Set of Support for Theory Reasoning

Giles Reger[1], Martin Suda[2]

[1]School of Computer Science, University of Manchester, UK
[2]TU Wien, Vienna, Austria

IWIL 2017 – Maun, May 7, 2017

Theory axioms in proofs

Consider the following toy theory problem

$$f(1+a) < a, \qquad \forall x.(x < f(x+1))$$

Theory axioms in proofs

Consider the following toy theory problem

$$f(1+a) < a, \qquad \forall x.(x < f(x+1))$$

can be refuted by Vampire via the following derivation:

$$\cfrac{\cfrac{\cfrac{x+y=y+x \qquad x < f(x+1)}{x < f(1+x)} \qquad \cfrac{\neg x < y \vee \neg y < z \vee x < z \qquad f(1+a) < a}{\neg(x < f(1+a)) \vee x < a}}{a < a} \qquad \neg(x < x)}{\bot}$$

Theory axioms in proofs

Consider the following toy theory problem

$$f(1+a) < a, \qquad \forall x.(x < f(x+1))$$

can be refuted by Vampire via the following derivation:

$$\dfrac{\dfrac{\dfrac{x+y=y+x \qquad x < f(x+1)}{x < f(1+x)} \qquad \dfrac{\neg x < y \vee \neg y < z \vee x < z \qquad f(1+a) < a}{\neg(x < f(1+a)) \vee x < a}}{a < a} \qquad \neg(x < x)}{\bot}$$

However, in the meantime, the theory axioms may also yield:

$$\neg(x < y) \vee \neg(y < x)$$

or (perhaps less usefully):

$$\neg(x_0 < x_1) \vee \neg(x_2 < x_0) \vee \neg(x_1 < x_3) \vee \neg(x_4 < x_5) \vee \neg(x_3 < x_4) \vee \neg(x_5 < x_2)$$

Inferences between axioms

Example problem `ARI176=1` from TPTP

$$3x + 5y \neq 22$$

can be shown unsatisfiable using axioms

$$x{+}y = y{+}x, \quad x{+}(y{+}z) = (x{+}y){+}z, \quad x{*}1 = x, \quad x{*}(y{+}z) = (x{*}y){+}(x{*}z)$$

Inferences between axioms

Example problem `ARI176=1` from TPTP

$$3x + 5y \neq 22$$

can be shown unsatisfiable using axioms

$x{+}y = y{+}x, \quad x{+}(y{+}z) = (x{+}y){+}z, \quad x{*}1 = x, \quad x{*}(y{+}z) = (x{*}y){+}(x{*}z)$

The derivation starts by:

$$\dfrac{\dfrac{x * 1 = x \qquad x * (y + z) = (x * y) + (x * z)}{x * (1 + y) = x + (x * y)} \qquad x + (y + z) = (x + y) + z}{(x * (1 + y)) + z = x + ((x * y) + z)}$$

Inferences between axioms

Example problem `ARI176=1` from TPTP

$$3x + 5y \neq 22$$

can be shown unsatisfiable using axioms

$x{+}y = y{+}x, \quad x{+}(y{+}z) = (x{+}y){+}z, \quad x{*}1 = x, \quad x{*}(y{+}z) = (x{*}y){+}(x{*}z)$

The derivation starts by:

$$\dfrac{\dfrac{x * 1 = x \qquad x * (y + z) = (x * y) + (x * z)}{x * (1 + y) = x + (x * y)} \qquad x + (y + z) = (x + y) + z}{(x * (1 + y)) + z = x + ((x * y) + z)}$$

The problem cannot be solved in Vampire in reasonable time without first combining axioms among themselves

- One useful technique for reasoning with theories and quantifiers is the addition of theory axioms

This talk in a nutshell

- One useful technique for reasoning with theories and quantifiers is the addition of theory axioms
- Quite successful in many cases.
  However, many axioms can be “explosive”.

This talk in a nutshell

- One useful technique for reasoning with theories and quantifiers is the addition of theory axioms
- Quite successful in many cases.
  However, many axioms can be "explosive".
- Set of support is a well known idea to prevent explosion

This talk in a nutshell

- One useful technique for reasoning with theories and quantifiers is the <u>addition of theory axioms</u>
- Quite successful in many cases.
  However, many axioms can be "explosive".
- <u>Set of support</u> is a well known idea to prevent explosion
- Idea 1: apply SOS for theory reasoning

This talk in a nutshell

- One useful technique for reasoning with theories and quantifiers is the addition of theory axioms
- Quite successful in many cases.
  However, many axioms can be “explosive”.
- Set of support is a well known idea to prevent explosion
- Idea 1: apply SOS for theory reasoning
- Idea 2: fine-tune this by allowing limited reasoning among theory axioms

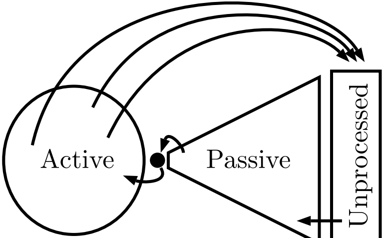
This talk in a nutshell

- One useful technique for reasoning with theories and quantifiers is the <u>addition of theory axioms</u>
- Quite successful in many cases.
  However, many axioms can be "explosive".
- <u>Set of support</u> is a well known idea to prevent explosion
- Idea 1: apply SOS for theory reasoning
- Idea 2: fine-tune this by allowing <u>limited reasoning</u> among theory axioms
- Preliminary evaluation of the technique

Outline

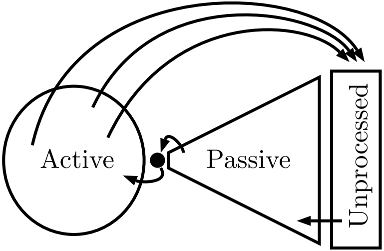
Saturation-based Theorem Proving

Compute deductive closure of the input N wrt inferences $\mathcal{I}$:



Saturation-based Theorem Proving

Compute deductive closure of the input N wrt inferences $\mathcal{I}$:



- clause selection schemes
- further aspects: literal selection, ordering restrictions, ...
- completeness considerations

Theory Reasoning in Vampire

Main focus

Reasoning with quantifiers and theories

Main focus

Reasoning with quantifiers and theories

Current arsenal:

Main focus

Reasoning with quantifiers and theories

Current arsenal:

- Evaluation of ground interpreted terms:
  $1 + 1 \Longrightarrow 2$, $1 < 1 \Longrightarrow \mathit{false}$, ...

Main focus

Reasoning with quantifiers and theories

Current arsenal:

- Evaluation of ground interpreted terms:
  $1 + 1 \Longrightarrow 2$, $1 < 1 \Longrightarrow \mathit{false}$, ...
- Interpreted operations treated specially by ordering

Main focus

Reasoning with quantifiers and theories

Current arsenal:

- Evaluation of ground interpreted terms:
  $1 + 1 \Longrightarrow 2$, $1 < 1 \Longrightarrow$ *false*, ...
- Interpreted operations treated specially by ordering
- Normalization of interpreted operations, i.e. only use $<$

Theory Reasoning in Vampire

Main focus

Reasoning with quantifiers and theories

Current arsenal:

- Evaluation of ground interpreted terms:
  $1 + 1 \Longrightarrow 2$, $1 < 1 \Longrightarrow \mathit{false}$, ...
- Interpreted operations treated specially by ordering
- Normalization of interpreted operations, i.e. only use $<$
- Theory axioms
  - hand-crafted set
  - either all added or none added (based on an option)

Theory Reasoning in Vampire

Main focus

Reasoning with quantifiers and theories

Current arsenal:

- Evaluation of ground interpreted terms:
  $1 + 1 \Longrightarrow 2$, $1 < 1 \Longrightarrow \mathit{false}$, ...
- Interpreted operations treated specially by ordering
- Normalization of interpreted operations, i.e. only use $<$
- Theory axioms
  - hand-crafted set
  - either all added or none added (based on an option)
- AVATAR with an SMT solver
  - Idea: Vampire only explores theory-consistent ground sub-problems

Main focus

Reasoning with quantifiers and theories

Current arsenal:

- Evaluation of ground interpreted terms:
  $1 + 1 \Longrightarrow 2$, $1 < 1 \Longrightarrow \mathit{false}, \ldots$
- Interpreted operations treated specially by ordering
- Normalization of interpreted operations, i.e. only use $<$
- Theory axioms
  - hand-crafted set
  - either all added or none added (based on an option)
- AVATAR with an SMT solver
  - Idea: Vampire only explores theory-consistent ground sub-problems
- Theory Instantiation and Unification with Abstraction

Main focus

Reasoning with quantifiers and theories

Current arsenal:

- Evaluation of ground interpreted terms:
  $1 + 1 \Longrightarrow 2$, $1 < 1 \Longrightarrow \mathit{false}$, ...
- Interpreted operations treated specially by ordering
- Normalization of interpreted operations, i.e. only use $<$
- Theory axioms
  - hand-crafted set
  - either all added or none added (based on an option)
- AVATAR with an SMT solver
  - Idea: Vampire only explores theory-consistent ground sub-problems
- Theory Instantiation and Unification with Abstraction

Some axioms

$$
\begin{array}{cc}
x + (y + z) = (x + y) + z & x + 0 = x \\
x + y = y + x & -(x + y) = (-x + -y) \\
- - x = x & x + (-x) = 0 \\
x * 0 = 0 & x * (y * z) = (x * y) * z \\
x * 1 = x & x * y = y * x \\
(x * y) + (x * z) = x * (y + z) & \neg(x < y) \vee \neg(y < z) \vee \neg(x < z) \\
x < y \vee y < x \vee x = y & \neg(x < y) \vee \neg(y < x + 1) \\
\neg(x < y) \vee x + z < y + z & \neg(x < x) \\
x < y \vee y < x + 1 \ (for\ ints) & x = 0 \vee (y * x)/x = y \ (for\ reals)
\end{array}
$$

Axioms can be "explosive"

ARI581=1.p

```
tff(mix_quant_ineq_sys_solvable_2,conjecture,(
  ! [X: $int] : ( $less(5,X) =>
   ? [Y: $int] : ( $less(Y,3) & $less(7,$sum(X,Y)))))).
```

- default strategy with all axioms: not solved in 60 s
- remove commutativity of $+$: solved instantly

Axioms can be “explosive”

ARI581=1.p

```
tff(mix_quant_ineq_sys_solvable_2,conjecture,(
  ! [X: $int] : ( $less(5,X) =>
   ? [Y: $int] : ( $less(Y,3) & $less(7,$sum(X,Y)))))).
```

- default strategy with all axioms: not solved in 60 s
- remove commutativity of $+$: solved instantly

SYN000=2.p

- “test tptp theory syntax” benchmark
- Vampire in default: 223 clauses (90 theory consequences, 1 used in the proof)
- negate the conjecture, run for 10 s:
  456 973 clauses (98 % are consequences of theory axioms)

Outline

The Set of Support Strategy

Basic idea:

- split the input clauses into a <u>set of support</u> and the rest
- restrict inferences to involve at least one premise from SOS
- new clauses are added to SOS

"Every inference must have an ancestor in the initial SOS."

The Set of Support Strategy

Basic idea:

- split the input clauses into a set of support and the rest
- restrict inferences to involve at least one premise from SOS
- new clauses are added to SOS

"Every inference must have an ancestor in the initial SOS."

In practice:

- just put non-SOS clauses directly to active

The Set of Support Strategy

Basic idea:

- split the input clauses into a <u>set of support</u> and the rest
- restrict inferences to involve at least one premise from SOS
- new clauses are added to SOS

"Every inference must have an ancestor in the initial SOS."

In practice:

- just put non-SOS clauses directly to active
- define SOS = clauses from the conjecture
  - Note: benchmarks without explicit conjecture SOS-suck

SOS in Vampire

Vampire's `-sos` option values:

- `off`: do not use SOS
- `on`: standard SOS
- `all`: SOS + select all literals of clauses in "initially active"

SOS in Vampire

Vampire's `-sos` option values:

- `off`: do not use SOS
- `on`: standard SOS
- `all`: SOS + select all literals of clauses in "initially active"

Experiment (relevant TPTP v6.4.0, 300 s)

	competition mode	competition mode with `sos=off`
Solved	11 948	11 613
Uniques	422	87

Outline

SOS for Theories

SOS and theory axioms

- the whole input problem is the SOS
- added theory axioms go directly to active
- new, fourth `-sos` option value: `theory`

SOS for Theories

SOS and theory axioms

- the whole input problem is the SOS
- added theory axioms go directly to active
- new, fourth `-sos` option value: `theory`
- Also applies to problems without explicit conjecture!

SOS for Theories

SOS and theory axioms

- the whole input problem is the SOS
- added theory axioms go directly to active
- new, fourth `-sos` option value: `theory`
- Also applies to problems without explicit conjecture!

Experiment (relevant SMTLIB, default strategy, 60 s)

	default mode	default mode + `sos=theory`
Solved	32 769	32 522
Uniques	641	394

How deep is theory reasoning?

Mining proofs for statistics:

- record maximum derivation depth
  of a pure theory consequence used in the proof

How deep is theory reasoning?

Mining proofs for statistics:

- record maximum derivation depth
  of a pure theory consequence used in the proof

Experiment (relevant SMTLIB, default strategy, 60 s)

Depth	count
0	31 959
1	209
2	304
3	200
4	49
5	21
6	27

What do useful pure theory consequences look like?

Example (deep pure theory consequences)

$$0 < x \vee x < 4$$

from `UFLIA/sledgehammer/TwoSquares/z3.637729.smt2`

$$\neg((x + (y + ((-x) + 2.0))) < y) \quad \text{and} \quad \neg(2.0 + x < x)$$

from `NRA/keymaera/ETCS-essentials-live-range2.proof-node1388.smt2`

What do useful pure theory consequences look like?

Example (deep pure theory consequences)

$$0 < x \vee x < 4$$

from `UFLIA/sledgehammer/TwoSquares/z3.637729.smt2`

$$\neg((x + (y + ((-x) + 2.0))) < y) \quad \text{and} \quad \neg(2.0 + x < x)$$

from `NRA/keymaera/ETCS-essentials-live-range2.proof-node1388.smt2`

Note that:

- large constants must be obtained by combining the basic axioms

What do useful pure theory consequences look like?

Example (deep pure theory consequences)

$$0 < x \lor x < 4$$

from `UFLIA/sledgehammer/TwoSquares/z3.637729.smt2`

$$\neg((x + (y + ((-x) + 2.0))) < y) \quad \text{and} \quad \neg(2.0 + x < x)$$

from `NRA/keymaera/ETCS-essentials-live-range2.proof-node1388.smt2`

Note that:

- large constants must be obtained by combining the basic axioms
- a clumsy search for a useful instance?

Explicitly liming depth of pure theory consequences

Depth	Count when threshold = 0	1	2	3	5	10	∞
0	32 522	32 253	32 130	32 061	32 162	32 040	31 959
1		552	237	209	216	208	209
2			551	314	310	307	304
3				312	254	212	200
4					69	48	49
5					61	21	21
6						26	27
total	32 522	32 805	32 918	32 896	**33 072**	32 863	32 769

Some further observations

Let us denote the depth threshold T

- solved with $T = n$ can still be solvable with $T = m < n$

Some further observations

Let us denote the depth threshold T

- solved with $T = n$ can still be solvable with $T = m < n$
- decreasing T can dramatically decrease the solution time and length of the found proof

Some further observations

Let us denote the depth threshold T

- solved with $T = n$ can still be solvable with $T = m < n$
- decreasing T can dramatically <u>decrease</u> the solution time and length of the found proof
- decreasing T can also dramatically <u>increase</u> the solution time and length of the found proof

Some further observations

Let us denote the depth threshold T

- solved with $T = n$ can still be solvable with $T = m < n$
- decreasing T can dramatically <u>decrease</u> the solution time and length of the found proof
- decreasing T can also dramatically <u>increase</u> the solution time and length of the found proof

Experiment (relevant SMTLIB, smtcomp mode, 1800 s)

	competition mode	`set sos=theory threshold=5`
Solved	37 009	36 821
Uniques	254	66

Conclusion

Summary

- adapted SOS for dealing with theory axioms
- tuned by a derivation depth parameter
- promising initial experiments

Conclusion

Summary

- adapted SOS for dealing with theory axioms
- tuned by a derivation depth parameter
- promising initial experiments

Ideas and plans for future work:

- better understand relations to other theory reasoning techniques

Conclusion

Summary

- adapted SOS for dealing with theory axioms
- tuned by a derivation depth parameter
- promising initial experiments

Ideas and plans for future work:

- better understand relations to other theory reasoning techniques
- what are the useful (deep) theory consequences?
  could they be precomputed?

Conclusion

Summary

- adapted SOS for dealing with theory axioms
- tuned by a derivation depth parameter
- promising initial experiments

Ideas and plans for future work:

- better understand relations to other theory reasoning techniques
- what are the useful (deep) theory consequences?
  could they be precomputed?
- distinguish "explosiveness" of axioms on case by case basis

Conclusion

Summary

- adapted SOS for dealing with theory axioms
- tuned by a derivation depth parameter
- promising initial experiments

Ideas and plans for future work:

- better understand relations to other theory reasoning techniques
- what are the useful (deep) theory consequences?
  could they be precomputed?
- distinguish "explosiveness" of axioms on case by case basis

Thank you for your attention!