calculus: extend $s\mathrm{DFL}_e$ with an intuitionistic implication $\rightarrow$

$$\frac{\Gamma \Rightarrow A \qquad \Sigma[B] \Rightarrow D}{\Sigma[\Gamma; A \rightarrow B] \Rightarrow D} \rightarrow l \qquad \frac{A; \Gamma \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B} \rightarrow r$$

- sBI has two implications: multiplicative $\multimap$ and $\rightarrow$, both defined wrt $\Rightarrow$
- Recall...

$$\frac{\Gamma \Rightarrow A \qquad \Sigma[B] \Rightarrow D}{\Sigma[\Gamma, A \multimap B] \Rightarrow D} \multimap l \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \multimap B} \multimap r$$

- Algebraic semantics: Heyting (intuitionistic) algebras with a commutative monoidal operation $\otimes$ and residuated implication $\multimap$

i.e. $x \otimes y \leq z$ iff $x \leq y \multimap z$ where $\leq$ is the Heyting partial order

A calculus for $BI + 1 \Rightarrow p \vee (p \rightarrow \bot)$ (BBI): an attempt (I)

- Boolean BI is the counterpart of BI with intuitionistic logic replaced by classical logic
- BBI is the propositional basis of separation logic (more widely used than BI)
- BBI is undecidable (Larchey-Wendling and Galmiche, 2010)
- We cannot extend BI by permitting multiple formulae in the succedent (analogous of LJ $\rightsquigarrow$ LK) because the standard cut-elimination fails due to the two types of structural connectives in the antecedent
- Idea: add hypersequent structure to *s*BI to interpret as before:

$$1 \Rightarrow p \vee (p \rightarrow \bot) \qquad 1 \Rightarrow p \,|\, 1 \Rightarrow (p \rightarrow \bot)$$

- However: the two right implication rules do not permit a (formula) interpretation of $\Rightarrow$
- If we cannot interpret $\Rightarrow$ then we cannot interpret $|$
- So the obvious extension of the $h\mathrm{DFL}_e$ method to BBI fails.

A calculus for $BI + 1 \Rightarrow p \vee (p \rightarrow \bot)$ (BBI): an attempt (II)

- Nevertheless we can consider the sequent consequences of the hypersequent calculus $h\mathrm{BI} + r$ for some structural rule r

$$\{\Gamma \Rightarrow A \quad | \quad \Gamma \Rightarrow A \text{ derivable in } h\mathrm{DFL}_e + r\}$$

- Our proof of cut-elimination extends to structural rule extensions of hBI
- Idea: add a structural rule which derives desired sequent, use the subformula property to check the consistency of structural rule extensions
- It remains to | interpret wrt the semantics of BI (future work)
- Aside. Recent work (Ciabattoni, Galatos, Terui 2016) interprets | for (non-commutative) FL as a special disjunction built from 'interated conjugates'
- Can we find interesting resource interpretations for such logics? Can we regain decidability for BBI-like logics?