On the stability of a modified Nyström method for Mellin convolution integral equations in weighted spaces

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Abstract

We consider second kind integral equations of Mellin convolution type having the form

$$f(y) + \int_0^1 k(x,y) f(x) dx + \int_0^1 h(x,y) f(x) dx = g(y), \quad y \in (0,1], \tag{1}$$

where $f(y)$ is the unknown, $h(x,y)$ and $g(y)$ are given sufficiently smooth functions and

$$k(x,y) = \frac{1}{x} \tilde{k} \left(\frac{y}{x} \right) \tag{2}$$

is a Mellin kernel with $\tilde{k}$ a given function on $[0,+\infty)$ satisfying proper assumptions.
The development of numerical methods for the solution of such kind of integral equations has a strong practical motivation due to the wide range of applications, particularly in engineering and physics.
The main difficulty in solving such equations is the proof of the stability of the chosen numerical method, being the noncompactness of the Mellin integral operator the chief theoretical barrier.
In this talk we address the concern over the stability of a numerical procedure for the solution of (1) in the case where the kernel $k(x,y)$ in (2) satisfies the following condition

$$\int_0^{+\infty} t^{-1+\sigma} |\tilde{k}(t)| dt < +\infty, \quad \text{for some} \quad \sigma > 0. \tag{3}$$

Under this assumption the Mellin integral operator

$$(\mathcal{K}F)(y) = \int_0^1 \frac{1}{x} \tilde{k} \left(\frac{y}{x} \right) F(x) dx, \tag{4}$$

is not necessarily bounded with respect to the uniform norm. We study the integral equation in a suitable weighted space of continuous functions. Then,

we consider an equivalent Mellin integral equation whose unknown is at least a continuous function. Finally, in order to approximate its solution, we apply a modified Nyström method.
Since the definition of the integral operators associated to the new equation involves a Jacobi weight, the proposed method uses a Gauss-Jacobi quadrature formula for their discretization. Unfortunately, due to the fixed singularity of the Mellin kernel at the point $x = y = 0$, such quadrature rule becomes inefficient for the approximation of the Mellin operator when y is very close to 0. Therefore, it becomes necessary to modify it in order to achieve stability and convergence results. This approach let us to reach our goal of proving theoretically the stability and the convergence of the proposed method. Furthermore, we are able to provide an error estimate in weighted uniform norm and to prove the well-conditioning of the involved linear systems which is crucial for the computation of the approximate solution.
Some numerical examples illustrate the efficiency of the proposed procedures.

Keywords: Mellin kernel, Mellin convolution integral equations, Nyström method, Gaussian rule

References

1. De Bonis, M.C., Laurita, C.: On the stability of a modified Nyström method for Mellin convolution equations, Submitted.