

Revisiting Edge-Based Non-Linear V4-Scheme for Turbulent Flows using Metric-Based Anisotropic Mesh Adaptation

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1. Introduction

Second-order vertex-centered Mixed-Element-Volume (MEV) MUSCL scheme [1, 2] has been very successful for solving steady Reynolds-Averaged Navier-Stokes (RANS) simulations when coupled with metric-based anisotropic mesh adaptation, notably for sonic boom prediction [3], high-lift prediction [4], vortex dominated flows [5], compressors [6], film-cooled turbine [7], and many other applications.

At the core of the MEV method lies in the edge-based linear V4-scheme and V6-scheme discretizing the convective terms. Generally, the V4-scheme is considered for steady applications thanks to its good iterative convergence property, and the V6-scheme is used for unsteady applications, in particular for eddy-resolved simulations because of its lower dissipation. In this work, we focus on the V4-scheme but the results extend easily to the V6-scheme. Theoretically, the linear V4 convective numerical scheme is third-order accurate for the linear advection on uniform unstructured or structured meshes [2]. If non-linear equations are considered and/or if the mesh is non-uniform then it becomes a low dissipation second-order scheme with a fourth-order numerical dissipation making it very accurate. On the numerical side, this scheme is very efficient because it reduces solving a 1D flux for each edge. It thus requires only one loop over the edges to compute the MUSCL extrapolation, the limiter and the approximate Riemann solver for convective fluxes. In comparison, more classical edge-based methods using stencil limiters [8, 9] require one loop to compute solution extrema in each vertex neighborhood, one loop to compute the limiter coefficient and one loop to compute the MUSCL extrapolation and convective fluxes.

In the quest of using even more accurate numerical scheme, Koobus et al. proposed a non-linear V6-scheme [10], but its formulation makes it very difficult to apply a limiter to the MUSCL extrapolation. Consequently, it was not usable for practical applications. Recent works on economical high-order flux-solution-reconstruction (FSR) schemes [11] and the third-order edge-based convective scheme [12, 13] give clues to solve that issue. The goal of this paper is to combine both approaches to propose a new non-linear V4-scheme. Theoretically, the non-linear V4 scheme should exhibit the same properties as linear V4 on uniform unstructured or structured meshes also for the Euler equations. If non-uniform meshes (such as adapted meshes) are considered, then it becomes a low dissipation second-order scheme with a fourth-order numerical dissipation. We expect this scheme to be more accurate than the linear one when solving RANS equations. This work will fairly compare the linear and the non-linear V4-schemes on realistic applications in aeronautics.

2. RANS modeling and discretization

We consider the Reynolds-Averaged Navier-Stokes (RANS) system written under vector form:

$$W_t + \nabla \cdot \mathcal{F}^E = \nabla \cdot \mathcal{F}^V + \mathcal{F}^S,$$

where W , $\mathcal{F}^E$, $\mathcal{F}^V$, and $\mathcal{F}^S$ are the conservative variables, the convective (Euler) fluxes, the viscous fluxes, and the source term vector, respectively. According to the standard approach to turbulence modeling based upon the Boussinesq hypothesis, the turbulence is modeled with an eddy viscosity μ_t , which is added to the laminar (or dynamic) viscosity μ . Here we choose the Spalart-Allmaras one equation turbulence model [14].

Let $\mathcal{H}$ be a mesh of domain Ω , the vertex-centered Edge-Based / Finite Volume formulation consists in associating with each vertex P_i of the mesh a finite volume cell, denoted C_i . The median cells are considered. The discretized domain Ω_h can be written as the union of the elements or the union of the finite volume cells: $\Omega_h = \bigcup_{i=1}^{N_K} K_i = \bigcup_{i=1}^{N_V} C_i$, where N_K is the number of elements and N_V the number of vertices.

The second-order accurate RANS discretized equations read:

$$|C_i| \frac{dW_i}{dt} + \mathbf{F}_i = \mathbf{V}_i + \mathbf{S}_i + \mathbf{\Gamma}_i, \quad (1)$$

where $\mathbf{F}_i$, $\mathbf{V}_i$, $\mathbf{S}_i$ and $\mathbf{\Gamma}_i$ are respectively the numerical convective, viscous, source flux and boundary terms:

$$\mathbf{F}_i = \int_{\partial C_i} F(W_i) \cdot \mathbf{n}_i d\gamma, \quad \mathbf{V}_i = \int_{\partial C_i} V(W_i) \cdot \mathbf{n}_i d\gamma, \quad \mathbf{S}_i = \int_{C_i} S(W_i) d\Omega, \quad \mathbf{\Gamma}_i = \int_{\partial C_i \cap \partial \Omega} G(W_i) d\gamma,$$

where $\mathbf{n}_i$ is the outer normal to the cell surface ∂C_i , and F , V and S are respectively the convective, viscous and source terms flux functions. G holds for the boundary flux function specified to impose the desired boundary condition. If a *Finite Volume scheme* is considered then W_i is the mean value of the solution W on cell C_i . If an *Edge-Based scheme* is considered then W_i is the pointwise of the solution W at vertex P_i . This paper only focuses on edge-based scheme and convective flux discretization.

The integration of the convective fluxes $\mathbf{F}$ of Equation (1) is done by decomposing the cell boundary ∂C_i into many facets. These facets are grouped and associated to the edges joining vertex P_i to its neighboring vertices, denoted $\{P_j\}$. The facets associated to each edge are averaged to yield a single mean face per edge:

$$\mathbf{F}_i = \int_{\partial C_i} F(W_i) \cdot \mathbf{n}_i d\gamma \approx \sum_{P_j \in \mathcal{V}(P_i)} \Phi(W_i, W_j, \hat{\mathbf{n}}_{ij}) |\mathbf{n}_{ij}|,$$

where $\hat{\mathbf{n}}_{ij} = \mathbf{n}_{ij}/|\mathbf{n}_{ij}|$ and $\mathcal{V}(P_i)$ is the set of all neighboring vertices linked by an edge to P_i . The computation of the convective fluxes is performed mono-dimensionally in the direction normal to the boundary of the finite volume cell. Therefore, the numerical calculation of the flux function Φ at the interface ∂C_{ij} is achieved by the resolution of a 1D Riemann problem in the direction of the normal $\mathbf{n}_{ij}$ by means of an approximate Riemann solver. Here, the numerical flux Φ is given by the Roe flux [15]:

$$\Phi(W_i, W_j, \hat{\mathbf{n}}_{ij}) = \frac{1}{2} [\hat{F}(W_i) + \hat{F}(W_j)] - \frac{1}{2} [\tilde{\mathbf{A}}(W_i, W_j, \hat{\mathbf{n}}_{ij}) | (W_j - W_i)],$$

with $\hat{F} = F \cdot \hat{\mathbf{n}}_{ij}$ and $[\tilde{\mathbf{A}}]$ is the dissipation matrix of the Roe flux. This discretization is first-order accurate. In order to increase the order of accuracy of the scheme, MUSCL type reconstruction methods have been designed [16].

3. Edge-based linear V4-scheme

To achieve second-order accuracy, the idea is to use extrapolated values W_{ij} and W_{ji} instead of W_i and W_j at the interface ∂C_{ij} to evaluate the flux with an approximate Riemann solver. That type of scheme are said *linear* because the extrapolation acts directly on the solution field W which is equivalent to extrapolate a linear flux. The numerical flux becomes:

$$\mathbf{F}_i \approx \sum_{P_j \in \mathcal{V}(P_i)} \Phi(W_{ij}, W_{ji}, \hat{\mathbf{n}}_{ij}) |\mathbf{n}_{ij}|,$$

where W_{ij} and W_{ji} are linearly extrapolated as:

$$W_{ij} = W_i + \frac{1}{2} (\nabla W)_i \cdot \overrightarrow{P_i P_j} \quad \text{and} \quad W_{ji} = W_j + \frac{1}{2} (\nabla W)_j \cdot \overrightarrow{P_j P_i}.$$

The linear V4-scheme is obtained using a combination of centered and upwind gradients. The centered gradient, which is related to edge $P_i P_j$, is implicitly defined along edge $P_i P_j$ by the relations:

$$(\nabla W)_{ij}^C \cdot \overrightarrow{P_i P_j} = W_j - W_i \quad \text{and} \quad (\nabla W)_{ij}^C \cdot \overrightarrow{P_j P_i} = W_i - W_j.$$

Upwind and downwind gradients, which are also related to edge $P_i P_j$, are computed according to the definition of upwind and downwind triangles/tetrahedra of edge $P_i P_j$. These triangles/tetrahedra are respectively denoted K_{ij} and K_{ji} . K_{ij} (resp. K_{ji}) is the unique triangle/tetrahedron of the ball of P_i (resp. P_j) the opposite face of which is crossed by the line defined by the edge $P_i P_j$. In contrast to most of the vertex-centered schemes for unstructured meshes where the upwind gradient is defined by a unique gradient computed at each vertex, the current scheme uses different upwind gradients defined for each edge, *i.e.* the P^1 Finite Element (FE) gradients of the considered elements. Upwind and downwind gradients are then defined for vertices P_i and P_j as:

$$(\nabla W)_{ij}^U = (\nabla W)|_{K_{ij}} \quad \text{and} \quad (\nabla W)_{ij}^D = (\nabla W)|_{K_{ji}}.$$

The linear V4-scheme uses the following high-order solution reconstruction [2, 17]

$$\begin{aligned} W_{ij}^{V4} &= W_i + (\Delta W)_i^{V4} = W_i + \frac{1}{2} \left[(1 - \beta) (\nabla W)_{ij}^C \cdot \overrightarrow{P_i P_j} + \beta (\nabla W)_{ij}^U \cdot \overrightarrow{P_i P_j} \right] \\ W_{ji}^{V4} &= W_j + (\Delta W)_j^{V4} = W_j + \frac{1}{2} \left[(1 - \beta) (\nabla W)_{ij}^C \cdot \overrightarrow{P_j P_i} + \beta (\nabla W)_{ij}^D \cdot \overrightarrow{P_j P_i} \right] \end{aligned}$$

leading to

$$\Phi^{V4} = \Phi \left(W_{ij}^{V4}, W_{ji}^{V4}, \hat{\mathbf{n}}_{ij} \right) = \frac{1}{2} \left[\hat{F}(W_{ij}^{V4}) + \hat{F}(W_{ji}^{V4}) \right] - \frac{1}{2} \left| \mathbf{A} \left(W_{ij}^{V4}, W_{ji}^{V4}, \hat{\mathbf{n}}_{ij} \right) \right| \left(W_{ji}^{V4} - W_{ij}^{V4} \right).$$

It can be demonstrated that this scheme is third-order accurate at $\beta = 1/3$ with fourth-order derivative dissipation only for linear advection on uniform unstructured or structured simplicial meshes. On non-uniform unstructured meshes or for non-linear equations, a second-order low-dissipation scheme is obtained. A fifth-order accurate scheme with sixth-order numerical dissipation scheme, the linear V6-scheme, is given in [2].

Note that the generalized slopes, $(\Delta W)_i^{V4}$ and $(\Delta W)_j^{V4}$, are the key quantities to high-order accuracy. Moreover, a limiter is directly applied to them for shock capturing. Our aim is to build genuinely high-order shock-capturing schemes based on these quantities with minimal additional computational cost.

4. New edge-based non-linear V4-scheme

Koobus et al. proposed a non-linear version of the V4 and V6 schemes in Ref [10].

In the non-linear V4-scheme, extrapolations are done directly on the fluxes:

$$\begin{aligned} \hat{F}_{ij}^{V4} &= \hat{F}_i + (\Delta \hat{F})_i^{V4} = \hat{F}_i + \frac{1}{2} \left[(1 - \beta) (\nabla \hat{F})_{ij}^C \cdot \overrightarrow{P_i P_j} + \beta (\nabla \hat{F})_{ij}^U \cdot \overrightarrow{P_i P_j} \right] \\ \hat{F}_{ji}^{V4} &= \hat{F}_j + (\Delta \hat{F})_j^{V4} = \hat{F}_j + \frac{1}{2} \left[(1 - \beta) (\nabla \hat{F})_{ij}^C \cdot \overrightarrow{P_j P_i} + \beta (\nabla \hat{F})_{ij}^D \cdot \overrightarrow{P_j P_i} \right] \end{aligned}$$

with $\beta = 1/3$, and the non-linear V4 flux is given by

$$\Phi^{NLV4} = \frac{1}{2} \left[\hat{F}_{ij}^{V4} + \hat{F}_{ji}^{V4} \right] - \frac{1}{2} \text{sign} \left(\tilde{\mathbf{A}}(W_{ij}^{V4}, W_{ji}^{V4}, \hat{\mathbf{n}}_{ij}) \right) \left(\hat{F}_{ji}^{V4} - \hat{F}_{ij}^{V4} \right).$$

The first term of the RHS is the central differenced flux and the second term of the RHS is the stabilisation term. This formulation makes it very difficult to apply if a limiter is needed, for instance for transonic, supersonic or hypersonic flows. Indeed, we do not know how to limit the flux reconstruction. Analogously, it is difficult to avoid the so-called expansion shock when the stabilisation term is expressed only in terms of flux jumps, *i.e.* $F_{ij} - F_{ji}$.

In this paper, we propose two simple modifications to solve the above-mentioned issues. First, the stabilisation term is built in the same way as the linear V4-scheme *i.e.* only in terms of solution jumps $W_{ij} - W_{ji}$. This is because the solution jump produces third-order (resp. second-order) error terms on regular (resp. irregular) meshes for linear and non-linear fluxes. Second, global third-order accuracy can be achieved by increasing the precision of the central differenced flux. To do that, we propose to use the same idea as the one considered in the FSR schemes [11] and the third-order edge-based scheme [12, 13] consisting in linearizing the flux in the extrapolation. Here, we propose the following modified fluxes

$$\Phi^{V4} = \Phi \left(W_{ij}^{V4}, W_{ji}^{V4}, \hat{\mathbf{n}}_{ij} \right) = \frac{1}{2} \left[\hat{F}_{ij}^{V4} + \hat{F}_{ji}^{V4} \right] - \frac{1}{2} \left| \mathbf{A} \left(W_{ij}^{V4}, W_{ji}^{V4}, \hat{\mathbf{n}}_{ij} \right) \right| \left(W_{ji}^{V4} - W_{ij}^{V4} \right),$$

where

$$\hat{F}_{ij}^{V4} = \hat{F}_i + \left(A \frac{\partial \hat{F}}{\partial W} \Big|_i + B \frac{\partial \hat{F}}{\partial W} \Big|_j \right) (\Delta W)_i^{V4} = \hat{F}_i + \frac{1}{2} \left(A \frac{\partial \hat{F}}{\partial W} \Big|_i + B \frac{\partial \hat{F}}{\partial W} \Big|_j \right) \left[(1 - \beta) (\nabla W)_{ij}^C \cdot \overrightarrow{P_i P_j} + \beta (\nabla W)_{ij}^U \cdot \overrightarrow{P_i P_j} \right], \quad (2)$$

$$\hat{F}_{ji}^{V4} = \hat{F}_j + \left(A \frac{\partial \hat{F}}{\partial W} \Big|_j + B \frac{\partial \hat{F}}{\partial W} \Big|_i \right) (\Delta W)_j^{V4} = \hat{F}_j + \frac{1}{2} \left(A \frac{\partial \hat{F}}{\partial W} \Big|_j + B \frac{\partial \hat{F}}{\partial W} \Big|_i \right) \left[(1 - \beta) (\nabla W)_{ij}^C \cdot \overrightarrow{P_j P_i} + \beta (\nabla W)_{ij}^D \cdot \overrightarrow{P_j P_i} \right], \quad (3)$$

and A and B are coefficients that can be set to guarantee third-order accuracy for general nonlinear fluxes on regular meshes. Moreover, (i) discontinuous solutions can be handled by limiting $(\nabla W)^{V4}$; (ii) entropy fix can be easily done by modifying the eigenvalues of $\tilde{\mathbf{A}}$.

4.1 Truncation error analysis on regular grids

Consider the following steady 1D EDP with general nonlinear flux

$$\operatorname{div}(F(u)) = \frac{dF(u)}{dx} = F_x = F_u u_x. \quad (4)$$

The above equation can be discretized with the new NLV4 scheme obtained with Eqs.(2-3), so that

$$\operatorname{div}(F)|_i \approx \frac{1}{V_j} \sum_{P_j \in \mathcal{V}(P_i)} \frac{1}{2} \left[F_{ij}^{V4} + F_{ji}^{V4} \right] + \frac{1}{2} |\tilde{\mathbf{A}}|_{i+1/2} \left(u_{ij}^{V4} - u_{ji}^{V4} \right). \quad (5)$$

As earlier explained, the stabilization part produces third-order error terms on regular meshes for linear and non-linear fluxes. For this reason, we can focus on the consistent part (purely-centered) and stabilization independently.

4.1.1 Purely-centered scheme

On an uniform mesh, the proposed flux reconstructions reduce for the edge $P_i P_{i+1}$ to

$$\begin{aligned} F_{ij}|_{i,i+1} &= F_i + \frac{1}{2} (A F_u + B F_u|_{i+1}) [(1-\beta)(u_{i+1} - u_i) + \beta(u_i - u_{i-1})] \\ F_{ji}|_{i,i+1} &= F_{i+1} - \frac{1}{2} (A F_u|_{i+1} + B F_u) [(1-\beta)(u_{i+1} - u_i) + \beta(u_{i+2} - u_{i+1})], \end{aligned}$$

and for the edge $P_i P_{i-1}$ to

$$\begin{aligned} F_{ij}|_{i,i-1} &= F_i + \frac{1}{2} (A F_u + B F_u|_{i-1}) [(1-\beta)(u_{i-1} - u_i) + \beta(u_i - u_{i+1})] \\ F_{ji}|_{i,i-1} &= F_{i-1} - \frac{1}{2} (A F_u|_{i-1} + B F_u) [(1-\beta)(u_{i-1} - u_i) + \beta(u_{i-2} - u_{i-1})]. \end{aligned}$$

The Taylor expansion of the edge-based purely-centered scheme for the conservation law results in

$$\begin{aligned} \frac{1}{V_j} \sum_{P_j \in \mathcal{V}(P_i)} \frac{1}{2} \left[F_{ij}^{V4} + F_{ji}^{V4} \right] &= F_x \\ &\quad - \frac{1}{288} \left[F_u 48 u_{xxx} [3\beta(A+B) - 1] \Delta x^2 \right. \\ &\quad + F_{uu} 144 u_{xx} u_x [A(\beta+1) + B(\beta-1) - 1] \Delta x^2 \\ &\quad + F_{uuu} 24 (u_x)^3 [3(A-B) - 2] \Delta x^2 \\ &\quad \left. + O(\Delta x^4, \Delta x^6) \right] = 0, \end{aligned} \quad (6)$$

with the following conditions returning a fourth-order purely-centered scheme:

$$\begin{aligned} 3\beta(A+B) - 1 &= 0 \\ A(\beta+1) + B(\beta-1) - 1 &= 0 \\ 3(A-B) - 2 &= 0. \end{aligned}$$

Surprisingly, the above equations are satisfied by

$$A = \frac{1}{3} \left(\frac{1}{2\beta} + 1 \right) \quad B = \frac{1}{3} \left(\frac{1}{2\beta} - 1 \right), \quad (7)$$

4.1.2 Stabilization

Consider the following stabilization terms

$$D_{i+1/2} = \frac{1}{2} |\tilde{\mathbf{A}}|_{i+1/2} (u_{ij} - u_{ji})|_{i,i+1} \quad D_{i-1/2} = \frac{1}{2} |\tilde{\mathbf{A}}|_{i-1/2} (u_{ij} - u_{ji})|_{i,i-1}$$

with the following standard V4 reconstruction for the edges $P_i P_{i+1}$ and $P_i P_{i-1}$

$$\begin{aligned} u_{ij}|_{i,i+1} &= u_i + \frac{1}{2} [(1-\beta)(u_{i+1} - u_i) + \beta(u_i - u_{i-1})] & u_{ji}|_{i,i+1} &= u_{i+1} - \frac{1}{2} [(1-\beta)(u_{i+1} - u_i) + \beta(u_{i+2} - u_{i+1})] \\ u_{ij}|_{i,i-1} &= u_i + \frac{1}{2} [(1-\beta)(u_{i-1} - u_i) + \beta(u_i - u_{i+1})] & u_{ji}|_{i,i-1} &= u_{i-1} - \frac{1}{2} [(1-\beta)(u_{i-1} - u_i) + \beta(u_{i-2} - u_{i-1})]. \end{aligned}$$

By expanding the dissipation matrices at the node P_i , the total amount of dissipation at the same node consists in

$$D_i = D_{i+1/2} - D_{i-1/2} = \beta \frac{\Delta x^3}{768} \left[384 u_{xxx} |\tilde{\mathbf{A}}|_i + O(\Delta x^2, \Delta x^4) \right] \quad (8)$$

As expected, dissipation produces third-order terms without the need of flux reconstruction.

In contrast with the linear V4, for which β must be set at $1/3$ to guarantee third-order accuracy for linear advection on regular meshes, the order of accuracy of the resulting nonlinear version is not affected by the value of β .

The same result can be proved on 2D and 3D regular meshes.

4.2 Flux-correction for general Riemann solvers

The above technique is restricted to the Roe flux function, because reconstruction is done separately on flux and solution. However, if one is interested in using a different flux function (*i.e.*, HLL, HLLC, etc.), it is possible to demonstrate that the NLV4 can be used for a generic second-order accurate scheme $F(W_{ij}^{V4}, W_{ji}^{V4})$, so that

$$\frac{1}{2} [F_{ij}^{V4} + F_{ji}^{V4}] = F(u_{ij}^{V4}, u_{ji}^{V4}) + \frac{1}{8} \Delta F_{ij}^{NLV4} + O(\Delta x^4), \quad (9)$$

with

$$\Delta F_{ij}^{NLV4} = (A F_u + B F_u|_{i+1})(\Delta u)_i^{V4} - (A F_u|_{i+1} + B F_u)(\Delta u)_j^{V4}. \quad (10)$$

However, the flux function must be a sufficiently-accurate approximation of the mid-edge point flux evaluation, in the sense that the β -scheme must provide an exact reconstruction for a quadratic solution. It is easy to demonstrate that such condition is verified for $\beta = 1/4$. Therefore, the above condition requires that for general flux functions β must be set at $1/4$.

5. Discretization error analysis

To verify accuracy, we use the method of manufactured solutions [18, 19], in which an *ad hoc* function can be made an exact solution of the Euler equations by including a suitable forcing term $\mathbf{s}(\mathbf{x})$ as

$$\text{div}(\mathcal{F}^{inv} - \mathcal{F}^{vis}) = \mathbf{s}(\mathbf{x}). \quad (11)$$

As an exact solution, we consider the following functions:

$$\rho = 1.3 + 0.20 \exp(-1.05x + 1.10y + 1.15z), \quad (12)$$

$$u = 0.5 + 0.10 \exp(1.05x - 1.10y + 1.15z), \quad (13)$$

$$v = -0.1 + 0.02 \exp(1.50x + 3.10y - 1.50z), \quad (14)$$

$$w = 0.2 + 0.05 \exp(1.70x + 1.10y + 1.15z), \quad (15)$$

$$p = 1.2 + 0.20 \exp(1.10x + 1.10y + 1.15z), \quad (16)$$

and compute the forcing term $\mathbf{s}(\mathbf{x})$ numerically at each vertex by evaluating the Euler equations with their exact gradients (analytically derived and evaluated at each vertex), as

$$\mathbf{s}(\mathbf{x}) = \begin{bmatrix} \nabla \cdot (\rho \mathbf{u}) \\ \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p \\ \nabla \cdot (\rho E + p) \mathbf{u} \end{bmatrix}. \quad (17)$$

The edge-based method approximates the differential form in a conservative manner at a vertex. Point-value evaluations of the source terms are sufficient for second-order schemes, but generally not so accurate to guarantee global third-order accuracy [20]. However, β -schemes, and in particular the V4 scheme, produce $O(h^3)$ -truncation error on regular grids. In this case, a point-value evaluation is exact and precise. Therefore, the spatial discretization at a vertex j in a tetrahedral results in

$$\frac{\mathbf{Res}_j}{V_j} = \frac{1}{V_j} \sum_{P_j \in \mathcal{V}(P_i)} \Phi(W_{ij}, W_{ji}, \hat{\mathbf{n}}_{ij}) |\mathbf{n}_{ij}| - \mathbf{s}(\mathbf{x}_j). \quad (18)$$

A family of consistently-refined regular tetrahedral grids in a unit cube domain is considered and the corresponding numerical solution W_j is computed (see Figure 1, on the left). These meshes are generated by subdividing a Cartesian grid of $n \times n \times n$ nodes, where $n = [16, 32, 48, 64, 80, 96]$. Numerical simulations are run with an implicit solver using exact first-order Jacobian. The CFL number is initially set at 0.1 and geometrically increases to 10^5 , according to different convergence criteria [21]. The linearized system is relaxed by a Symmetric Gauss-Seidel (SGS) relaxation scheme until the linear system residual is reduced by two orders of magnitude or until it reaches the maximum of 40 relaxations. Strong boundary conditions imposing the exact solution are used at boundaries. Each calculation is considered converged when residuals stagnate between 10^{-14} and 10^{-16} . The discretization error is measured in the following norm [22],

$$DE_p = \left(\frac{1}{N} \sum_j |W_j - W_j^{ex}|^p \right)^{\frac{1}{p}}, \quad (19)$$

where W_j and W_j^{ex} are the numerical and the exact solution at the vertex j respectively, N is the number of vertices and p is the order of the norm considered. Here, we employed $p = 2$, for which the above indicator corresponds to the root mean square. For consistently refined meshes, one can consider the following effective mesh size

$$h_{eff} = \frac{1}{N} \sum_{j=1}^N V_j^{1/3}. \quad (20)$$

The L^2 norm of the discretization error for the density field is plotted against the effective mesh size in Figure 1.

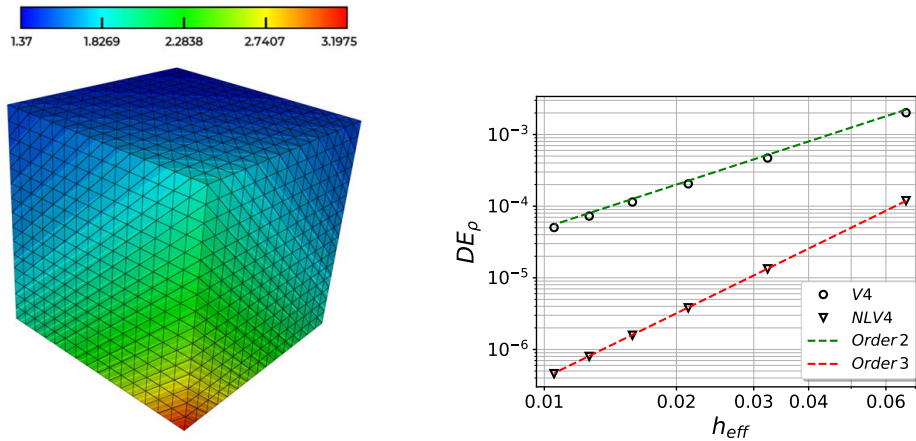


Figure 1: Accuracy verification for the Euler equations on 3D uniform tetrahedral meshes. Left, mesh and numerical density field. Right, L^2 norm of the discretization error on density field: comparison between V4 and the new NLV4 scheme accuracy trend at interior vertices for strong Dirichlet boundary conditions.

In the picture on the right, we can see that the standard V4 scheme (circles) produces second-order accurate solutions for Euler equations, as expected. On the other hand, the new NonLinear V4 (diamonds) exhibits third-order accuracy. This results shows that the new scheme recovers, for Euler equations, the same high-order property that standard V4 exhibits "only" for linear advection.

5.1 Inviscid Steady Subsonic Flow over a 3D Smooth Bump

The first case considered is the steady subsonic flow over a smooth bump through a duct. The Mach number consider is $M = 0.3$. The smooth bump is defined by

$$z_b = 0.05 \sin^4 \left[\frac{\pi (10x - 3)}{9} \right], \quad x \in [0.3, 1.2],$$

and it is placed on the bottom boundary of a rectangular duct with size $(x, y, z) = [-0.25, 1.75] \times [0, 1] \times [0, 1]$

All boundary conditions are imposed weakly through the right state using the linearly exact formula (see Appendix B of Ref [23]): free stream boundary condition is imposed at the inflow; back pressure condition with unit static pressure ratio is imposed at the outflow; slip condition on all the other boundaries. As pointed out in Ref [24], any boundary state involving normals to boundaries (*e.g.*, slip condition) requires sufficiently accurate nodal normals to preserve the designed accuracy. In this case, this is achieved by the knowledge of the exact tangent vector, as

$$\mathbf{n}_i^{BC} = (dz_b/dx, 0, 1),$$

and $\hat{\mathbf{n}}_i^{BC} = \mathbf{n}_i^{BC} / |\mathbf{n}_i^{BC}|$. A mesh-convergence study using a *multi-scale* error estimate based on the L_4 norm of the Mach number field is run on eight anisotropic meshes with mesh sizes:

$$C = \{ 5000, \quad 10\,000, \quad 20\,000, \quad 40\,000, \quad 80\,000, \quad 160\,000, \quad 320\,000 \}.$$

The final adapted mesh and pressure field solution is shown in Figure 3.

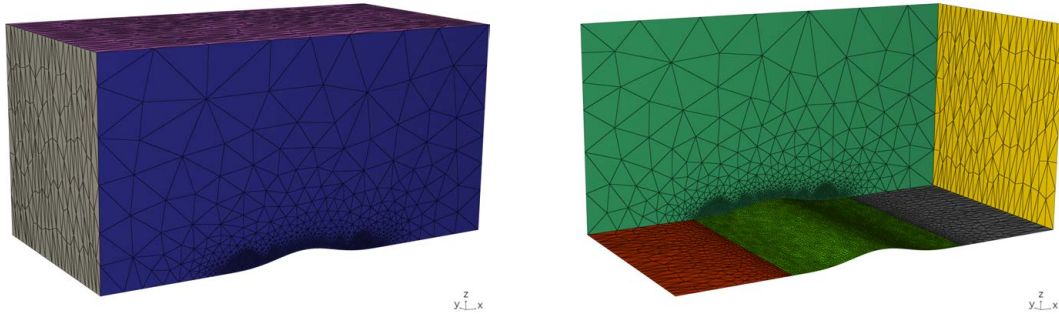


Figure 2: 3D square duct with smooth bump: Initial coarse mesh only composed of 10 026 vertices, 42 244 tetrahedra and 11 262 triangles on the boundaries. Exterior (left) and interior (right) view.

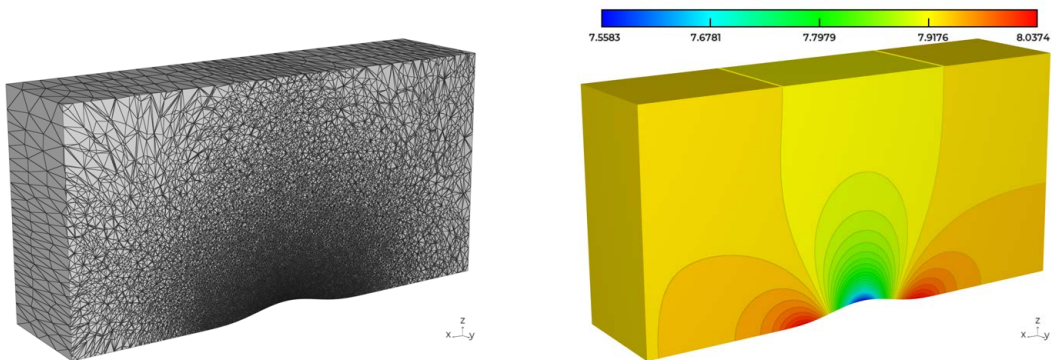


Figure 3: 3D subsonic flow over a smooth bump. Multi-scale anisotropic metric-based adaptation: adapted mesh composed of 350 000 vertices, 110 000 boundary triangles and 19 000 tetrahedra (left) and pressure field $\hat{p}$ (right).

For each calculation, we selected a maximum number of 40 symmetric Gauss-Seidel (SGS) iterations and 2 orders of magnitude by which the relative linear system residual must be decreased at each iteration. The local CFL is increased with a growth rate of 1.05, from 0.1 to 1×10^5 . Convergence is considered achieved when non-linear residuals fall below $R_{lg} = 1 \times 10^{-11}$. In order to evaluate the accuracy, the following entropy error is used

$$\epsilon_e = \frac{p}{\rho^\gamma} - \frac{p_\infty}{\rho_\infty^\gamma} = \frac{\hat{p}}{\hat{\rho}^\gamma} - \frac{1}{\gamma M_\infty^2}, \quad \text{with} \quad \hat{p} = \frac{p}{\rho_\infty v_\infty^2}, \quad \text{and} \quad \hat{\rho} = \frac{\rho}{\rho_\infty},$$

and measured in the RMS norm using Eq.(19). The L_2 norm of entropy error is plotted against the effective mesh size h_{eff} in Figure 4. We can see that the NLV4 scheme tailored for Roe flux (diamonds) is slightly more accurate on the finest adapted meshes, when compared with the baseline V4 scheme (circles). The difference between both schemes are no so brutal as occurring on uniform grids, simply because the NLV4 is formally second-order accurate on irregular tetrahedral meshes. However, its high-order accuracy on smooth meshes makes it slightly less dissipative on irregular ones.

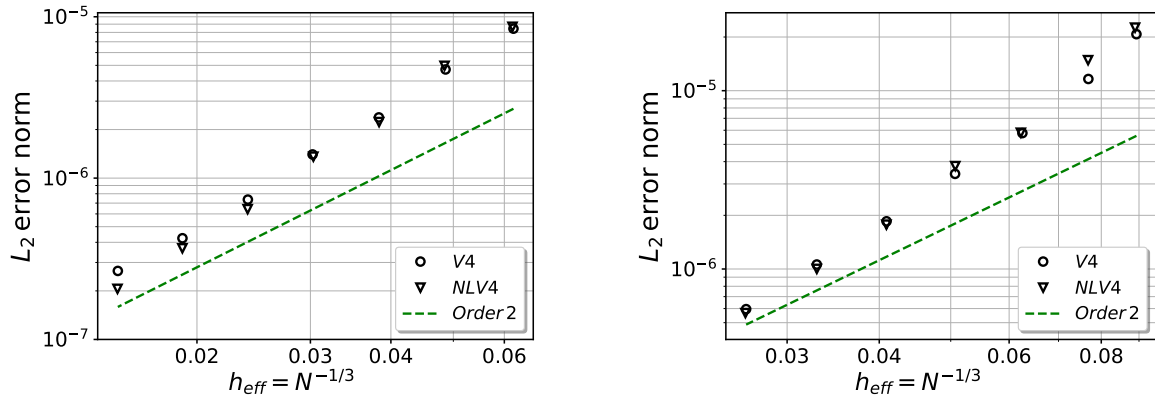


Figure 4: 3D subsonic flow over a smooth bump. Solutions obtained during the mesh-adaptive process. NLV4 and standard V4 schemes. L_2 norms of entropy error for interior points (left) and boundary vertices (right).

5.2 Laminar Steady Subsonic Flow past a 3D delta wing at low Reynolds number

As described in Section 4.1, the first version of the new NLV4 scheme here proposed is specifically tailored for the Roe flux function. In this case, in contrast with the baseline V4 scheme, β is a free-parameter and can be adjust to control the amount of dissipation.

To analyze the effect of β on practical simulations, we considered the laminar flow past a delta wing [25] with $M_\infty = 0.3$, zero angle of attack (AoA), $Re = 4 \times 10^3$ based on the root chord c , and the reference area $A_{ref} = 0.133974596$. The initial mesh is shown in Figure 5. It is only composed of 7919 vertices, 37 391 tetrahedra and 6776 triangles on the boundaries. No-slip boundary condition is set on the wing's surface. Free-stream boundary condition is imposed on all the other boundaries, except for the one the wing is attached to, where a symmetry boundary condition is prescribed.

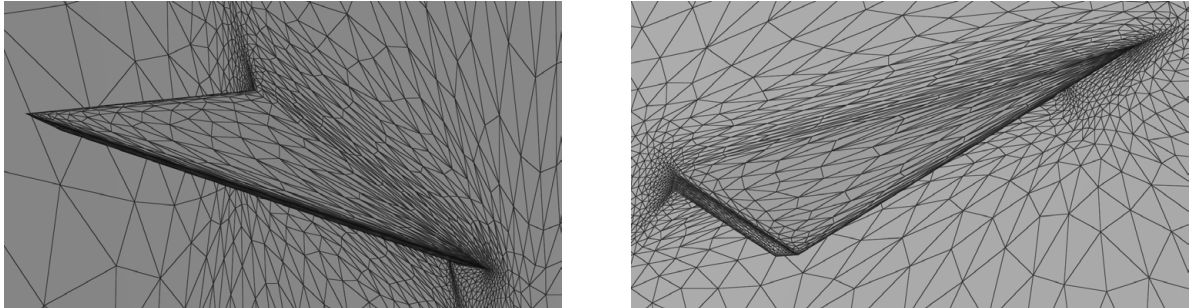


Figure 5: 3D laminar delta wing: $M = 0.3$, $Re = 4 \times 10^3$, $AoA=0^\circ$. Initial coarse mesh only composed of 7919 vertices, 37 391 tetrahedra and 6776 triangles on the boundaries.

A mesh-convergence study using a goal-oriented error estimate based on the total lift coefficient C_L is run on the

following eight mesh sizes:

$$C = \{ 4000, \quad 8000, \quad 16\,000, \quad 32\,000, \quad 64\,000, \quad 128\,000, \quad 256\,000, \quad 1\,024\,000 \}.$$

The convergence of aerodynamic coefficients is plotted in Figure 6. As regards the drag coefficient, one can easily note that the mesh-adaptive simulations using three different values of β gives the same result on 128 K-vertices adapted mesh. The same C_L is obtained with 256 K-vertices. The first observation is that when a robust mesh-adaptive process is used, the value of β does not affect the final mesh-solution. On the other side, low values of β give different solution on coarse meshes.

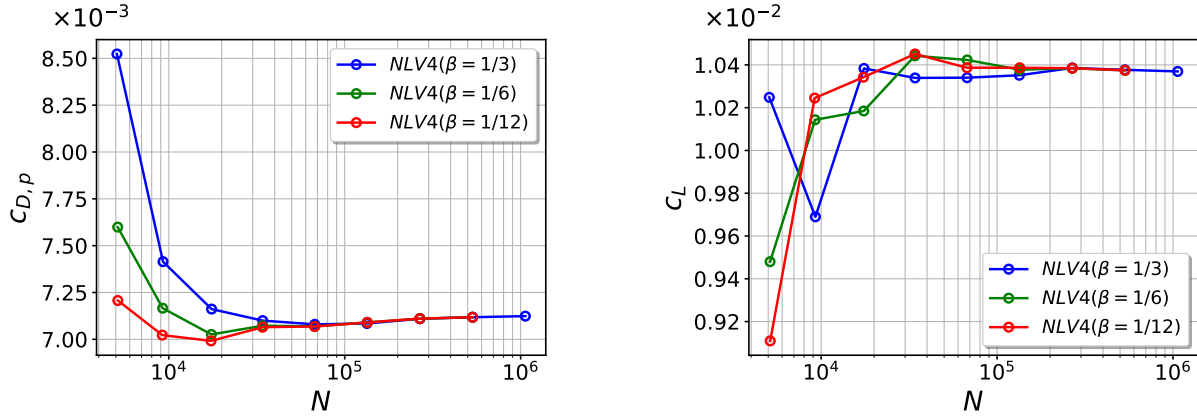


Figure 6: 3D laminar delta wing: $M = 0.3$, $Re = 4 \times 10^3$, $AoA=0^\circ$. Goal-oriented metric-based anisotropic mesh adaptation based on the total lift coefficient C_L . Pressure drag $C_{D,p}$ and total lift C_L coefficients against the number of vertices N . Roe flux function. Comparison between mesh-adaptive results obtained with different value of β for the new NLV4 scheme.

In order to make a quantitative analysis, we take as reference the drag and lift coefficient obtained on 1024 K-vertices adapted mesh using the scheme with $\beta = 1/3$. A relative error is computed with respect these reference values, and listed in Table 1. The reader can see that reducing the value of β gives much better C_D prediction on very coarse meshes (i.e., 4 K and 8 K); slightly worse results on low-medium resolved meshes (i.e., 16 K and 32 K); slightly better result on refined meshes (i.e., 64 K and 128 K); same results on sufficiently refined meshes close to mesh-convergence. This result confirms that low-values of β in NLV4 scheme act essentially on the numerical dissipation, which is less and less important with increasing level of refinement.

C	$C_{D,p}$ relative error (%)			C_L relative error (%)		
	$\beta = 1/3$	$\beta = 1/6$	$\beta = 1/12$	$\beta = 1/3$	$\beta = 1/6$	$\beta = 1/12$
4 K	19.65	6.67	1.17	1.17	8.58	12.15
8 K	4.08	0.59	1.42	6.55	2.18	1.20
16 K	0.53	1.37	1.86	0.13	1.78	0.26
32 K	0.33	0.72	0.83	0.29	0.70	0.79
64 K	0.61	0.77	0.78	0.28	0.52	0.17
128 K	0.55	0.47	0.47	0.17	0.08	0.16
256 K	0.19	0.19	0.19	0.16	0.14	0.15
512 K	0.08	0.09	0.08	0.07	0.05	0.05

Table 1: Error on $C_{D,p}$ and C_L . $C_{D,p}(1024\text{ K})$ and $C_L(1024\text{ K})$ for $\beta=1/3$ is considered as reference.

Another mesh-adaptive calculation is performed at $AoA = 4^\circ$, to investigate the effects of the NLV4 in presence of larger separations. In this case, we analyze the behavior of the NLV4 scheme with Flux-Correction (FC) -see Section 4.2- applied to a different Riemann solver. Specifically, the HLLC flux function is used [26]. The evolution of the aerodynamic coefficients are plotted against the number of vertices in Figure 7. We can see that both pressure and viscous components of drag are moderately earlier captured. Moreover, the improvements provided by the NLV4 scheme are even greater when observing the lift coefficient C_L : the same lift is almost recovered with 30 % few vertices.

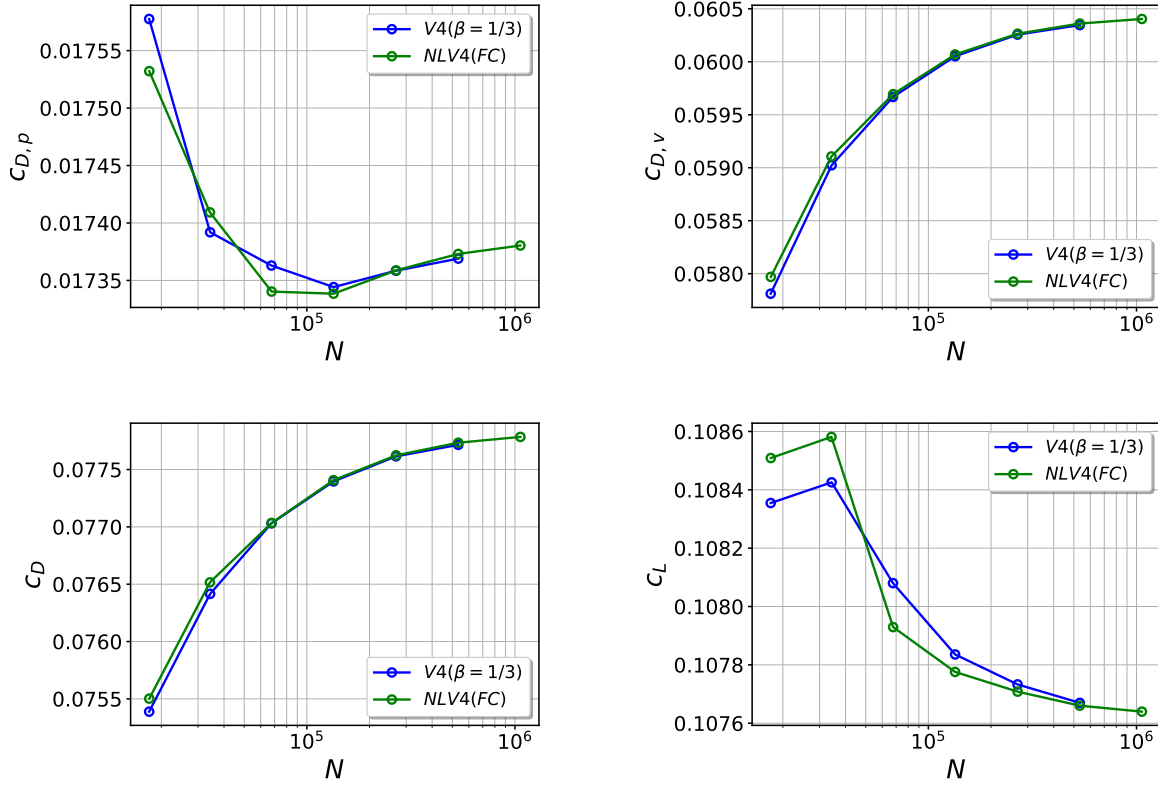


Figure 7: 3D laminar delta wing: $M = 0.3$, $Re = 4 \times 10^3$, $AoA = 4^\circ$. Goal-oriented metric-based anisotropic mesh adaptation based on the total lift coefficient C_L . Pressure drag $C_{D,p}$, viscous drag $C_{D,v}$, total drag C_D and total lift C_L coefficients against the number of vertices N . HLLC flux function. Comparison between mesh-adaptive results obtained with the baseline V4 and the new NLV4 scheme with Flux-Correction (FC).

5.3 Turbulent Steady subsonic flow past 2D NACA0012 airfoil

The third test case considered is the subsonic flow past a 2D NACA0012 airfoil, with $M_\infty = 0.15$, $T_\infty = 300$ K, angle of attack $AoA = 10^\circ$, $Re = 6 \times 10^6$

A mesh-convergence study using a goal-oriented error estimate based on the total drag coefficient C_L is run on the following six mesh sizes:

$$C = \{ 2000, \quad 4000, \quad 8000, \quad 16\,000 \quad 32\,000, \quad 64\,000 \}.$$

The schemes considered in this calculation are baseline V4 scheme and the NLV4 with flux-correction. The flux function used is the HLLC. The final adapted mesh and solution are shown in Figure 8. We can see that the separation bubble and the shear layers at the trailing edge are meshed very properly thanks to the action of the error estimate.

The evolution of the aerodynamic coefficients during the adaptive process is illustrated in Fig.10. Again, as already observed in the case of delta wing at $AoA = 4^\circ$, the evolution of drag is globally improved (both in pressure $C_{D,p}$ and viscous $C_{D,f}$ components). The lift coefficient is lightly captured earlier as well. This result confirms that the lower dissipation of the NLV4 scheme is emphasized at high angles of attack, when larger separation and detached vortex are more likely to happen.

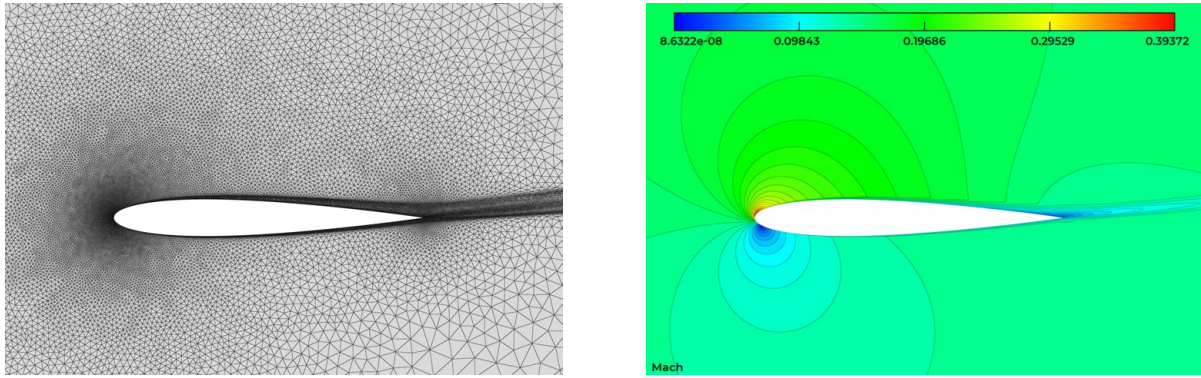


Figure 8: 2D NACA0012 airfoil: $M_\infty = 0.15$, $T_\infty = 300$ K, $Re = 6 \times 10^6$, $AoA=10^\circ$. Goal-oriented metric-based anisotropic mesh adaptation based on the total lift coefficient C_L . Adapted mesh (left) and Mach number field (right) obtained for a mesh size of $C = 64\,000$.

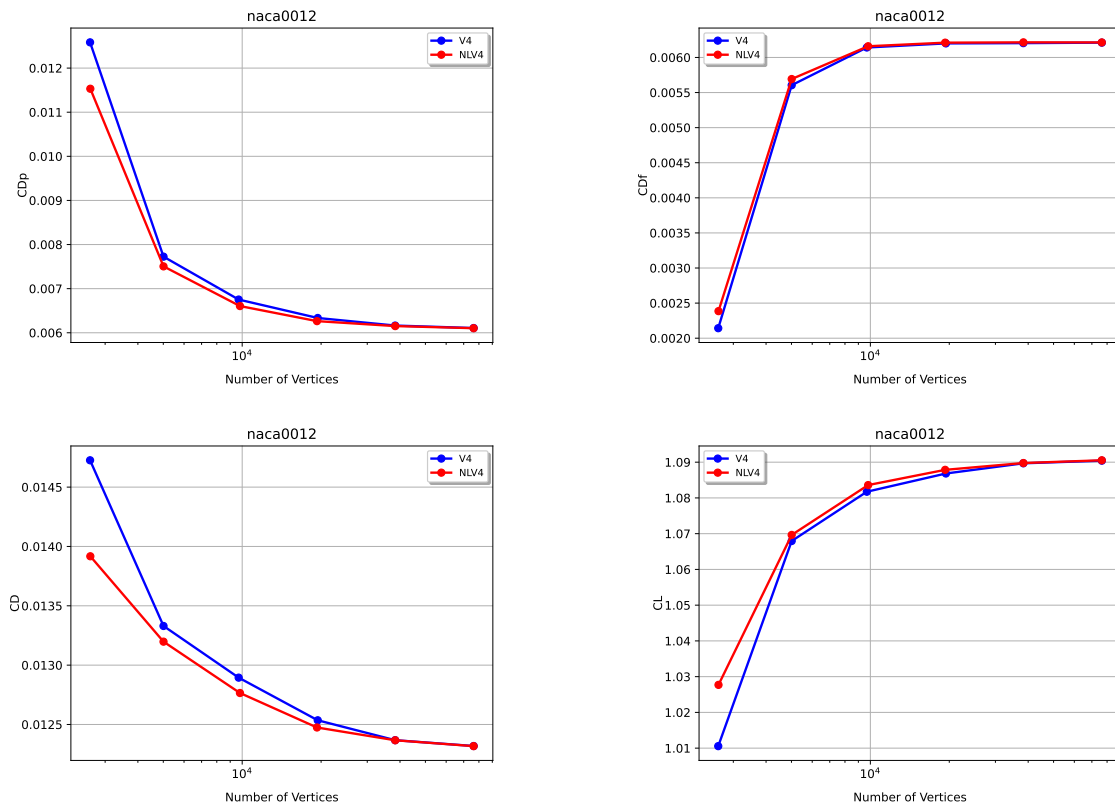


Figure 9: 2D NACA0012 airfoil: $M_\infty = 0.15$, $T_\infty = 300$ K, $Re = 6 \times 10^6$, $AoA=10^\circ$. Goal-oriented metric-based anisotropic mesh adaptation based on the total lift coefficient C_L . Pressure drag $C_{D,p}$, viscous drag $C_{D,f}$, total drag C_D , and total lift C_L coefficients against the number of vertices N . HLLC flux function. Comparison between mesh-adaptive results obtained with baseline V4 (blue) and NLV4 (red) schemes.

5.4 Turbulent Steady subsonic flow past 2D multi-element airfoil

The considered case is the subsonic flow past a 2D multi-element airfoil, with $M_\infty = 0.2$, angle of attack $AoA = 16^\circ$, $Re = 5 \times 10^6$, $T_\infty = 272.1$ K. This is a well-known case test that is often required as a validation in the context of the AIAA CFD High-Lift Prediction Workshop.

A mesh-convergence study using a *goal-oriented* error estimate based on the total lift coefficient C_L is run on the following four mesh sizes:

$$C = \{ 8000, \quad 16\,000, \quad 32\,000, \quad 64\,000 \}.$$

The schemes considered in this calculation are baseline V4 scheme and the NLV4 with flux-correction. The flux function used is the HLLC. The final adapted mesh and solution are shown in Figure 11. We can see that the separated boundary layer and shear layers are automatically captured and meshed accordingly. A transonic bubble is well visible on the first element.

The evolution of the aerodynamic coefficients during the adaptive process is illustrated in Fig.10. The reader can observe that the evolution of drag is improved only slightly with the NLV4 scheme. Conversely, the viscous drag $C_{D,f}$ and lift C_L coefficients are highly improved. This result confirms that the NLV4 scheme is more accurate at high angles of attack, when large separation is more likely to happen.

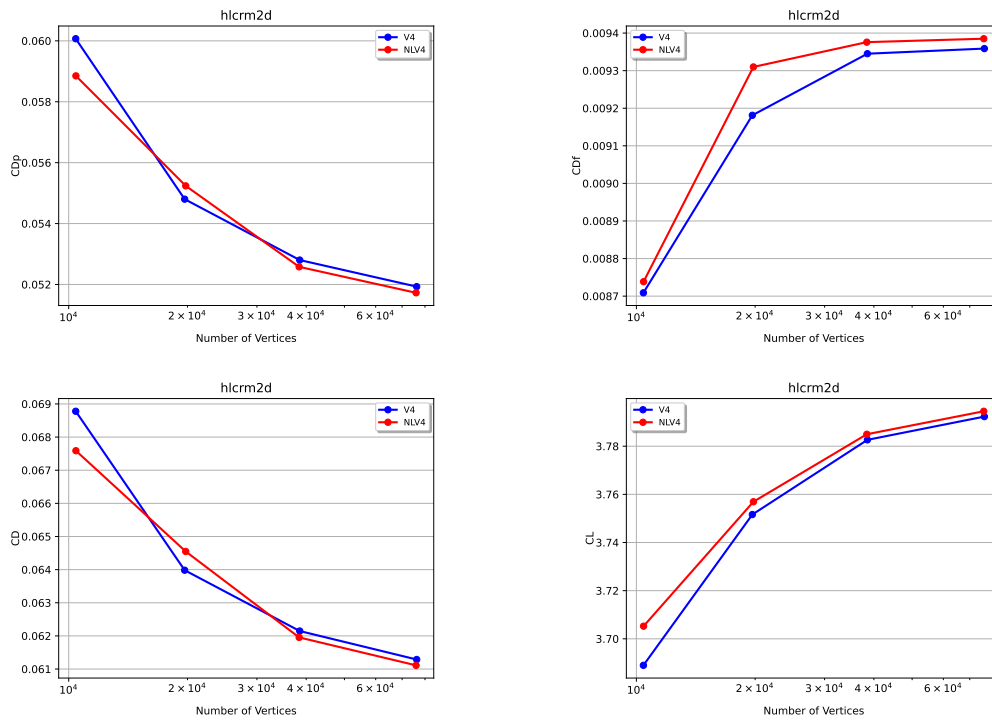


Figure 10: 2D multi-element airfoil: $M_\infty = 0.2$, $T_\infty = 272.1$ K, $Re = 5 \times 10^6$, $AoA=16^\circ$. Goal-oriented metric-based anisotropic mesh adaptation based on the total lift coefficient C_L . Pressure drag $C_{D,p}$, viscous drag $C_{D,f}$, total drag C_D , and total lift C_L coefficients against the number of vertices N . HLLC flux function. Comparison between mesh-adaptive results obtained with baseline V4 (blue) and NLV4 (red) schemes.

5.5 Laminar hypersonic flow past 2D cylinder

The last test case is the hypersonic flow past a 2D cylinder of unity diameter, with $M_\infty = 5.73$, angle of attack $AoA = 0^\circ$, $T_\infty = 39.6$ K. The Reynolds number is $Re = 2050$ for a reference radius of 0.5. The wall temperature is set to $T_w = 210$ K for the isothermal wall boundary condition. This is a well-known verification case test for hypersonic flows.

A mesh-convergence study using a *goal-oriented* error estimate based on the heat flux coefficient C_H is run on the following six mesh sizes:

$$C = \{ 4000, \quad 8000, \quad 16\,000, \quad 32\,000, \quad 64\,000, \quad 128\,000 \}.$$

We present the results with the NLV4 with flux-correction using the HLLC approximate Riemann solver in order

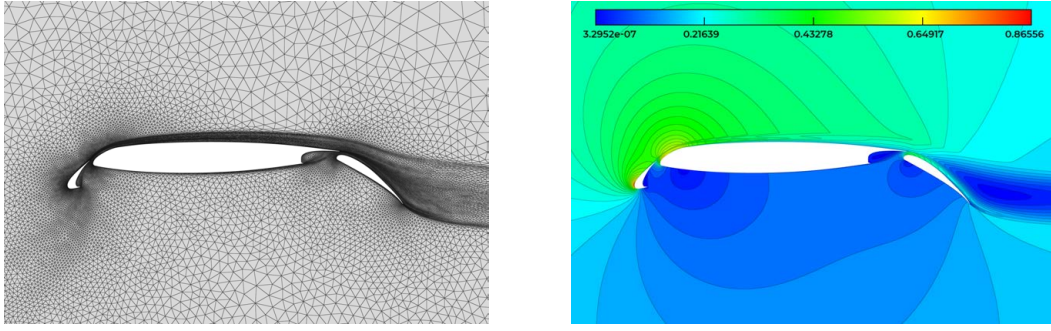


Figure 11: 2D multi-element airfoil: $M_\infty = 0.2$, $T_\infty = 272.1$ K, $Re = 5 \times 10^6$, $AoA=16^\circ$. Goal-oriented metric-based anisotropic mesh adaptation based on the total lift coefficient C_L . Adapted mesh (left) and Mach number field (right) obtained for a mesh size of $C = 64\,000$.

to demonstrate that the new formulation enables us to deal with shock. Here, the Piperno limiter has been considered [21].

Figures 13 and 14 show the final adapted mesh composed of 32 000 vertices and the associated Mach and temperature fields. The bow shock is automatically captured by the mesh-adaptive process, as is the thermal boundary layer. We observe that the bow shock is partially refined by the goal-oriented error estimate, because in the farfield, it no longer has an impact on the heat flux calculation.

The mesh convergence of the heat flux is shown in Figure 12. It is almost achieved at 32 000 vertices. We note that, on the left picture, the heat flux profile at 32 000 and 128 000 are almost identical. They are compared with the experiment of Tewfik [27] and the numerical simulation of Kopriva [28].

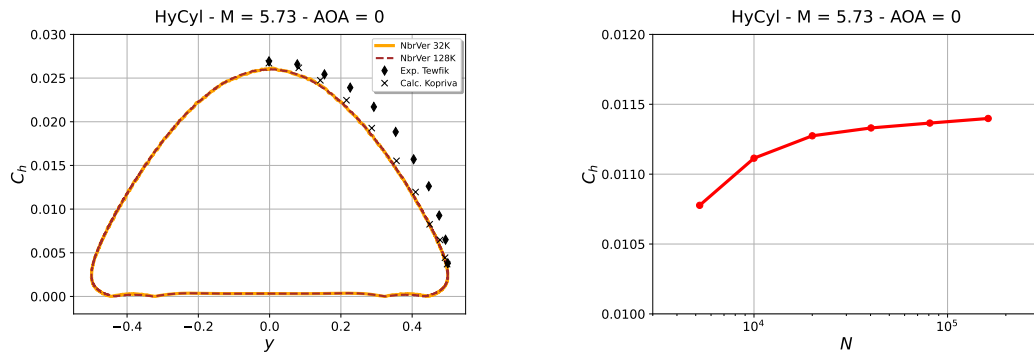


Figure 12: Hypersonic cylinder: $M_\infty = 5.73$, $Re = 2050$, $T_\infty = 39.6$ K, $T_w = 210$ K. Left, convergence of the heat flux profile and comparison with experimental data and reference numerical computation. Right, mesh-convergence of the heat flux.

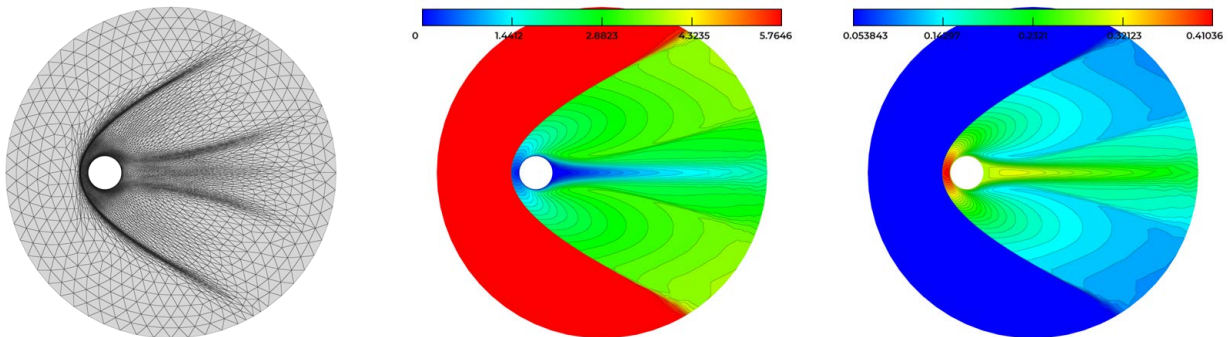


Figure 13: Hypersonic cylinder: $M_\infty = 5.73$, $Re = 2050$, $T_\infty = 39.6$ K, $T_w = 210$ K. Goal-oriented metric-based anisotropic mesh adaptation based on the heat flux. Global view of adapted mesh (left), Mach number field (middle), and temperature field (right) for a mesh size of $C = 32\,000$.

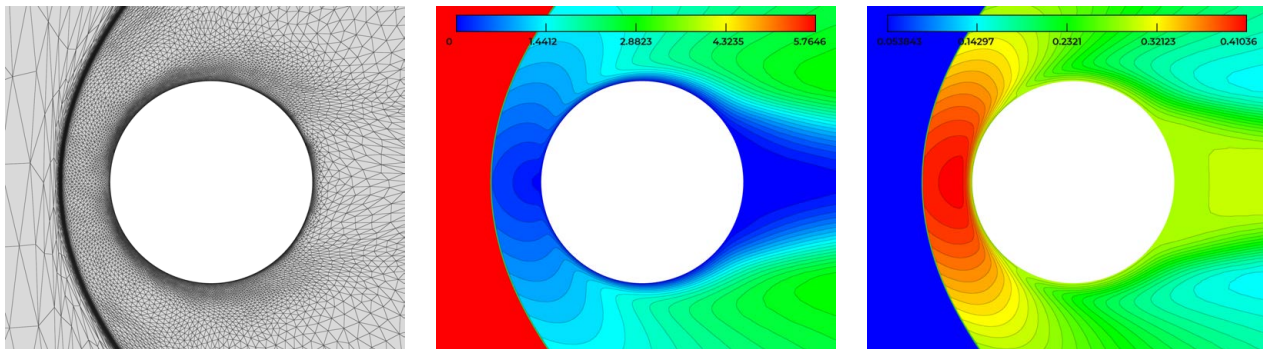


Figure 14: Hypersonic cylinder: $M_\infty = 5.73$, $Re = 2050$, $T_\infty = 39.6$ K, $T_w = 210$ K. Goal-oriented metric-based anisotropic mesh adaptation based on the heat flux. Close up view on adapted mesh (left), Mach number field (middle), and temperature field (right) for a mesh size of $C = 32\,000$.

6. Conclusions

In this paper, we present a small step forward in the development of a more accurate second-order V4-scheme for robust metric-based anisotropic mesh adaptation with minimum modifications in the solver. The original version of the V4-scheme is third-order accurate only on uniform meshes for the linear advection. Its non-linear version is third-order accurate only on uniform meshes but for general non-linear fluxes. We begin with the original idea of Koobus et al. [29] which was not practical for flows with discontinuities, then we proposed a new version which is compatible with the use of limiters. To this end, we followed the same idea of FSR schemes [11] by developing a suitable linearization for the V4-scheme reconstruction. Two different formulations were proposed. The first one is based on the Roe approximate Riemann solver, which allows us to adjust the dissipation while maintaining the order of convergence. We then proposed a second one to allow the use of any type of approximate Riemann solvers. But, this constrains the value of beta to $1/4$.

Third-order convergence is obtained for a manufactured Euler solution on uniform unstructured meshes where the NLV4-scheme is way more accurate than the V4-scheme. On the 3D Euler bump with adapted meshes, it achieves a slight better accuracy but drops to second-order convergence. Finally, the NLV4-scheme has been validated on several mesh-adaptive laminar and turbulent cases. For each case, it achieved slight improvement with respect to the V4-scheme.

The next step is to extend this methodology to the V6-scheme [30] and to analyze the accuracy of both non-linear schemes on unsteady problems.

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