Blocked Clauses in First-Order Logic

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Motivation

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- We give a polynomial time algorithm for deciding whether a clause is blocked.
- We implement blocked-clause elimination as a preprocessing technique for first-order theorem provers.
  - We evaluate the effectiveness for various provers on the TPTP benchmark library.

Outline

1. Background:
   - Overview on preprocessing techniques.
   - Blocked clauses in propositional logic.
2. Blocked clauses in first-order logic without equality
3. Blocked clauses in first-order logic with equality (*equality-blocked clauses*)
4. Complexity of detecting blocked clauses.
5. Evaluation results for first-order blocked-clause elimination.

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  - As we will see, blocked clauses are redundant too.

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- Example:

C: $x \vee y \vee b$

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$\neg b \vee \neg y$

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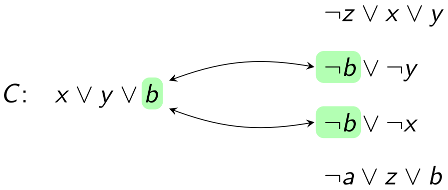

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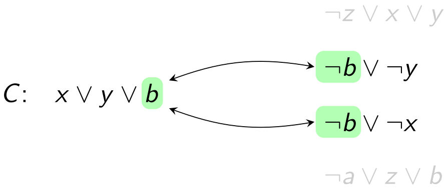

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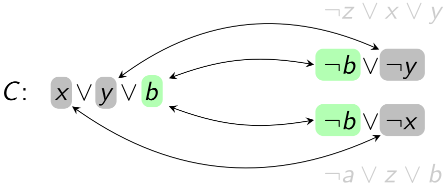

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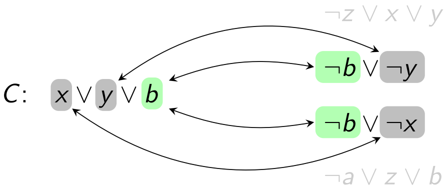

- Blocked clauses are redundant, so they can be safely removed/added.

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$\neg z \vee x \vee y$

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$C\colon \quad x \vee y \vee b$

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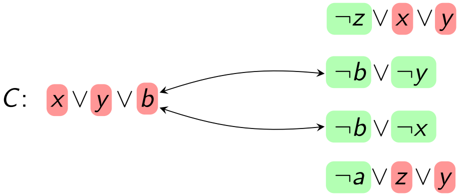

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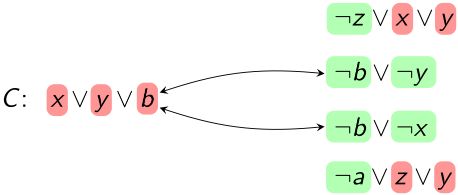

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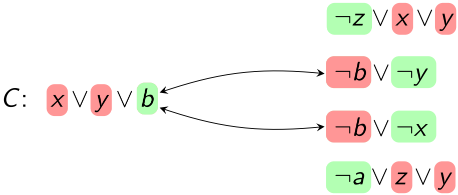

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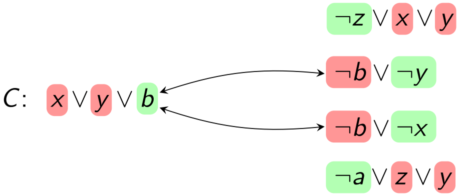

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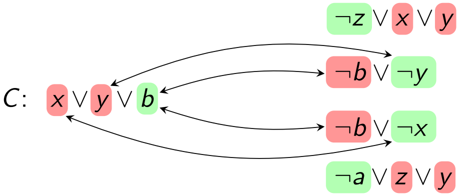

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➥ The problem is factoring: $\neg P(u, v)$ and $\neg P(v, u)$ unify.

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- Better approach: Take care of factoring.

Definition (L-resolvent)

Let $C = L \lor C$ and $D = N_1 \lor \cdots \lor N_m \lor D$ be clauses where $L, \bar{N}_1, \ldots, \bar{N}_m$ are unifiable by an mgu σ.

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  3. $P(x, x)$ (not a tautology!) resolution with both $\neg P(u, v)$ and $\neg P(v, u)$.

Definition

A clause $C \vee L$ is blocked by L in a formula F if all L-resolvents of $C \vee L$ with clauses in $F \setminus \{C \vee L\}$ are tautologies.

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  - Every model of F must falsify $P(a)$.

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- Let $C = P(a)$ and $F = \{a \approx b, \neg P(b)\}$
  - There are no resolvents of C since $P(a)$ and $P(b)$ are not unifiable (a and b are constants)
  - ➥ C is blocked w.r.t. F.
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  - Every model of F must falsify $P(a)$.

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$$x \not\approx a \vee x \not\approx b,$$

is not valid.

Blocked Clauses in First-Order Logic With Equality

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Let $C = L(t_1, \ldots, t_n) \vee C'$. Flattening the literal $L(t_1, \ldots, t_n)$ in C yields the clause $C^- = \bigvee_{1 \leq i \leq n} x_i \not\approx t_i \vee L(x_1, \ldots, x_n) \vee C'$, with $x_i, \ldots, x_n$ being fresh variables not occurring in C.

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➥ Equality-blocked clauses are redundant in first-order logic with equality.

Outline

1. Background:
    - Overview on preprocessing techniques.
    - Blocked clauses in propositional logic.
2. Blocked clauses in first-order logic without equality
3. Blocked clauses in first-order logic with equality
   (*equality-blocked clauses*)
4. **Complexity of detecting blocked clauses.**
5. Evaluation results for first-order blocked-clause elimination.

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The Main Challenge

Given $C = L \vee C'$ and $D = N_1 \vee \cdots \vee N_n \vee D'$ such that $L, \bar{N}_1, \ldots, \bar{N}_n$ unify, there are $2^n - 1$ L-resolvents whose validity should be checked!

Testing Validity of all L-resolvents

Algorithm 1:

Input:

Candidate $C = L \vee C'$ and a partner $D = N_1 \vee \ldots \vee N_n \vee D'$, where $N_1, \ldots, N_n$ are all the literals of D which pairwise unify with $\bar{L}$

Output:

Are all L-resolvents of $L \vee C'$ and D valid?

for $k \leftarrow 1, \ldots, n$ **do**
  $N \leftarrow \{N_k\}$
  while L unifiable with literals $\bar{N}$ via an *mgu* σ **do**
    $K \leftarrow$ all pairs of complementary literals in $C'\sigma \vee (D \setminus N)\sigma$
    if $K = \emptyset$ **then**
      return NO
    if every pair of complementary literals in K contains a literal $N_i\sigma$ **then**
      $N \leftarrow N \cup \{N_i \mid N_i\sigma$ is part of a complementary pair$\}$
    else
      break (the while loop)
return YES

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Blocked Clause Elimination (BCE) – Implementation

Implemented in automated theorem prover Vampire

- as an optional preprocessing step: `-bce on`
- blocked / equality-blocked based on the presence of $\approx$
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The main loop

- index clauses by predicate and polarity
- priority queue of candidates (C, L); the least effort first
- if candidate (C, L) and partner D do not yield a valid L-resolvent store (C, L) with D for potential later "resurrection"

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Single candidate-partner check

- cheap but safe approximation of Algorithm 1
- cheap approximation of congruence closure reasoning

Do Blocked Clauses Occur in Practice?

The setup:

- All first-order benchmarks from TPTP 6.4.0:
  - 15 942 problems: 8044 general FO, 7898 CNF
  - 73 % with equality
- 300 s for parsing, (clausification), and blocked-clause elimination

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Results

- Blocked-clause elimination finished on all but one problems.
- Average time: 0.238 s, median 0.001 s.
- 11.72 % of the collective total of 299 379 591 clauses are blocked.
- 59 % of problems contain a blocked clause.
- In more than 1000 problems, 25 % of the clauses could be eliminated.
- 113 satisfiable formulas directly solved by blocked-clause elimination.

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Strategies for proving theorems (CASC 2016 FOF champions)

	unsatisfiable			satisfiable			total		
Vampire	**3172**	−28	+40	**458**	−0	+5	**3630**	−28	+45
E	**3097**	−20	+27	**363**	−1	+9	**3460**	−21	+36
CVC4	**2930**	−18	+37	**9**	−0	+68	**2939**	−18	+105

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Satisfiability checking strategies (CASC 2016 FNT champions)

	satisfiable			unsatisfiable			total		
Vampire	**531**	−0	+24	**719**	−4	+5	**1250**	−4	+29
iProver	**558**	−0	+1	**755**	−6	+4	**1313**	−6	+5
CVC4	**489**	−1	+28	**1724**	−24	+20	**2213**	−25	+48

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Results

- w/o BCE: 6766 successes, with BCE: **8414** successes
- only w/o BCE: 148, only with BCE: **1796**

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  (such as predicate elimination [KK2016]).
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- Beyond elimination: addition and decomposition in FO?

BCE and Unused Definition Elimination

Propositionally, BCE simulates UDE:

$$\neg p \lor \varphi \qquad p \lor \neg\varphi \qquad \psi[p]$$

for any $\neg p \lor C \in CNF(\varphi)$ and $p \lor D \in CNF(\neg\varphi)$ the resolvents

$$C \lor D$$

are tautologies (unless we name "asymmetrically" inside φ)

This breaks down with quantifiers:

Example

Given a definition $p(x) \equiv \exists y.q(x,y)$, trying to block

$$\neg p(x) \lor q(x, sk_{\exists y.q(x,y)}(x)) \qquad p(x) \lor \neg q(x,y)$$