Many-Fold Fuzzy Semantics of Many-Place Sequent Calculi with Enlargement

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Abstract. In this paper we propose and study a semantics of many-place sequent calculi with Enlargement as well as implicit Permutation and Contraction based upon the conception of many-fold fuzzy set being a natural extension of that of two-fold one. In the propositional case, we come to the conception of fuzzy many-place matrix that is a fuzzification of the conception of many-place matrix proposed by us in earlier papers and provides semantics of propositional calculi of the kind involved.

1. Introduction

Since appearance of the conception of fuzzy set [62], its applications to various branches of Mathematics and Computer Science have become more than miscellaneous and, in many cases, even rather unexpected. Just recently, the idea of "L-fuzzification" (cf. [12]) has been used in [28]/[31] [and advanced in [39]]/ for providing semantics of derivable rules [schemas] going back to [21, 22] (instead of that of merely derivable axioms, following the paradigm of [24, 25, 30, 34, 36, 46, 47, 49, 48, 50]) of two-side (viz., ordinary Gentzen-style; cf. [10]) multiple-conclusion sequent calculi with structural/"weak (viz., ortho-)structural" rules upon the basis of L-fuzzification of the notions of ordinary (viz., bi-valential) valuation and interpretation of sequents in it (in its turn, going back to [25, 30, 34]) based upon natural treatment of two-side propositional sequents as clauses (cf. [53]) of the first-order signature with single unary assertion/truth predicate. It is remarkable that, therein, fuzzification has been involved to study crisp objects — sequent calculi, in their turn, having substantial applications to Automated Reasoning (cf. [9]) and, more generally, to such advanced branches of Computer Science as Artificial Intelligence, especially when involving minimality/optimality issues like in [44, 45, 42] as well as those of either program implementation like in [33, 37, 38, 43] or many-sorted framework (like in [50]) going back to [39].

Nevertheless, the universal framework of [28] and [31] has proved, in principle, too restrictive to cover the following generic classes of sequent calculi:

1) any kind of many-place sequent calculi (cf. [54], [55]) including Tait-style (viz., one-place; cf. [61]) calculi — viz., signed sequent calculi according to the equivalent signed sequent formalism/paradigm going back to [59];

2) two-side sequent calculi without Cut and/or Sharing;

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1In this connection, recall that the Sequent approach to Automated Deduction, though being equivalent to the Resolution one [53, 52] within the context of classical logic, is equally applicable to paraconsistent (more generally, relevance) logics (such as [1, 60]), while the Resolution rule (more precisely, its instance — the Ex Contradictione Quodlibet one — is not derivable in them.
The primary goal of the present paper is to cover these two classes. It appears that this is coherently fulfilled by means of involving the conception of many-fold \( L \)-fuzzy set, being a tuple of \( L \)-fuzzy sets with same basic set and grading lattice, that naturally extends the conception of two-fold fuzzy set \([6, 7]\), being sufficient for the second class. And what is more, involving either Cut or Sharing makes two-fold fuzzy sets under consideration couples of measures of necessity and possibility, as in \([6, 7, 8]\). In that case, results of \([28]\) become particular cases of those to be proved below but with essentially different argumentation based upon generic advanced results, some of which actually extend those of \([34]\) and \([41]\) to non-propositional case. Among other things, new argumentation discloses atomic Booleanity of \( L \)-fuzzy sets involved in addition to their lattice completeness implicitly discovered in \([28]\).

Throughout the paper, we mainly follow the formalism of \([28]\) and \([31]\) except that, for simplifying the overall exposition, sequent places (that is, sides in the two-side case) are treated here as rather finite sets than finite sequences of formulas, in which case Permutation and Contraction become trivial rules and, for this reason, are not considered at all.

The rest of the paper is as follows. In Section 2, we mainly specify basic notions and notations and argue some underlying issues to be used further. Section 3 incorporates main generic results concerning sequent calculi. In Section 4, we exemplify our general elaboration by studying a cut and/or sharing-free (and so being beyond the scopes of \([28]\) at all) multiplicative two-side sequent calculi resulted from Gentzen’s calculus \([10]\) by adding rules inverse to logical ones, the empty-sequent-less fragment of the former having been studied in \([40]\), as well as both three multiple-conclusion Gentzen-style axiomatizations of FDE \([3]\) going back to \([22, 24, 25]\) and two ones of the implication-less fragment of Gödel’s three-valued logic \([11]\) going back to \([21, 32]\).
any $\bar{a} \in S^{+}/\ast$, put:

$$
\phi^{l}/\bar{a} \triangleq \begin{cases} 
\alpha_{0}/b & \text{if } l = (1,0), \\
(\phi^{l}(\bar{a} \upharpoonright (l-1))) \circ \alpha_{l-1} & \text{otherwise.}
\end{cases}
$$

In particular, given any $f : S \rightarrow S$ and any $n \in \omega$, we have $f^{n} \triangleq (\circ \upharpoonright S^{n}) \Delta_{S}(n \times \{\bar{f}\}) : S \rightarrow S$. Finally, given also an indexed system $\{T_{j}\}_{j \in J}$ of sets, any $\bar{f} \in \prod_{j \in J} S^{T_{j}}$ determines the mapping $\langle \prod_{j \in J} \bar{f}_{j} \rangle = (\prod_{j \in J} \bar{f}_{j}) : S \rightarrow (\prod_{j \in J} T_{j})$, $a \mapsto \langle f_{j}(a) \rangle_{j \in J}$.

Let $\mathcal{A}$ be a set. Given any $S \subseteq \wp(\mathcal{A})$, an $M \in S$ is said to be maximal, provided $\{T \in S | M \subseteq T \} \subseteq \{M\}$, the set of all them being denoted by $\operatorname{max}(S)$. A $U \subseteq \wp(\mathcal{A})$ is said to be upward-directed, provided, for every $S \in \wp(\mathcal{U})$, there is some $T \in U$ such that $(\bigcup S) \subseteq T$, in which case $U \neq \emptyset$, when taking $S = \emptyset$. An $S \subseteq \wp(\mathcal{A})$ is said to be inductive, provided, for every upward-directed $U \subseteq S$, it holds that $(\bigcup U) \in S$, in which case, by Zorn’s Lemma (cf. [19, 9]), $(S \neq \emptyset) \Rightarrow (\operatorname{max}(S) \neq \emptyset)$. A closure system over $A$ is any $\mathcal{C} \subseteq \wp(A)$ such that, for every $S \subseteq \mathcal{C}$, it holds that $(\bigwedge \cap S) \in \mathcal{C}$, in which case any $\mathcal{B} \subseteq \mathcal{C}$ is called a closure basis of $\mathcal{C}$, provided $\mathcal{C} = \{A \cap \cap S | S \subseteq \mathcal{B}\}$. An operator over $A$ is any unary operation $O \in \wp(\mathcal{A})$. This is said to be “monotonic/idempotent/transitive” “inductive/finitary”, provided, for “all $\mathcal{B} \in \wp(A)(2^{1/\ast})” any upward-directed $U \subseteq \wp(A)^{\ast}$, it holds that $(O^{1/\ast}(B)) \subseteq O((B \cup U)) \subseteq U \cup O(U)$. A closure operator over $A$ is any monotonic idempotent transitive operator over $A$. Given any inductive and monotonic operator $O$ over $A$, a $B \in \wp(A)$ is said to be $O$-closed, provided $O(B) \subseteq B$, the set $\operatorname{Cl}(O)$ of all $O$-closed elements of $\wp(A)$ being a[n inductive] closure system over $A$. Finally, for any closure operator $O$ over $A$, we have $\operatorname{Cl}(C) = \wp(\mathcal{O}(A))$, while any $\mathcal{B} \subseteq \operatorname{Cl}(C)$ is a closure basis of $\wp(A)$ iff $\mathcal{O}(D) = (A \cap \cap (B \in \mathcal{B}[D \subseteq B]))$, for all $D \in \wp(A)$.

Let $\mathcal{P} = (P, \leq^\mathcal{P})$ be a poset. Given any $S \subseteq P$, a/an lower/upper bound of $S$ is any $a \in (P|\cap S)$ such that $a(\leq / \geq)^{\mathcal{P}} b$, for each $b \in S$ [then called the least/greatest element of $S$], the greatest/least one (if any) being denoted by $(\bigvee / \bigwedge)^{\mathcal{P}} S$ and called the meet/join of $S$. Then, $\mathcal{P}$ is referred to as a [complete] lattice, provided every $S \subseteq (P|\cup \mathcal{P})$ has both meet and join. In that case, we, as usual, write $\langle (1/0) \rangle$ for the unit/zero $(\bigvee / \bigwedge)^{\mathcal{P}} \emptyset$ of $\mathcal{P}$ and $\langle a(\bigvee / \bigwedge)^{\mathcal{P}} b \rangle$ for $(\bigvee / \bigwedge)^{\mathcal{P}} \{a, b\}$, where $a, b \in P$, while $S$ is called a [prime] filter/ideal of $\mathcal{P}$, whenever, for all $a, b \in P$,

$$
(a(\bigvee / \bigwedge)^{\mathcal{P}} b) \in S \iff \{(a, b) \subseteq S\} \text{ (whereas } P \setminus S \text{ is an/a ideal/filter of } \mathcal{P}) \text{, as well } \mathcal{P} \text{ is referred to as [complemented, whenever, each } a \in P \text{ has a complement in } \mathcal{P} \text{ (viz., some } b \in P \text{ such that } (a(\bigvee / \bigwedge)^{\mathcal{P}} b) = (0/1)^{\mathcal{P}} \text{ and } [\text{completely}] \text{ distributive, whenever for all } A \in (P|\cup \mathcal{P})^{\mathcal{P}}(P) \text{ and all } B \in (P|\cup \mathcal{P})^{\mathcal{P}}(P)^{A},

\text{it holds that } (\bigvee / \bigwedge)^{\mathcal{P}}_{a \in A}(V / \bigwedge)^{\mathcal{P}} = (V / \bigwedge)^{\mathcal{P}}_{f \in E} \bigcup \bigcap (\bigvee / \bigwedge)^{\mathcal{P}}_{a \in A} f(a). \text{ In general, any mention of } \mathcal{P} \text{ (including the superscript) is often omitted, unless any confusion is possible. Likewise, as usual, “Boolean” stands for “complemented distributive bounded”. We equally follow the conventional [infinitary] algebraic representation of [complete] lattices (cf., e.g., [2, 13, 14, 57, 58]) tacitly, according to which, in particular, direct products of indexed families of them are defined in the standard algebraic manner (i.e., by setting $\prod_{i \in I} (P_{i}, \leq_{P_{i}}) = (\prod_{i \in I} P_{i}, \leq_{\prod_{i \in I} P_{i}}(\{(a, b) \in \prod_{i \in I} P_{i} | \forall i \in I : a_{i} \leq_{P_{i}} b_{i}\}) \}$). Given any set $S$, $\wp(S), \in \wp(S)^{2}$ is a complemented completely distributive complete lattice called the power one of $S$ and identified with $\wp(S)$, in which case any two-element one is isomorphic to $\wp(1) = 2$, and so [finite] ones are exactly isomorphic copies of direct [finite] powers of 2 (cf. [57]).

### 2.2. Formal languages and calculi

A (formal) language is a couple of the form $L = (Fm_{L}, Sb_{L})$, where $Fm_{L}$ is a set, whose elements are referred to as $L$-formulas,
and $Sb_L$ is a set of unary operations on $Fm_L$ closed under composition and containing $ι_L ≜ Δ_{Fm_L}$, whose elements are referred to as $L$-substitutions.

Elements of $Ru_L^{[ω]} ≜ (\wp(\wp(Fm_L) × Fm_L))$ are referred to as $[\text{finitary}]$ $L$-rules, any $(Γ, Φ) ∈ Ru_L$ being normally written in either of conventional forms $Γ → Φ$ or $Γ_Ψ^{Φ}$. $Φ$ elements of $Γ$ being referred to as its conclusion/premises, $L$-rules of the form $Ψ → Φ$, where $Ψ ∈ Γ$, being referred to as inverse to $Γ$. $L$-rules of the form $σ(Γ → Φ) ≜ (σ[Γ] → σ(Φ))$, where $σ ∈ Sb_L$, being referred to as (substitutional) $L$-instances of $Γ → Φ$. $L$-Rules with(out) premises are said to be proper or non-axiomatic (resp., called $L$-axioms and identified with their conclusions). Rules with conclusion being one of premises are said to be trivial.

An $L$-calculus is any $C ⊆ Ru_L$, in which case we set $(C ∣ ω) ≜ (C ∩ Ru_L^{[ω]}).$ Then, $C$ is said to be finitary, whenever $(C ∣ ω) = C$. Further, $C$ is said to be schematic, provided it contains every $L$-instance of each of its elements.

An $L$-valuation is any $v ∈ Fm_L$, in which case $(Γ → Φ) ∈ Ru_L$ is said to be true/valid/satisfied in $v$ under $σ ∈ Sb_L$ ($v ⊨ (Γ → Φ)[σ]$, in symbols), if $(σ[Γ] ⊆ v) ⇒ (σ(Φ) ∈ v)$, and true/valid/satisfied in $v$, ($v ⊨ (Γ → Φ)$, in symbols), whenever it is true in $v$ under each $L$-substitution. Next, $v$ is said to be total/proper, if $v = / ≠ Fm_L$. Further, we also have the $L$-valuation $C_L(v) ≜ (Fm_L \setminus v)$ said to be complementary to $v$, in which case $C_L(C_L(v)) = v$. Finally, given any $L$-calculus $C$, the class of all proper $L$-valuations satisfying each member of $C$ is denoted by $Val^ω(C)$.

An $L$-consequence (relation) is any $L$-calculus $⊢$ satisfying the following consequence conditions:

(Reflexivity) $Φ ⊨ Φ$,  
(Monotonicity) $(Γ ⊨ Φ & Γ ⊆ Ξ) ⇒ Ξ ⊨ Φ$,  
(Transitivity) $(Γ ⊨ Ξ & Ξ ⊨ Φ) ⇒ Γ ⊨ Φ$,

for all $Γ, Ξ ∈ ϕ(Fm_L)$ and $Φ ∈ Fm_L$. (We adopt the following natural abbreviations: $Γ ⊨ Φ$ is used for $(Γ → Φ) ∈ ⊢$, $Γ ⊨ Ξ$ means $∀Ψ ∈ Ξ : Γ ⊨ Ψ.$) Further, we have the $L$-consequence relation $⊢^ω ⊆ ⊢$ defined as follows: for all $(Γ → Φ) ∈ Ru_L$, set:

$$(Γ ⊨ \bar{ω} Φ) ⇔ ∃Ξ ∈ ϕ^ω(Γ) : (Ξ ⊨ Φ).$$

(Notice that $\bar{ω}$ is schematic, whenever $\bar{ω}$ is so.) Then, $\bar{ω}$ is said to be compact, provided $\bar{ω} ⊆ ⊢^ω$. Further, $\bar{ω}$ is said to be inconsistent, if it is [not] distinct from $Ru_L$. In view of the reflexivity, monotonicity and transitivity of $\bar{ω}$, we have the closure operator $Cn_\barω$ over $Fm_L$, defined by $Cn_\barω(Γ) ≜ \set{Φ ∈ Fm_L | Γ \bar{ω} Φ}$, for all $Γ ⊆ Fm_L$. (Note that $Cn_\barω$ is inductive iff $\bar{ω}$ is compact.) It is routine checking that:

(1) $Val(\bar{ω}) = \text{Cl}(Cn_\barω),$

provided $\bar{ω}$ is schematic.

An $L$-semantics is any set $S$ of $L$-valuations. In that case, the set $\bar{ω} S$ of all $L$-rules true in $S$, that is, true in each member of it, is a schematic $L$-consequence (for $Sb_L$ is closed under composition), said to be the semantic one of or defined by $S$. As the consequence of the total $L$-valuation is inconsistent, we have:

(2) $\bar{ω} S = \bar{ω} \text{Pr}(S),$

where $\text{Pr}(S)$ denotes the class of all proper members of $S$. Further, set:

$S^{-1}(S) ≜ \set{σ^{-1}[v] | v ∈ S, σ ∈ Sb_L},$

$M^{[ω]}(S) ≜ \set{Fm_L \cap S | ∅ ≠ S ⊆ S}.$
Lemma 2.1. Any schematic \( L \)-consequence \( \vdash \) is defined by any closure basis \( \mathcal{B} \) of \( \text{Cl}(C_{n-}) \) (in particular, by \( \text{Val}(\vdash) \)).

Proof. Consider any \( (\Gamma \rightarrow \Phi) \in (\text{Ru}_L \setminus \vdash) \). Then, \( \Phi \notin C_{n-}(\Gamma) \), in which case there is some \( v \in \mathcal{B} \subseteq \text{Cl}(C_{n-}) \) such that \( \Gamma \rightarrow \Phi \) is not true in \( v \) under \( \iota_L \). Then, (1) completes the argument. \( \blacksquare \)

Let \( \mathcal{C} \) be an \( L \)-calculus. An extension of \( \mathcal{C} \) is any schematic \( L \)-consequence including \( \mathcal{C} \). The least extension of \( \mathcal{C} \) is denoted by \( \vdash_{\mathcal{C}} \) and said to be the consequence of (or axiomatized by) \( \mathcal{C} \).\(^2\) (When \( \mathcal{C} \) is a schematic \( L \)-consequence, we simply have \( \vdash_{\mathcal{C}} = \mathcal{C} \); in particular, \( \vdash_{\mathcal{C}} = \vdash_{\mathcal{C}} \), in any case.) In case \( \mathcal{C} \) is finitary, \( \vdash_{\mathcal{C}} \) is an extension of \( \mathcal{C} \), and so \( \vdash_{\mathcal{C}} \) is compact. Conversely, given any schematic \( L \)-consequence \( \vdash_{\mathcal{C}}, \vdash_{\mathcal{C}} \) is axiomatized by \( \vdash_{\mathcal{C}} \), in which case any compact schematic \( L \)-consequence is axiomatized by a finitary \( L \)-calculus. An extension of \( \mathcal{C} \) is said to be relatively axiomatized by an \( L \)-calculus \( \mathcal{R} \), provided it is axiomatized by \( \mathcal{C} \cup \mathcal{R} \) or, equivalently, by \( \vdash_{\mathcal{C}} \cup \mathcal{R} \), and finitary, whenever it is relatively axiomatized by a finitary \( L \)-calculus.

Lemma 2.2. Let \( \mathcal{C} \) be an \( L \)-calculus. Then, \( \text{Val}(\mathcal{C}) \) is closed under \( S^{-1} \) and \( M \).

Proof. Notice that an \( L \)-valuation \( v \) belongs to \( \text{Val}(\mathcal{C}) \) iff, for every \( \sigma \in \text{Sb}_L \), it holds that \( \sigma^{-1}[v] \in \text{Cl}(C_{n-}) \).

Consider any \( \sigma \in \text{Sb}_L \). Then, for any \( S \subseteq \varphi(\text{Fm}_L) \), we have \( \sigma^{-1}[\bigcap S] = \bigcap\{\sigma^{-1}[T] \mid T \in S\} \), so \( \text{Val}(\mathcal{C}) \) is closed under \( \mathcal{M} \). Moreover, for any \( \sigma' \in \text{Sb}_L \), we have both \( (\sigma \circ \sigma') \in \text{Sb}_L \) and \( (\sigma \circ \sigma')^{-1} = (\sigma'^{-1} \circ \sigma^{-1}) \), so \( \text{Val}(\mathcal{C}) \) is closed under \( S^{-1} \), as required. \( \blacksquare \)

Theorem 2.3. Let \( \mathcal{S} \) be an \( L \)-semantics. Then,
\[
\text{Val}^\mathcal{S}(\vdash_{\mathcal{S}}) = \mathcal{M}^\mathcal{S}([\text{Pr}(S^{-1}([\text{Pr}(\mathcal{S}^\mathcal{S})])])].
\]

Proof. The inclusion from right to left is by Lemma 2.2 and the inclusion \( \mathcal{S} \subseteq \text{Val}(\vdash_{\mathcal{S}}) \).

Conversely, in view of Lemma 2.2 and (1), \( \mathcal{S}^{-1}(\mathcal{S}) \subseteq \text{Cl}(C_{n\mathcal{S}}) \) is a closure basis of \( \text{Cl}(C_{n\mathcal{S}}) \). In this way, (1) completes the argument of the non-optimal case[, from which the optional one ensues immediately]. \( \blacksquare \)

Let \( \mathcal{C} \) be a finitary \( L \)-calculus. Given any \( \Gamma \subseteq \text{Fm}_L \) and any \( \Phi \in \text{Fm}_L \), a \( \mathcal{C} \)-derivation [of \( \Phi \)] from \( \Gamma \) is any \( \partial \in \text{Fm}_L^* \) [with \( \Phi \in \text{img}(\partial) \)] such that, for every \( k \in (\text{dom} \partial) \), either \( \partial(k) \in (\Gamma \cup \partial[k]) \) or there is some \( C \)-instance \( \Xi = \Psi \) of a rule in \( \mathcal{C} \) such that \( \Psi = \partial(k) \), whereas \( \partial \upharpoonright k \) is a \( \mathcal{C} \)-derivation of each element of \( \Xi \) from \( \Gamma \) [in which case \( \Phi \) is said to be derivable in \( \mathcal{C} \) from \( \Gamma \)]. An \( L \)-rule \( \Gamma \rightarrow \Phi \) is said to be derivable in \( \mathcal{C} \), whenever \( \Phi \) is derivable in \( \mathcal{C} \) from \( \Gamma \). The set of all \( \mathcal{L} \)-rules derivable in \( \mathcal{C} \) is denoted by \( \vdash_{\mathcal{C}} \).

Proposition 2.4. Let \( \mathcal{C} \) be a finitary \( L \)-calculus. Then, \( \vdash_{\mathcal{C}} = \vdash_{\mathcal{C}} \).

Proof. The reflexivity of \( \vdash_{\mathcal{C}} \) is by taking \( \partial = \{0, \Phi\} \), where \( \Phi \in \text{Fm}_L \). The monotonicity is evident. For proving the transitivity, consider any \( \Gamma, \Delta \in \varphi(\text{Fm}_L) \), any \( \Phi \in \text{Fm}_L \), any \( \mathcal{C} \)-derivations \( \partial \) of \( \Phi \) from \( \Delta \) and \( \partial \Phi \) of each \( \Psi \in \Delta \) from \( \Gamma \). Then, \( \partial \) is a \( \mathcal{C} \)-derivation of \( \Phi \) from \( \Xi = (\Delta \cap (\text{img} \partial)) \subseteq \varphi(\Delta) \), in which case there is some bijection \( \Upsilon \) from \( n \in |\Xi| \in \omega \) onto \( \Xi \), and so \( \langle \langle \partial \Upsilon \rangle \rangle_{n_\mathcal{C}, \partial} \) is a \( \mathcal{C} \)-derivation of \( \Phi \) from \( \Gamma \). Thus, \( \vdash_{\mathcal{C}} \) is an \( L \)-consequence. Next, for any \( \sigma \in \text{Sb}_L \), any \( \mathcal{C} \)-derivation \( \partial \) from any \( \Gamma \subseteq \text{Fm}_L \), \( \partial \sigma \) is a \( \mathcal{C} \)-derivation from \( \sigma(\Gamma) \), for \( \text{Sb}_L \) is closed under \( \sigma \), so \( \vdash_{\mathcal{C}} \) is schematic. Further, for any \( (\Gamma \rightarrow \Phi) \in \mathcal{C} \), there is

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\(^2\)When dealing with scripts (e.g., those of \( Cn \) or \( \vdash \)), we normally write \( O \) for \( \vdash_{\mathcal{O}} \), where \( O \) is either \( C \) or \( S \).
some bijection $\widehat{\Omega}$ from $|\Gamma| \in \omega$ onto $\Gamma$, in which case $(\widehat{\Omega}, \Phi)$ is a $\mathbb{C}$-derivation of $\Phi$ from $\Gamma$, for $\epsilon_L \in \text{Sb}_{L}$, and so $\vdash_{\mathbb{C}}$ is an extension of $\mathbb{C}$. Finally, by induction on the length of $\mathbb{C}$-derivations, it is routine checking that $\vdash_{\mathbb{C}} \subseteq \vdash$, for any extension $\vdash$ of $\mathbb{C}$, as required.

Proposition 2.4 is often used tacitly throughout the rest of the paper. The issue of derivation is equally applicable to infinitary calculi but with involving the apparatus of transfinite ordinal arithmetics (cf. [19]). However, such an extension would be no matter for arguing main results of the paper. For this reason, we have refrained from redundant complication of the overall exposition.

2.2.1. Propositional languages and logical matrices. Let $\Sigma$ be a propositional (viz., sentential or functional) signature, that is, a set of function symbols of finite arity to be viewed as propositional (viz., sentential) connectives, and $\mathfrak{m}_{\Sigma}$ the absolutely-free $\Sigma$-algebra, freely generated by the countable set of propositional (viz., sentential) variables $V_{\omega} \triangleq \{p_k\}_{k \in \omega}$. Then, we have the propositional (viz., sentential) language $P_{\Sigma} \triangleq \langle \text{Fm}_{\Sigma}, \text{hom}(\mathfrak{m}_{\Sigma}, \mathfrak{m}_{\Sigma}) \rangle$ over $\Sigma$ (and $V_{\omega}$). (When dealing with indices, we normally write $\Sigma$ for $P_{\Sigma}$.) A (logical) $\Sigma$-matrix (cf., e.g., [17]) is any couple of the form $A = \langle \mathfrak{A}, D^A \rangle$, where $\mathfrak{A}$ is a $\Sigma$-algebra, called the underlying one of $\mathfrak{A}$, and $D^A \subseteq A$. In general, matrices are denoted by Calligraphic letters (possibly, with indices), their underlying algebras being denoted by corresponding Fraktur letters (with same indices, if any).3

2.3. Disjunctive calculi. Fix any binary operation $\delta$ on $\text{Fm}_{L}$. Given any $\Gamma, \Delta \subseteq \text{Fm}_{L}$, put $\delta(\Gamma, \Delta) \triangleq \delta[\Gamma \times \Delta]$.

An $L$-calculus $\mathbb{C}$ is said to be $\delta$-disjunctive, provided

\begin{align*}
\text{(3)} & \quad (\text{Cn}_{\mathbb{C}}(\Gamma \cup \{\Phi\}) \cap \text{Cn}_{\mathbb{C}}(\Gamma \cup \{\Psi\})) = \text{Cn}_{\mathbb{C}}(\Gamma \cup \{\delta(\Phi, \Psi)\}), \\
\text{(4)} & \quad ((\{\Phi, \Psi\} \cap v) \neq \emptyset) \iff (\delta(\Phi, \Psi) \in v),
\end{align*}

for all $\Phi, \Psi \in \text{Fm}_{L}$. Further, an $L$-valuation $v$ is said to be (strongly)/weakly $\delta$-disjunctive, provided

\begin{align*}
\text{(5)} & \quad \delta(\Phi, \Psi) \in \text{Cn}_{\mathbb{C}}(\{\Phi\}), \\
\text{(6)} & \quad \delta(\Phi, \Psi) \in \text{Cn}_{\mathbb{C}}(\{\Psi\}), \\
\text{(7)} & \quad \delta(\text{Cn}_{\mathbb{C}}(\Gamma \cup \{\Phi\}), \Psi) \subseteq \text{Cn}_{\mathbb{C}}(\Gamma \cup \{\delta(\Phi, \Psi)\}), \\
\text{(8)} & \quad \delta(\Psi, \Phi) \subseteq \text{Cn}_{\mathbb{C}}(\{\delta(\Phi, \Psi)\}), \\
\text{(9)} & \quad \Phi \in \text{Cn}_{\mathbb{C}}(\{\delta(\Phi, \Psi)\});
\end{align*}

(iii) either (5) or (6) as well as each of (7), (8) and (9) hold.

Proposition 2.5. Let $\mathbb{C}$ be an $L$-calculus. Then, the following are equivalent:

(i) $\mathbb{C}$ is $\delta$-disjunctive;

(ii) for all $\Gamma \subseteq \text{Fm}_{L}$, and all $\Phi, \Psi \in \text{Fm}_{L}$, it holds that:

\begin{align*}
\text{(5)} & \quad \delta(\Phi, \Psi) \in \text{Cn}_{\mathbb{C}}(\{\Phi\}), \\
\text{(6)} & \quad \delta(\Phi, \Psi) \in \text{Cn}_{\mathbb{C}}(\{\Psi\}), \\
\text{(7)} & \quad \delta(\text{Cn}_{\mathbb{C}}(\Gamma \cup \{\Phi\}), \Psi) \subseteq \text{Cn}_{\mathbb{C}}(\Gamma \cup \{\delta(\Phi, \Psi)\}), \\
\text{(8)} & \quad \delta(\Psi, \Phi) \subseteq \text{Cn}_{\mathbb{C}}(\{\delta(\Phi, \Psi)\}), \\
\text{(9)} & \quad \Phi \in \text{Cn}_{\mathbb{C}}(\{\delta(\Phi, \Psi)\});
\end{align*}

(iii) either (5) or (6) as well as each of (7), (8) and (9) hold.

Proof. First, assume (i) holds. Then, by (3) with $\Gamma = \emptyset$ (and $\Psi = \Phi$), we get (5), (6), (8)(and (9)). Further, consider any $\Upsilon \in \text{Cn}_{\mathbb{C}}(\Gamma \cup \{\Phi\})$, in which case, by (5), we have $\delta(\Upsilon, \Psi) \in \text{Cn}_{\mathbb{C}}(\Gamma \cup \{\Phi\})$. Moreover, by (6), we also have $\delta(\Upsilon, \Psi) \in \text{Cn}_{\mathbb{C}}(\Gamma \cup \{\Phi\})$. Hence, by (3), we eventually get $\delta(\Upsilon, \Psi) \in \text{Cn}_{\mathbb{C}}(\Gamma \cup \{\delta(\Phi, \Psi)\})$, so (7) holds, and so does (ii).

Next, (ii)$\Rightarrow$(iii) is trivial. Conversely, (iii)$\Rightarrow$(ii) is by the equivalence of (5) and (6) under (8).

3This convention equally concerns [many-place] {fuzzy} matrices to be defined below.
Finally, assume (ii) holds. Then, (5) and (6) yield the inclusion from from right to left in (3). Conversely, consider any \( T \in (C\mathcal{C}_C(\Gamma \cup \{\Phi\}) \cap C\mathcal{C}_C(\Gamma \cup \{\Psi\})) \). Then, by (7), (8) and (9), we have, respectively, both \( \delta(\Psi, T) \in C\mathcal{C}_C(\Gamma \cup \{\delta(\Phi, \Psi)\}) \) and \( \delta(\Phi, T) \in C\mathcal{C}_C(\Gamma \cup \{\delta(\Psi, \Phi)\}) \), in which case we get \( T \in C\mathcal{C}_C(\Gamma \cup \delta(\Phi, \Psi)) \), so the inclusion from from left to right in (3) holds, and so does (i), as required.

**Lemma 2.6.** Let \( \mathcal{C} \) be a finitary \( \delta \)-disjunctive \( L \)-calculus. Then, the set of all \( \delta \)-disjunctive elements of \( \text{Cl}(C\mathcal{C}_C) \) is a closure basis of \( \text{Cl}(C\mathcal{C}_C) \).

**Proof.** Consider any \( X \in \text{Cl}(C\mathcal{C}_C) \). Let \( \mathcal{S}_X \) be the set of all \( \delta \)-disjunctive elements of \( \text{Cl}(C\mathcal{C}_C) \) including \( X \). Then, \( X \subseteq \bigcap \mathcal{S}_X \). For proving the converse, consider any \( T \in (\text{Fn}_L \setminus X) \). Set \( \mathcal{H}_X \triangleq \{ Z \in \text{Cl}(C\mathcal{C}_C) | X \subseteq Z \neq T \} \). Then, as \( \mathcal{C} \) is finitary, \( C\mathcal{C}_C \) is inductive, in which case \( \text{Cl}(C\mathcal{C}_C) \) is inductive as well, and so is \( \mathcal{H}_X \). Moreover, \( X \in \mathcal{H}_X \). Hence, by Zorn’s Lemma, there is some \( W \in \max(\mathcal{H}_X) \).

Let us prove, by contradiction, that \( W \) is \( \delta \)-disjunctive. For suppose \( W \) is not \( \delta \)-disjunctive. On the other hand, \( W \in \text{Cl}(C\mathcal{C}_C) \) is weakly \( \delta \)-disjunctive, in view of Proposition 2.5(ii) \( \Rightarrow \) (ii)(5), (6). Therefore, there are some \( \Phi, \Psi \in (\text{Fn}_L \setminus W) \) such that \( \delta(\Phi, \Psi) \neq W \). Then, in view of the maximality of \( W \), for every \( \Omega \in \{\Phi, \Psi\} \), we have \( C\mathcal{C}_C(W \cup \{\Omega\}) \notin \mathcal{H}_X \), in which case \( \mathcal{C} \in C\mathcal{C}_C(W \cup \{\Omega\}) \), for \( X \subseteq W \subseteq C\mathcal{C}_C(W \cup \{\Omega\}) \). Therefore, by the \( \delta \)-disjunctivity of \( \mathcal{C} \), we get \( \mathcal{Y} \in W \), contrary to the fact that \( W \in \mathcal{H}_X \). Therefore, \( W \) is \( \delta \)-disjunctive, in which case it belongs to \( \mathcal{S}_X \). Thus, \( T \notin \bigcap \mathcal{S}_X \), as required.

As a consequence of Lemmas 2.1, 2.6 and (1), we then get:

**Corollary 2.7.** The consequence of any finitary \( \delta \)-disjunctive \( L \)-calculus \( \mathcal{C} \) is defined by the set of all \( \delta \)-disjunctive elements of \( \text{Val}(\mathcal{C}) \).

### 2.4. Many-fold \( L \)-fuzzy sets.

Fix any \( n \in (\omega \setminus 1) \).

A \([\text{complemented}/"\{\text{completely}\}]/\text{distributive}/\text{crisp}]/(\text{completely} \langle \mathcal{L} \rangle \text{-fuzzy set})\) is a triple of the form \( F = (B^F, \mathcal{L}^F, \mu^F) \), where \( B^F \) is a set, said to be the basic one of \( F \), while \( \mathcal{L}^F \) is a \([\text{complemented}/"\{\text{completely}\}]/\text{distributive}]/(\text{two-element})\) lattice (more specifically, complete lattice), referred to as the grading one of \( F \), whereas \( \mu^F : B^F \rightarrow \mathcal{L}^F \) called the membership function of \( F \) if in which case it is uniquely determined — up to isomorphism of two-element lattices — and, for this reason, naturally identified with \( \mu^{-1}(\{1_{\mathcal{L}^F}\}) \subseteq B^F \). Then, given any set \( A \) and any \( g : A \rightarrow B^F \), we have the \([\text{complemented}/"\{\text{completely}\}]/\text{distributive}/\text{crisp}]/(\text{completely} \langle \mathcal{L} \rangle \text{-fuzzy set}) \) \( g^{-1}[F] \) with the same grading lattice, the basic set \( A \) and the membership function \( g \circ \mu^F \). Further, given sets \( A \) and \( J \) as well as a \( \mathcal{J} \)-tuple \( \vec{F} \) of \([\text{complemented}/"\{\text{completely}\}]/\text{distributive}]/(\text{completely} \langle \mathcal{L} \rangle \text{-fuzzy set}) \) with basic set \( A \), its fuzzy direct product \( \prod_i \vec{F}_i \) over \( A \) is defined to be the \([\text{complemented}/"\{\text{completely}\}]/\text{distributive}]/(\text{completely} \langle \mathcal{L} \rangle \text{-fuzzy set}) \) with the same basic set \( A \), the grading lattice \( \prod_{i \in \mathcal{I}} \mathcal{L}^F_i \) and the membership function \( \prod_{i \in \mathcal{I}} \mu^F_i \).

A \([\text{complemented}/"\{\text{completely}\}]/\text{distributive}/\text{crisp}]/(\text{n-fold} \langle \mathcal{L} \rangle \text{-fuzzy set})\) is any \( \mathcal{J} \)-tuple \( \vec{F} \) of \([\text{complemented}/"\{\text{completely}\}]/\text{distributive}/\text{crisp}]/(\text{completely} \langle \mathcal{L} \rangle \text{-fuzzy set}) \) with same basic set and grading lattice, said to be the ones of \( \vec{F} \), that is uniquely determined and, for this reason, naturally identified with the \((n + 2)\)-tuple \((B^{\vec{F}}, \mathcal{L}^{\vec{F}}, \mu^{\vec{F}})_{i \in \mathcal{N}} \) in \( N \subseteq (\mathcal{P}(n) \setminus 1) \), provided, for each \( a \in B^{\vec{F}} \) and all \( n \in N \), it holds that \( (\forall \bar{a} \in \mathcal{L}^{\vec{F}})(\forall \mu^{\vec{F}}_i(a) = (1|\bar{a})^{\mathcal{L}^{\vec{F}}}_i \). Then, given any sets \( A \) and \( J \) as well as a \( \mathcal{J} \)-tuple \( \vec{F} \) of \([\text{N-reflexive}/"\text{transitive}]/[\text{complemented}/"\{\text{completely}\}]/\text{distributive}]/(\text{n-fold} \langle \mathcal{L} \rangle \text{-fuzzy set}) \) with basic set \( A \), we have its fuzzy direct product over \( A \) defined point-wise \( \prod^n \vec{F} \triangleq (\prod^n \vec{F}_i)_{i \in \mathcal{N}} \) and being an \([\text{N-reflexive}/"\text{transitive}]/\text{fuzzy set} \)
[complemented/"(completely) distributive"] n-fold (completely) fuzzy set with basic set $A$.

It is remarkable that a 2-fold crisp fuzzy set $\tilde{F}$ is $\{2\}$-reflexive/-transitive iff $(B^{F_0} \setminus F_0) \subseteq \| \subseteq F_1$. This justifies the following notion.

A 2-fold (crisp) fuzzy set $\tilde{F}$ is said to be left-dual reflexive/transitive, provided, for each $a \in B^{F_0}$, it holds that $\mu^{F_0}(a)(\leq / \geq)\cup\mu^{F_1}(a)$ (that is, $(B^{F_0} \setminus F_0, F_1)$ is $\{2\}$-reflexive/-transitive). {Note that both left-dual reflexive and transitive 2-fold fuzzy sets are actually ordinary — 1-fold — ones.} This well fits [6, 7], when treating $F_{0/1}$ as possibility measure, while treating $F_{1/0}$ as possibility one.

3. Advanced Applications to sequent calculi

Fix any formal language $L$ and any $n \in (\omega \setminus \{1\})$. Elements of $\text{Seq}_n^L \triangleq \varphi_n(\text{Fm}_L)^n$ are referred to as $n$-place $L$-sequents. Any $\sigma \in \text{Sb}_L$ determines the equally-denoted unary operation on $\text{Seq}_n^L$ by setting $\sigma(\Phi) \triangleq \langle \sigma[\Phi_i] \rangle_{i \in n}$, for all $\Phi \in \text{Seq}_n^L$. In this way, we get the formal language $S_n^L \triangleq (\text{Seq}_n^L, \text{Sb}_L)$, called the $n$-place sequentialization of $L$. Then, $S_n^L$-rules/axioms/calculi/valuations/semantics/consequences are referred to as $n$-place $L$-sequent rules/axioms/calculi/valuations/semantics/consequences.

Given any $N \subseteq n$, any $\Gamma \in \varphi_n(\text{Fm}_L)$ and any $\Phi, \Psi \in \text{Seq}_n^L$, put $[\Gamma/\Psi] \triangleq ((N \times \{1\}) \times ((n \setminus N) \times 1)) \in \text{Seq}_n^L \ni \emptyset \triangleq \{n/\emptyset\}$ and define $(\Phi \uplus \Psi) \in \text{Seq}_n^L$ component-wise, as follows: $\pi_i(\Phi \uplus \Psi) \triangleq (\Phi_i \cup \Psi_i)$, for all $i \in n$. We adopt the conventions of Subsection 2.3 as for $\delta$ with regard to the binary operation $\uplus$ on $\text{Seq}_n^L$.

An $n$-place $L$-sequent calculus $G$ is said to be multiplicative, provided, for every $S_n^L$-instance $\Gamma \rightarrow \Phi$ of any rule of $G$ and all $\Psi \in \text{Seq}_n^L$, it holds that $(\Gamma \uplus \Psi) \vdash^G (\Phi \uplus \Psi)$, in which case, when $G$ is finitary, for any $G$-derivation $\delta$ from any $\Delta \subseteq \text{Seq}_n^L$, $\delta \circ (\uplus \Psi)$ is a $(\vdash \Gamma \uplus \Psi)$-derivation from $\Delta \uplus \Psi$, and so, by Proposition 2.4, we get:

\begin{equation}
(Cn_G(\Delta \uplus \Psi) \subseteq Cn_G(\Delta \uplus \Psi)).
\end{equation}

Non-trivial $n$-place $L$-sequent rules of the form $\Phi \rightarrow (\Psi \uplus \Phi)$, where $\Phi, \Psi \in \text{Seq}_n^L \ni \Psi \neq \emptyset$, are referred to as "basic structural"/"Enlargement instances", the set of all them being denoted by $B_n^L$.

Let $N \subseteq (\varphi(n) \setminus \{1\})$. Then, $n$-place $L$-sequent axioms/"proper rules" of the form $((N[\varphi] \uplus \Phi) / (\{i[\varphi] \varphi \circ \Phi \in \text{Seq}_n^L})) \rightarrow \Phi$, where $N \in N, \varphi \in \text{Fm}_n$ and $\Phi \in \text{Seq}_n^L$, are called $N$-Sharings/-Cuts, the set of all them being denoted by $(\mathcal{S}/\mathcal{C})^L_n(N)$.

**Proposition 3.1.** Let $N, M \in \varphi(\varphi(n) \setminus \{1\})$. Then, $B_n^L \cup S_n^L(N) \cup C_n^L(M)$ is multiplicative.

**Proof.** Let $G$ be either $B_n^L$ or $S_n^L(N)$ or $C_n^L(M)$. Then, $G$ is schematic. Moreover, for every $(\Gamma \rightarrow \Phi) \in G$ and each $\Psi \in \text{Seq}_n^L$, $((\Gamma \uplus \Psi) \rightarrow (\Phi \uplus \Psi)) \in G$. Therefore, $G$ is multiplicative, as required. \hfill $\Box$

An $[N$-reflexive/-transitive, where $N \subseteq (\varphi(n) \setminus \{1\})$ $n$-place (complemented) $\{(completely) distributive\}$ fuzzy/crisp $L$-valuation is any $n$-fold $[N$-reflexive/-transitive $(complemented)$ $\{(completely) distributive\}$ fuzzy/crisp set $\tilde{F}$ with basic set $\text{Fm}_L$. This determines the $n$-place $L$-sequent valuation $(\tilde{F}) \triangleq \{\Phi \in \text{Seq}_n^L | \forall \mu^{F_0}(\Phi_i) = 1^{\mu^{F_0}}\}$. Any $n$-place $L$-sequent semantics consisting of $n$-place $L$-sequent valuation[s] of such a kind is referred to as $[N$-reflexive/-transitive] $(complemented)$-$\{(completely) distributive\}$ fuzzy/crisp-composable.

---

\[^4\text{This covers all miscellaneous systems of many-place sharings and cuts (cf. [54], [55], [34], [41]).} \]
Remark 3.2. It is routine checking that the consequence of any fuzzy-/crisp-decomposable $n$-place $L$-sequent semantics /"is multiplicative and" contains all basic structural rules.

Lemma 3.3. Any finitary multiplicative $n$-place $L$-sequent calculus $G$ with derivable basic structural rules is $\psi$-disjunctive.

Proof. With using Propositions 2.4 and 2.5(iii)$\Rightarrow$(i). First, (6) is by basic structural rules. Next, (8) and (9) are by the fact that $(\Phi \lor \Psi) = (\Psi \lor \Phi)$ and $(\Phi \lor \Psi) = \Phi$, for all $\Phi, \Psi \in \text{Seq}_L^n$. Finally, consider any $((\Gamma \cup \{\Phi\}) \rightarrow \Upsilon) \in \forall G$ and any $\Psi \in \text{Seq}_L^n$. Then, by basic structural rules and (10), we have $(\Gamma \cup \{\Phi \lor \Psi\}) \in \forall G \rightarrow (\Upsilon \lor \Psi) \in \forall G$. (The converse is as well, in view of the fact that $\forall \in \text{Seq}_L^n$-sequent semantics /"is multiplicative and" contains all basic structural rules is $\psi$-disjunctive.

Lemma 3.4. Any proper $\psi$-disjunctive $n$-place $L$-sequent valuation $v$ is crisp-decomposable.

Proof. Define the crisp $n$-place $L$-valuation $\bar{v}$ as follows: for every $i \in n$, put $c_i \triangleq \{\varphi \in \text{Fm}_L^1 | [i/\varphi] \in v\}$. Consider any $\Phi \in \text{Seq}_L^n$. First, assume $\Phi \in (\bar{v})$. Then, there are some $i \in n$ and some $\varphi \in (\Phi \cap c_i)$, in which case we have $[i/\varphi] \in v$, and so by the $v$-disjunctivity of $v$, we get $\bar{v} = ([i/\varphi]) \in v$. Conversely, assume $\bar{v} \notin (\bar{v})$. Then, for every $i \in n$ and each $\varphi \in \Phi_i$, it holds that $\varphi \notin c_i$, that is $[i/\varphi] \notin v$. Moreover, $\emptyset \notin v$, for, otherwise, in view of the $v$-disjunctivity of $v$, for every $\Psi \in \text{Seq}_L^n$, we would have $\Psi = (\Psi \lor \emptyset) \in v$, and so $v$ would be total. Hence, for each $i \in n$, we have some bijection $i \in m_i \triangleq |\Phi_i| \in \omega$ onto $\Phi_i$, in which case, by the $v$-disjunctivity of $v$, we have $[i/\Phi_i] = \bar{v}^{[i/\Phi_i]_{j \in m_i},} \notin v$, and so we get $\bar{v} = \bar{v}^{[i/\Phi_i]_{j \in m_i},} \notin v$. Thus, $v = (\bar{v})$, as required.

Lemma 3.5. Let $\bar{F}$ be an $n$-place fuzzy $L$-valuation and $N \subseteq (\varphi(n) \setminus 1)$. Suppose $\bar{F}$ is proper. Then, $(\bar{F}) \in \text{Val}(\langle \mathbb{S}|\mathbb{C}_n^1(N) \rangle)$ if/then $\bar{F}$ is $\mathbb{N}$-reflexive-$\mathbb{M}$-transitive.

Proof. The "if" part is immediate. The converse is so as well, in view of the fact that $\emptyset \notin (\bar{F})$, taking Remark 3.2 into account.

Now, we are in a position to prove the key result of the paper:

Corollary 3.6. Let $G$ be an $n$-place $L$-sequent calculus and $N, M \in \varphi(\varphi(n) \setminus 1)$. Suppose $G$ is finitary. Then, $G$ is multiplicative and $(\mathbb{B}_n^1 \cup \mathbb{S}_n^1(N) \cup \mathbb{C}_n^1(M)) \subseteq \forall G$ if/then $G$ is defined by an $\mathbb{N}$-reflexive-$\mathbb{M}$-transitive crisp-decomposable $n$-place $L$-sequent semantics.

Proof. The "if" part is by Remark 3.2 and Lemma 3.5. (The converse is by Lemmas 3.3, 3.4, 3.5, Corollary 2.7, Proposition 2.4 and (2)).

As an immediate consequence of Corollary 3.6 and Proposition 3.1, we first get:

Corollary 3.7. Let $N, M \in \varphi(\varphi(n) \setminus 1)$. Then, the consequence of $\mathbb{B}_n^1 \cup \mathbb{S}_n^1(N) \cup \mathbb{C}_n^1(M)$ is defined by the set of all $\mathbb{N}$-reflexive-$\mathbb{M}$-transitive crisp-decomposable $n$-place $L$-sequent valuations.

Given any $\sigma \in \text{Sb}_L$ and any $N \subseteq (\varphi(n) \setminus 1)$ as well as any (complemented/"{completely}") $\mathbb{N}$-reflexive-$\mathbb{M}$-transitive $n$-place (completely) fuzzy $L$-valuation $\bar{F}$, we have one $\sigma^{-1}[\bar{F}] \triangleq |\sigma^{-1}[\bar{F}]|_{i \subseteq n}$ such that:

$$(\sigma^{-1}[\bar{F}]) = \sigma^{-1}[\bar{F}].$$

Likewise, given any set $J$ and any $J$-tuple $\bar{F}_J$ of (complemented/"{completely}") $\mathbb{N}$-reflexive-$\mathbb{M}$-transitive $n$-place (completely) fuzzy $L$-valuations, we have one $\prod_{i \in n} \bar{F}_i$ such that:

$$(\prod_{i \in n} \bar{F}_i) \Rightarrow (\text{Seq}_L^n \cap \bigcap_{j \subseteq J} (\bar{F}_i)).$$
In this way, combining Corollary 3.7, Theorem 2.3, Lemma 2.1, (11) and (12), we get:

**Corollary 3.8.** Let \( N, M \in \wp(\wp(n) \setminus 1) \). Then, any element of \( \text{Val}(\mathcal{B}_1^N \cup S_2^N(N) \cup C_1^N(M)) \) is \( N \)-reflexive \( M \)-transitive \{complemented-\}[\{completely-\}]\{distributive-\}[\{completely-\}]\{fuzzy-decomposable\}. In particular, any extension of \( \mathcal{B}_1^N \cup S_2^N(N) \cup C_1^N(M) \) is defined by an \( N \)-reflexive \( M \)-transitive \{complemented-\}[\{completely-\}]\{distributive-\}[\{completely-\}]\{fuzzy-decomposable\} \( n \)-place \( L \)-sequent semantics.

After all, Remark 3.2, Lemma 3.5 and Corollaries 3.6 and 3.8 yield the main generic result of the paper:

**Theorem 3.9.** Let \( \mathcal{G} \) be an \( n \)-place \( L \)-sequent calculus and \( N, M \in \wp(\wp(n) \setminus 1) \).\[\text{Suppose } \mathcal{G} \text{ is finitary.}\] Then, \(" \mathcal{G} \) is multiplicative and" \( \mathcal{B}_1^N \cup S_2^N(N) \cup C_1^N(M) \subseteq \vdash_{\mathcal{G}} \) iff/iff/iff is defined by an \( N \)-reflexive \( M \)-transitive \("\{\text{complemented-}\}\{\text{completely-}\}\{\text{distributive-}\}\{\text{completely-}\}\{\text{fuzzy-}\}/\text{crisp-decomposable} \) \( n \)-place \( L \)-sequent semantics.

3.1. **The propositional case and many-place fuzzy matrices.** Fix any propositional signature \( \Sigma \). We write “propositional \( \Sigma \)-sequent” for “\( P_\Sigma \)-sequent”.

An \( \{\text{\( N \)-reflexive-\}\}-transitive, where \( N \subseteq (\wp(n) \setminus 1) \) \( n \)-place \{complemented\} \{\{completely-\}\}\{distributive\} \{\{completely-\}\}\{fuzzy\}/\text{crisp} \Sigma \)-matrix is any couple of the form \( A = (\mathfrak{A}, \vec{F}^A) \), where \( \mathfrak{A} \) is a \( \Sigma \)-algebra, called the underlying algebra of \( A \), and \( \vec{F}^A \) is an \( N \)-reflexive/\( M \)-transitive \{complemented\} \{\{completely-\}\}distributive \{\{completely-\}\}fuzzy/\text{crisp} set with basic set \( A \), called the one of \( A \).\[\text{Suppose } \mathcal{G} \text{ is finitary.}\] Then, given any subalgebra \( \mathfrak{B} \) of \( \mathfrak{A} \), put \( (A \upharpoonright B) = (\mathfrak{B}, \vec{F}^A \upharpoonright B)_{i \in \mathcal{I}} \).

Given any class \( M \) of \{\{\text{\( N \)-reflexive-\}\}-transitive\} \( n \)-place \{complemented\} \{\{completely-\}\}\{distributive\} \{\{completely-\}\}fuzzy/\text{crisp} \Sigma \)-matrices, we have the \{\{\text{\( N \)-reflexive-\}\}-transitive\} \{\{completely-\}\}distributive \{\{completely-\}\}fuzzy/\text{crisp}\ \{\text{completely-}\}\{\text{fuzzy-}\}/\text{crisp-decomposable} \( n \)-place propositional \( \Sigma \)-semantics \( S_1(M) \equiv \{h^{-1}[\vec{F}^A] \upharpoonright A \in M, h \in \text{hom}(\vec{S}_M, \mathfrak{A})\} \), in which case \( \vdash_M \upharpoonright S_1(M) \) is said to be defined by \( M \).

Conversely, given a set \( S \) of \{\{\text{\( N \)-reflexive-\}\}-transitive\} \( n \)-place \{complemented\} \{\{completely-\}\}distributive \{\{completely-\}\}fuzzy/\text{crisp} \( \Sigma \)-valuations, we have the class \( M(S) \equiv \{\vec{S}_M, \vec{F} \mid \vec{F} \in S\} \) of \{\{\text{\( N \)-reflexive-\}\}-transitive\} \( n \)-place \{\{completely-\}\}distributive \{\{completely-\}\}fuzzy/\text{crisp} \Sigma \)-matrices, in which case \( S_1(M(S)) = S^{-1}(\{S\}) \), in view of (11), and so, by Lemma 2.2, we get:

\[
\vdash_{M(S)} = \vdash_{\{S\}} \upharpoonright ,
\]

as \( S^{-1} \) is idempotent, because \( \Delta_{FnL} \in S_{bL}, \) for any formal language \( L \). In this way, Theorem 3.9 and (13) immediately yield:

**Corollary 3.10.** Let \( \mathcal{G} \) be an \( n \)-place propositional \( \Sigma \)-sequent calculus and \( N, M \in \wp(\wp(n) \setminus 1) \).\[\text{Suppose } \mathcal{G} \text{ is finitary.}\] Then, \(" \mathcal{G} \) is multiplicative and" \( \mathcal{B}_1^N \cup S_2^N(N) \cup C_1^N(M) \subseteq \vdash_{\mathcal{G}} \) iff/iff is defined by a class of \( N \)-reflexive \( M \)-transitive \( n \)-place \("\{\text{complemented}\}\{\text{completely-}\}\{\text{distributive-}\}\{\text{completely-}\}\{\text{fuzzy-}\}/\text{crisp}\ \Sigma \)-matrices.

The \[\{\text{\( \Sigma \)-optional}\}\] “crisp” case of this corollary has been actually due to [34] and [41].

3.2. **Peculiarities of the two-side case.** Here, it is supposed that \( n = 2 \). In that case, \("\{\text{left/right side}\}\) means \("\text{place} \( 0/1 \)\)”. Any two-side sequent \( \vec{F} \) is written in the conventional form \( \Phi_0 \Rightarrow \Phi_1 \) involving the binary infix ”side-separator” symbol \( \Rightarrow \) (instead of the traditional ones \( \upharpoonright \) and \( \rightarrow \) just to avoid a confusion with equally traditional notations of consequence relations and rules, respectively).

\[\text{The crisp case corresponds to } [34] \text{ and } [41].\]
Likewise, given any set $J$ consisting of (completely) distributive- (completely) fuzzy-left-dual (reflexive) (transitive) crisp-decomposable $L$-valuations that fits both [28] and [31].

Any (left-dual reflexive)-transitive 2-side (complemented) $L$-sequent valuation determines the 2-side $L$-sequent valuation $(\vec{F}) \triangleq \{ \vec{F} \in \text{Seq}_L^2 | \bigwedge_{\sigma} \vec{F}^\sigma (\mu_{\bar{J}}(\Phi_0)) \leq \varepsilon^\sigma \lor \vec{F}^\sigma (\mu_{\bar{J}}(\Phi_1)) \}/“$in which case:

\[
(\vec{F})_j = (\langle C \|_{\Phi_0}, F_1 \rangle)
\]

In view of (14), we then get the following "left-dual" version of the 2-side case of Corollary 3.6.

**Corollary 3.11.** Let $G$ be a finitary 2-side $L$-sequent calculus. Then, $G$ is multiplicative and $(\text{B}_1^2(\cup \text{S}_L^2) \cup \text{C}_L^2) \subseteq \Gamma_G$ iff $\Gamma_G$ is defined by a left-dual (reflexive) (transitive) crisp-decomposable 2-side $L$-sequent semantics.

As an immediate consequence of Corollary 3.11 and Proposition 3.1, we then get:

**Corollary 3.12.** The consequence of $B_1^2(\cup \text{S}_L^2) \cup \text{C}_L^2$ is defined by the set of all left-dual (reflexive) (transitive) crisp-decomposable 2-side $L$-sequent valuations.

Given any $\sigma \in \text{Sb}_L$ and any left-dual reflexive-transitive 2-side \{completely\} distributive (completely) fuzzy-crisp L-valuation $\vec{F}$, $\sigma^{-1}[\vec{F}]$ is one such that:

\[
(\sigma^{-1}[\vec{F}]_j) = \sigma^{-1}[\vec{F}]_j.
\]

Likewise, given any set $J$ and any $J$-tuple $\vec{F}$ of left dual reflexive-transitive 2-side \{completely\} distributive (completely) fuzzy $L$-valuations, $\prod_{\vec{F}}$ is one such that:

\[
(\prod_{\vec{F}}) = (\text{Seq}_L^2 \cap \bigwedge_{\vec{F}}(\vec{F}_j)).
\]

In this way, combining Corollary 3.12, Theorem 2.3, Lemma 2.1, (15) and (16), we get:

**Corollary 3.13.** Any element of Val($\text{B}_1^2(\cup \text{S}_L^2) \cup \text{C}_L^2$) is left-dual (reflexive) (transitive) (complemented) \{completely\} distributive-\{completely\} fuzzy-decomposable. In particular, any extension of $\text{B}_1^2(\cup \text{S}_L^2) \cup \text{C}_L^2$ is defined by a left-dual (reflexive) (transitive) (complemented) \{completely\} distributive-\{completely\} fuzzy-crisp-decomposable 2-side $L$-sequent semantics.

After all, Remark 3.2, Lemma 3.5 and Corollaries 3.11 and 3.13 yield the main generic result of the paper concerning two-side sequent calculi:

**Theorem 3.14.** Let $G$ be a finitary 2-side $L$-sequent calculus. Then, \{G is multiplicative and \} $(\text{B}_1^2(\cup \text{S}_L^2) \cup \text{C}_L^2) \subseteq \Gamma_G$ iff $\Gamma_G$ is defined by a left-dual (reflexive) (transitive) \{complemented\} \{completely\} distributive-\{completely\} fuzzy-crisp-decomposable 2-side $L$-sequent semantics.

The optional case of this theorem (dealing with both Sharings and Cuts) has been due to [28]. However, the argumentation found therein is not applicable to proving the theorem under consideration in the general case, because it was essentially based
3.2.1. The propositional case. Here, we properly follow Subsection 3.1. A 2-side fuzzy Σ-matrix $A = (\mathfrak{A}, \vec{F}^A)$ is said to be left-dual reflexive/-transitive, whenever $\vec{F}^A$ is so. Given any class $M$ of {left-dual reflexive|transitive} 2-side [complemented] $\{((\text{completely}) \text{-} \text{distribution}) \text{"}(\text{completely}) \text{-} \text{fuzzy"}/\text{crisp}\}$-matrices, we have the left-dual {reflexive|transitive} [complemented] $\{((\text{completely}) \text{-} \text{distribution}) \text{"}(\text{completely}) \text{-} \text{fuzzy"}/\text{crisp}\}$-decomposable 2-side propositional Σ-semantics $S_1(M) \triangleq \{h^{-1}[\vec{F}^A] \upharpoonright |A \in M, h \in \text{hom}(\mathfrak{A}_{\mathbb{C}}, \mathfrak{A})\}$, in which case $\vdash_{M(S)} \triangleq \vdash_{S_1(M)}$ is said to be left-dual defined by $M$, while, for any set $S$ of {left-dual reflexive|transitive} 2-side [complemented] $\{((\text{completely}) \text{-} \text{distribution}) \text{"}(\text{completely}) \text{-} \text{fuzzy"}/\text{crisp}\}$-matrices, whereas $S_1(M(S)) = S^{-1}([S])$, in view of (15), and so, by Lemma 2.2, we get:

$$\vdash_{M(S)} \vdash_{S_1},$$

as $S^{-1}$ is idempotent, because $\Delta_{\text{Fm}} \in Sb_L$, for any formal language $L$. In this way, Theorem 3.14 and (17) immediately yield:

**Corollary 3.15.** Let $G$ be a (finitary) 2-side propositional Σ-sequent calculus. Then, "G is multiplicative and" $(B^G_2(\cup \cup \cup C_2^2)) \subseteq \vdash_G \text{iff/iff(iff)} \vdash_G$ is defined by a class of {left-dual reflexive|left-dual transitive} 2-side [complemented] $\{((\text{completely}) \text{-} \text{distribution}) \text{"}(\text{completely}) \text{-} \text{fuzzy"}/\text{crisp}\}$-valuations, every member of $M(S)$ is a {left-dual reflexive|transitive} 2-side [complemented] $\{((\text{completely}) \text{-} \text{distribution}) \text{"}(\text{completely}) \text{-} \text{fuzzy"}/\text{crisp}\}$-matrix, whereas $S_1(M(S)) = S^{-1}([S])$, in view of (15), and so, by Lemma 2.2, we get:

$$\vdash_{M(S)} \vdash_{S_1},$$

as $S^{-1}$ is idempotent, because $\Delta_{\text{Fm}} \in Sb_L$, for any formal language $L$. In this way, Theorem 3.14 and (17) immediately yield:

**Corollary 3.15.** Let $G$ be a (finitary) 2-side propositional Σ-sequent calculus. Then, "G is multiplicative and" $(B^G_2(\cup \cup \cup C_2^2)) \subseteq \vdash_G \text{iff/iff(iff)} \vdash_G$ is defined by a class of {left-dual reflexive|left-dual transitive} 2-side [complemented] $\{((\text{completely}) \text{-} \text{distribution}) \text{"}(\text{completely}) \text{-} \text{fuzzy"}/\text{crisp}\}$-matrices.

### 4. Examples

Here, we deal with the propositional languages $\Sigma_{[01]}(\mathfrak{A}) \triangleq \{\wedge, \vee, \sim, [(\sim), \top, \bot]\}$, where $\wedge$ is conjunction and $\vee$ is disjunction — are binary, while $\sim$ is negation — is unary [whereas $\top$ and $\bot$ are true and falsehood constants — are nullary].

By $\mathfrak{A}_{[01]}$ denote the $\Sigma_{[01]}$-algebra with carrier $\mathfrak{A}$ and operations given as follows: put, for all $a, b \in \mathfrak{A}$, $(a \wedge \vee b) \mathfrak{A}_{[01]}(a, b) \triangleq (\min \max) (a, b, c)$, and $(\sim \{\top\}) \mathfrak{A}_{[01]}(a) \triangleq a = (a, 1, 0) / (a, 0, 1)$ as well as $(\bot \top) \mathfrak{A}_{[01]}(a) \triangleq (a, 0, 1)$. In this connection, we use the following standard abbreviations:

$$t \triangleq (1, 1) \quad f \triangleq (0, 0) \quad b \triangleq (0, 1) \quad n \triangleq (1, 0)$$

Let $\Sigma'$ be a signature containing a nullary/ unary/ binary $b / t / 0$. Then, a 2-side crisp $P_{\Sigma'}$-valuation $\hat{C}$ is said to be $b$-true/false, whenever $b \in |C_{\Sigma'}$, for all $i \in 2$. Likewise, it is said to be $i$-(mutually-)negative, whenever $(\varphi \in C_1) \iff (\varphi \notin C_{1-i})$, for all $i \in 2$ and all $\varphi \in \text{Fm}_{\Sigma'}$. Finally, it is said to be $\circ$-conjunctive/disjunctive, whenever, for all $i \in 2$ and all $\varphi \in \text{Fm}_{\Sigma'}$, $(\varphi \circ \psi) \in C_{1-i}$, if both either $\varphi \in C_i$ and/or $\psi \in C_i$. Likewise, a 2-side propositional $\Sigma'$-sequent calculus $S$ is said to be []-, $\circ$-conjunctive/disjunctive, whenever, for all $\varphi \in \text{Fm}_{\Sigma'}^+$, it holds that $(((1|0) : \Gamma) \cup ((0|1) : [(\text{true} \varphi)])$ $\vdash_{S} (((1|0) : \Gamma) \cup ((0|1) : [(\text{true} \varphi)]))$.

---

6This is nothing but FDE [1, 3] (cf. [22, 25, 51]) [expanded by truth and falsehood constants (as well as classical negation)].
of Gentzen’s calculus versus
First-Degree Entailment and its extensions

By \( \mathbb{L}^{(S)(C)} \) we denote the two-side propositional \( \Sigma_{[01]} \)-sequent calculus constituted by basic structural rules \((\text{and Sharings})\) \{as well as Cuts\} collectively with both the following rules and inverse to these [unless they are basic structural ones]:

\[
\begin{align*}
\text{Left} & \quad (\wedge) \quad (\Gamma \cup \{\phi, \psi\}) \rightarrow \Delta & (\wedge) \quad (\Gamma \rightarrow (\Delta \cup \{\phi, \psi\})) \\
& \quad (\Gamma \cup \{\phi \land \psi\}) \rightarrow \Delta & (\Gamma \rightarrow (\Delta \cup \phi \land \psi)) \\
& \quad (\Gamma \cup \{\phi \lor \psi\}) \rightarrow \Delta & (\Gamma \rightarrow (\Delta \cup \phi \lor \psi)) \\
& \quad \downarrow \quad \vdash & \quad \vdash \\
\text{Right} & \quad (\top) & (\top) \\
& \quad (\Gamma \top) \rightarrow \Delta & (\Gamma \rightarrow (\top \cup \phi)) \\
\end{align*}
\]

where \( \phi, \psi \in \text{Fm}_{\Sigma} \) and \( \Delta, \Gamma \in \mathcal{P}_{\Sigma}(\text{Fm}_{\Sigma}) \). Note that \( \mathbb{L}_{[01]}^{(SC)} \) is the propositional fragment of Gentzen’s calculus [10] supplemented with rules inverse to the above logical ones, which are derivable in the original calculus, so they have same derivable logical fragment of Gentzen’s calculus [10] supplemented with rules inverse to the above.

Lemma 5.1. \( \mathbb{L}^{(S)(C)}_{[01]} \) is multiplicative.

Proof. Note that \( \mathcal{G} \triangleq \mathbb{L}^{(S)(C)}_{[01]} \setminus (B_{\Sigma}^2 \cup S_{\Sigma}^2 \cup C_{\Sigma}^2) \) is schematic. Moreover, for any \((\Gamma \vdash \Phi) \in \mathcal{G}\) and all \( \Psi \in \text{Seq}_{\Sigma} \), it holds that \((\Gamma \vdash \Phi \rightarrow \Psi) \in \mathcal{G}\).

Hence, \( \mathcal{G} \) is multiplicative. Then, Proposition 3.1 completes the argument. \( \blacksquare \)

Given any \( \bar{c} \in \{b, n\}^* \), we have the subalgebra \( \mathfrak{B}_{[4]_{[01]}-\varepsilon} \) with carrier \( B_{4-\varepsilon} \triangleq 2^2 \) (image \( \bar{c} \)), in which case \( \mathfrak{B}_{[4]_{[01]}-\varepsilon} \) corresponds to “[the bounded version of] Kleene’s three-valued logic [16]”/“[the bounded version of] Priest’s logic of paradox [20] (cf. [23, 27])”/“the classical logic”. Then, we have the 2-side crisp \( \Sigma_{[01]} \)-matrix \( \mathfrak{B}_{[4]_{[01]}} \triangleq \{ \mathfrak{B}_{[4]_{[01]}}, \{t, n\}, \{t, b\} \} \). Put \( \mathfrak{B}_{[4]_{[01]}-\varepsilon} \triangleq \{ \mathfrak{B}_{[4]_{[01]}}, B_{4-\varepsilon} \} \). This is clearly left-dual reflexive/transitive crisp-decomposable 2-side propositionnal \( \Sigma_{[01]} \)-sequent semantics \( \mathfrak{S} \).

Theorem 5.2 (Completeness Theorem). The consequence of \( \mathbb{L}^{(S)(C)}_{[01]} \) is left-dual defined by \( \mathfrak{B}_{[4]_{[01]}-\varepsilon} \).

Proof. Note that both \( \{t, n\} \) and \( \{t, b\} \) are prime filters of \( \mathfrak{B}_{[4]} \). Hence, \( (\wedge) \) and \( (\lor) \) are true in \( \mathfrak{S} \). Moreover, \( \sim \mathfrak{B} \) \( \{t, b\} \) \( \{t, b\} \), and so \( \sim \mathfrak{B} \) \( \{t, b\} \) \( \{t, b\} \). Therefore, \( (\sim) \) are true in \( \mathfrak{S} \). Finally, \( (\top) \) is clearly true in \( \mathfrak{S} \), for \( \vdash \) is not so. Thus, by Corollary 3.12, we get \( \vdash \mathbb{L}^{(S)(C)}_{[01]} \subseteq \vdash \mathfrak{B}_{[4]_{[01]}-\varepsilon} \).

Conversely, by Lemma 5.1 and Corollary 3.11, \( \vdash \mathbb{L}^{(S)(C)}_{[01]} \) is left-dual defined by a left-dual (reflexive) \{transitive\} crisp-decomposable 2-side propositionnal \( \Sigma_{[01]} \)-sequent semantics \( \mathfrak{S} \). Consider any \( v \in \mathfrak{S} \). Then, there is some left-dual reflexive \{left-dual transitive\} 2-place crisp \( \mathfrak{P}_{\Sigma} \) valuation \( \bar{C} \) such that \( v = (\bar{C}) \). In that case, taking \( \sim \) into account, we see that it is \( \sim \)-negative. Likewise, by the rules \( (\wedge) \) and \( (\lor) \) as well as inverse to these, we conclude it is both \( \wedge \)-conjunctive and \( \lor \)-disjunctive. Finally, as \( \sim \) \( v \), by the rules \( (\top) \) \( (\top) \) and \( (\top) \), we eventually conclude that it is both \( \bot \)-false and \( \top \)-true. Therefore, by the following immediate observation, we conclude that

\[
h \triangleq \left( \prod_{i \in \mathbb{N}} \lambda_{\text{Fm}_{\Sigma}}^{\wedge} \right) \in \text{hom}(\text{m}_{\Sigma}, \mathfrak{B}_{[4]_{[01]}}).
\]
while $\tilde{C} = h^{-1}[\{(t, n), (t, b)\}]$:

**Claim 5.3.** Let $\Sigma' \supseteq \Sigma_{[01]}$ be a signature and $\tilde{C}$ a 2-side crisp both $\sim$-negative, $\land$-conjunctive and $\lor$-disjunctive $[\text{as well as both } \bot \text{-false and } \top \text{-true}]/ P_{2\Sigma}$-valuation. Then, $h \triangleq (\prod_{i \in \mathbb{E}} C_i^{\Sigma'}) \in \text{hom}(\mathfrak{Fm}_{\Sigma'} \upharpoonright \Sigma_{[01]}, \mathfrak{B}_{4, [01]})$, while $\tilde{C} = h^{-1}[\{(t, n), (t, b)\}]$.

And what is more, once $C_0 \subseteq \mid \supseteq C_1$, we also have $(n|b) \not\in h[\mathfrak{Fm}_{\Sigma_{[01]}}]$. In this way, $v \in S_t(\mathfrak{B}_{4, [01]}\upharpoonright \omega (t))$. Thus, $S \subseteq S_t(\mathfrak{B}_{4, [01]}\upharpoonright \omega (t))$, and so $\vdash_{\text{LK}_{[01]}} \supseteq \vdash_{\text{LK}_{[01]}}^\text{bd} \mathfrak{B}_{4, [01]}\upharpoonright \omega (t)$, as required.

This strengthens [40] and, in general, demonstrates the power of generic results obtained in the work.

**Corollary 5.4.** There are Cuts/Sharings not derivable in $\text{LK}_{[01]}^{\text{SC/C}}$, and so in $\text{LK}_{[01]}$.

**Proof.** Let $h \in \text{hom}(\mathfrak{Fm}_{\Sigma_{[01]}}, \mathfrak{B}_{4, [01]}\upharpoonright \omega (n|b))$ extend $V_\omega \times \{b/n\}$. Then, $((\varnothing \mapsto p_0, p_0 \mapsto \varnothing) \mapsto \emptyset)/(p_0 \mapsto p_0)$ is not true in $h^{-1}[\{(t, n), (t, b)\}]$ under $\text{LK}_{[01]}$. In this way, Theorem 5.2 completes the argument.

On the other hand, $\text{LK}_{[01]}^{\text{SC}}$ and $\text{LK}_{[01]}$ are well-known to have same derivable axioms, and so admissible rules.\(^7\) This can equally be shown with using Theorem 5.2 as follows:

**Corollary 5.5** ($\text{Cut Elimination Theorem}$). $\text{Cu}_{\text{LK}_{[01]}^{\text{SC}}} (\varnothing) = \text{Cu}_{\text{LK}_{[01]}^{\text{SC}}} (\varnothing)$.

**Proof.** The inclusion from right to left is trivial, for $\text{LK}_{[01]}^{\text{SC}} \subseteq \text{LK}_{[01]}$. Conversely, consider any $(\Gamma \to \Delta) \in \text{Cu}_{\text{LK}_{[01]}^{\text{SC}}} (\varnothing)$. Then, by Theorem 5.2, for each $h \in \text{hom}(\mathfrak{Fm}_{\Sigma_{[01]}}, \mathfrak{B}_{4, [01]}\upharpoonright \theta n)$, either $(h[\Gamma] \cap \{f\}) \neq \emptyset$ or $(h[\Delta] \cap \{t\}) \neq \emptyset$. Consider any $g \in \text{hom}(\mathfrak{Fm}_{\Sigma_{[01]}}, \mathfrak{B}_{4, [01]}\upharpoonright \theta n)$. Let $\mathfrak{R}_{4, [01]}$ be the $\Sigma_{[01]}$-algebra with carrier 4 and operations defined as follows: put $[(\bot \lor \top) \mathfrak{R}_{4, [01]} \triangleq (0/3)$ as well as $\text{both } (a \land \lor \mathfrak{R}_{4, [01]} b) \triangleq (\text{min } \max (a, b)$ and $\sim \mathfrak{R}_{4, [01]} a \triangleq (3 - a)$, for all $a, b \in 4$. Then, we have the surjective $e \in \text{hom}(\mathfrak{R}_{4, [01]}, \mathfrak{B}_{4, [01]}\upharpoonright \theta n)$ defined by:

\[
\begin{align*}
e(0) & \triangleq f, \\
e(1) & \triangleq b, \\
e(2) & \triangleq b, \\
e(3) & \triangleq t.
\end{align*}
\]

Therefore, there is some $f \in \text{hom}(\mathfrak{Fm}_{\Sigma_{[01]}}, \mathfrak{R}_{4, [01]})$ such that $g = (f \circ e)$. Moreover, we have the $e' \in \text{hom}(\mathfrak{R}_{4, [01]}, \mathfrak{B}_{4, [01]}\upharpoonright \theta nb)$ defined by:

\[
\begin{align*}
e'(0) & \triangleq f, \\
e'(1) & \triangleq f, \\
e'(2) & \triangleq t, \\
e'(3) & \triangleq t,
\end{align*}
\]

in which case $h \triangleq (f \circ e') \in \text{hom}(\mathfrak{Fm}_{\Sigma_{[01]}}, \mathfrak{B}_{4, [01]}\upharpoonright \theta nb)$. Assume $g[\Gamma] \subseteq \{t\}$. Then, $f[\Gamma] \subseteq \{3\}$, in which case $h[\Gamma] \subseteq \{t\}$, and so there is some $\psi \in \Delta$ such that $h(\psi) = t$. Thus, $f(\psi) \in \{2, 3\}$, in which case $g(\psi) \in \{t, b\}$, and so $(\Gamma \to \Delta) \in (g^{-1}[\{(t, \{t, b\})\}]t)$. In this way, Theorem 5.2 completes the argument.

\(^7\)This clarifies the peculiarity of semantics of rather derivable rules than merely derivable axioms, studied here as well as in [28], [31], [34] and [41].
This demonstrates the power of the algebraic technique provided by the conception of fuzzy matrix as well as of the generic semantic approach elaborated here.

6. **Multiple-conclusion Gentzen-style axiomatizations of FDE and its basic expansions**

6.1. **Multiplicative calculi.** By $\mathbb{LB}_{01}^{[-]}$ we denote the two-side propositional $\Sigma_{01}^{[-]}$-sequent calculus constituted by basic structural rules, Sharings, Cuts, the above rules $\langle \land \rangle$ and $\langle \lor \rangle$ [as well as both Left ($\bot$) and Right ($\top$)] collectively with the following ones:

\[
\begin{align*}
\text{Left} & : \quad \land : \quad (\Gamma \cup \{\sim \phi, \sim \psi\}) \Rightarrow \Delta \quad \rightarrow \quad (\Gamma \Rightarrow (\Delta \cup \{\sim \psi\})); \Gamma \Rightarrow (\Delta \cup \{\sim \psi\}) \\
\text{Right} & : \quad \lor : \quad (\Gamma \Rightarrow (\Delta \cup \{\sim \phi\})); \Gamma \Rightarrow (\Delta \cup \{\phi \land \psi\}) \\
\end{align*}
\]

where $\phi, \psi \in \text{Fm}_{\varphi}^{[-]}$ and $\Gamma, \Delta \in \varphi_\omega(\text{Fm}_{\varphi}^{[-]})$. These are the calculi introduced and studied in [25].

**Lemma 6.1.** $\mathbb{LB}_{01}^{[-]}$ is multiplicative.

**Proof.** Note that $G \overset{\triangle} ={\mathbb{LB}_{01}^{[-]} \setminus (\mathbb{B}_1^{(\sim)} \cup \mathbb{G}_2^{(\sim)} \cup \mathbb{C}_2^{(\sim)})}$ is schematic. Moreover, for any $(\Gamma \Rightarrow \Phi) \in G$ and all $\Psi \in \text{Seq}_{\varphi}^{[-]}$, it holds that $(\Gamma \Rightarrow (\Phi \lor \Psi)) \in G$.

Hence, $G$ is multiplicative. Then, Proposition 3.1 completes the argument. $\blacksquare$

Let $D \overset{\triangle} ={\{t, b\}}$ and $\mathbb{B}_{G,01}^{[-]} \overset{\triangle} ={(\mathbb{B}_1^{[-]}, (D, D))}$.

**Theorem 6.2** (Completeness Theorem; cf. [25]). The consequence of $\mathbb{LB}_{01}^{[-]}$ is left-dual defined by $\mathbb{B}_{G,01}^{[-]}$.

**Proof.** First, the following identities

\[(\sim \forall) \quad [(\sim \sim)]_{x_0} \Rightarrow (x_0 \approx (x_0 \sim \sim x_0))\]

are true in $\mathbb{B}_{G,01}^{[-]}$. Therefore, $\langle \sim \rangle$ are true in $\mathbb{S}_G^{[-]}(\mathbb{B}_{G,01}^{[-]}; \sim \sim)$. Next, $D$ is a prime filter of the Boolean lattice $\mathbb{B}_{G,01}^{[-]}$, while $\sim \sim \mathbb{B}_{G,01}^{[-]}(D) = \{f, b\}$ is a prime ideal of it [whereas $\langle \top \mathbb{B}_{G,01}^{[-]}/\mathbb{B}_{G,01}^{[-]} \rangle \not\in D$ is the zero/unit of it $\langle \sim \mathbb{B}_{G,01}^{[-]} \rangle \not\in D$ being the complement of any $a \in 2^2]$. Hence, all rules in $\mathbb{LB}_{01}^{[-]}$ other than basic structural ones as well as both Cuts and Sharings are true in $\mathbb{S}_G^{[-]}(\mathbb{B}_{G,01}^{[-]}; \sim \sim)$. Thus, by Corollary 3.12, we get $\vdash_{\mathbb{LB}_{01}^{[-]}} \subseteq \vdash_{\mathbb{B}_{G,01}^{[-]}}$.

Conversely, by Lemma 6.1 and Corollary 3.11, $\vdash_{\mathbb{LB}_{01}^{[-]}}$ is left-dual defined by a left-dual reflexive left-dual transitive crisp-decomposable 2-side propositional $\Sigma_{01}^{[-]}$-sequent semantics $S$. Let $\Sigma' \overset{\triangle} ={\Sigma_{01}^{[-]} \setminus \Sigma_{01}^{[-]}}$. Consider any $v \in S$, in which case there is some $C_0 \subseteq \text{Fm}_{\Sigma'}$ such that $v = (C_0, C_0 \cup t)$, and so, by $\langle \sim \rangle$, $C_0, C_1 \sim [-]$ negative, where $C_1 \approx \{C_1 \sim \sim \mathbb{B}_{G,01}^{[-]}(1)\}$. Likewise, by $\langle \sim \rangle$ and $\langle \sim \rangle$ as well as both Left/Right ($\bot/\top$) and ($\sim \bot \bot$)), it is both $\land$-conjunctive and $\lor$-disjunctive.
[as well as both \( \bot \)-false and \( \top \)-true]. [[Finally, by \((\sim)\sim\), \( (C_{0(\xi)}), C_{0(\eta)} \) is \( \sim \)-negative.]] Then, taking Claim 5.3 into account, we conclude that

\[
h \triangleq (\prod_{i=2}^{C_i} \chi_t) \in \text{hom}(\mathfrak{F}_{\Sigma^i}, \mathfrak{B}_{[\xi,0]}^{\sim})
\]

is such that \( C_0 = h^{-1}[D] \). Thus, \( v \in \mathcal{S}_1(\mathcal{B}_{[\xi,0]}^{\sim}) \), in which case

\[
\mathcal{S} \subseteq \mathcal{S}_1(\mathcal{B}_{[\xi,0]}^{\sim}),
\]

and so \( \vdash_{\mathcal{L}_{\mathcal{B}}^{\sim}} \models \vdash_{\mathcal{B}_{[\xi,0]}^{\sim}}^{\mathcal{B}_{[\xi,0]}^{\sim}} \), as required.

This provides a much more immediate and transparent insight into the Completeness Theorem for \( \mathcal{L}_{\mathcal{B}}^{\sim} \), originally proved in [25], thus once more demonstrating the power and usefulness of the generic elaboration presented here.

Since \( D \) is a prime filter of \( \mathcal{B}_{[\xi,0]}^{\sim} \), \( \models \{ \wedge, \vee \} \), taking the truth of distributive lattice and De Morgan identities (cf. [26]):

\[
(19)
\sim (x_0(\wedge|x_1) \approx (\sim x_0(\vee|x_1) \sim x_1)
\]

in \( \mathcal{B}_{[\xi,0]}^{\sim} \) into account, by Theorem 6.2, we also get:

**Corollary 6.3.** \( \mathcal{L}_{\mathcal{B}}^{\sim} \) is both \( (\wedge/\vee) \)-conjunctive/-disjunctive and \( \sim \)-relatively \((\wedge/\vee)\)-conjunctive/-disjunctive.

### 6.2. Non-multiplicative calculi

Let \( \mathcal{L}_{\mathcal{B}C}^{\sim} \) be the two-side propositional \( \Sigma^{\sim} \)-sequent calculus constituted by basic structural rules, Sharings, Cuts, \((\wedge/\vee)\) and \( (\sim) \) [as well as \( (\sim) \) with \( \Gamma = \Delta = \emptyset \)"/"Left/Right \((\bot/\top)\) (together with \( (\sim) \))"] collectively with the following rules:

**Weak Contraposition**

\[
\phi \sim \psi \implies \sim \phi \equiv \psi.
\]

where \( \phi, \psi \in \mathcal{F}_{\mathcal{B}_{[\xi,0]}^{\sim}} \), in which case \( (\sim \langle \vee \rangle \vee \{ \bot/\top \}) \) [and \( (\sim) \) with \( \Gamma = \Delta = \emptyset \) are derivable in it, and so, by Corollary 6.3, its consequence, being thus an extension of \( \{ \text{the consequence of} \} \ \mathcal{L}_{\mathcal{B}}^{\sim} \), contains the following rules:

\[
(20)
\Gamma \implies \Delta \implies \sim (\Delta \implies \sim \Gamma),
\]

where \( \Gamma, \Delta \in \mathcal{P}(\mathcal{F}_{\mathcal{B}_{[\xi,0]}^{\sim}}) \), those with \( \Gamma, \Delta \in \mathcal{P}(\mathcal{F}_{\mathcal{B}_{[\xi,0]}^{\sim}}) \) \( \{ \wedge/\vee \} \) and \( |\Gamma| = |\Delta| \) being \{ either trivial, when \( \Gamma = \Delta = \emptyset \), or \( \} \) in it.

Now, let \( \mathcal{B}_{[\xi,0]}^{\sim} \) be the 2-side fuzzy \( \mathcal{F}_{[\xi,0]}^{\sim} \)-matrix with underlying algebra \( \mathfrak{B}_{[\xi,0]}^{\sim} \) and 2-fold fuzzy set having grading lattice \( \mathfrak{B}_4 \{ \wedge, \vee \} \) and diagonal membership functions.

**Theorem 6.4.** Let \( \Xi \subseteq \mathcal{P}(\mathcal{F}_{\mathcal{B}_{[\xi,0]}^{\sim}}) \). Then, \( \Xi \vdash_{\mathcal{L}_{\mathcal{B}}^{\sim}} \Phi \iff \Xi \vdash_{\mathcal{L}_{\mathcal{B}}^{\sim} \mathfrak{B}_{[\xi,0]}^{\sim}}^{\mathcal{B}_{[\xi,0]}^{\sim}} \Phi \). [In particular, the consequence of \( \mathcal{L}_{\mathcal{B}C}^{\sim} \) is left-dual defined by \( \mathcal{B}_{[\xi,0]}^{\sim} \).]

**Proof.** The “only-if” part is by the immediate fact that

\[
\mathcal{L}_{\mathcal{B}}^{\sim} \subseteq \vdash_{\mathcal{B}_{[\xi,0]}^{\sim}} \Phi,
\]

in view of the truth of distributive \{bounded \{ more specifically, Boolean\}\} lattice and De Morgan identities (19) as well as (18) in \( \mathcal{B}_{[\xi,0]}^{\sim} \).
Conversely, assume \((\Gamma \rightarrow \Phi) \not\subseteq \vdash_{\text{LBWC}_{-[\sim]_01}}\), in which case, by (20), 
\(((\Gamma \cup \Gamma') \rightarrow \Phi) \not\subseteq \vdash_{\text{LBWC}_{-[\sim]_01}}\), where \(\Gamma' \equiv \{ \sim | \Delta \rightarrow \Theta | (\Theta \rightarrow \Delta) \in \Gamma\}\), and so, by Theorem 6.2, there is some \(h \in \text{hom}(\mathfrak{m}_{\Sigma|\{\sim\}-01}, \mathbb{B}_{[\sim]}_{4,01})\) such that \((\Gamma \cup \Gamma') \subseteq (h^{-1}[\mathbb{F}_{\text{LBWC}_{-[\sim]_01}}]) \not\subsetneq \Phi\). Then, since \(D\) and \((2^2 \setminus (\sim F_4)^{-1}[D]) = \{t, n\}\) are exactly all non-empty proper prime filters of the distributive lattice \(\mathfrak{B}_4 \rightarrow \{\wedge, \vee\}\), by the Prime Ideal Theorem for distributive lattices, we have \((\Phi | \Gamma) \not\subseteq \subseteq (h^{-1}[\mathbb{F}_{\text{LBWC}_{-[\sim]_01}}]) \in S_4(\mathbb{B}_{[\sim]}_{4,01(-\sim)}))\), as required.

Since \(D\) is a prime filter of the lattice \(\mathfrak{B}_4 \rightarrow \{\wedge, \vee\}\), in which case axioms of \(\vdash^{\text{ld}}_{\mathbb{B}_{[\sim]}_{4,01(-\sim)}}\) are those of \(\vdash^{\text{ld}}_{\mathbb{B}_{[\sim]}_{4,01(-\sim)}}\) by Theorems 6.2 and 6.4, we, first, get:

**Corollary 6.5.** Derivable axioms of \(\text{LB}_{/01}\) are exactly those of \(\text{LBWC}_{/[\sim]_01}\). In particular, \(\text{LB}(\text{WC})_{/[\sim]-1}\) is a Gentzen-style axiomatization of FDE.

Nevertheless, such is not, generally speaking, the case for proper rules. More precisely, we have:

**Corollary 6.6.** \((\{p_0, p_2\} \rightarrow p_1) \rightarrow (\{\sim p_1, p_2\} \rightarrow \sim p_0)\) \(\not\subseteq \vdash_{\text{LBWC}_{/[\sim]_01}}\), in which case \((p_0 \rightarrow p_1) \rightarrow (\sim p_1 \rightarrow \sim p_0)\) \(\subseteq \vdash_{\text{LBWC}_{/[\sim]_01}}\) is not derivable in \(\text{LB}_{/[\sim]_01}\), and so \(\vdash_{\text{LBWC}_{/[\sim]_01}} \supseteq \vdash_{\text{LB}_{/[\sim]_01}}\). In particular, \(\text{LBWC}_{/[\sim]_01}\), being non-multiplicative, is \{left-dual\} defined by no crisp-decomposable 2-side propositional \(\Sigma_{[\sim]_01}\)-sequent semantics. Likewise, neither of \(\sim\) with \(\Gamma = \Delta = \emptyset\) belonging to \(\text{LBWC}_{\sim}\) is derivable in \(\text{LBWC}\), in which case the consequence of the former is a distinct extension of the one of the latter, and so the consequences of \(\text{LBWC}_{\sim}\), \(\text{LBWC}\) and \(\text{LB}\) are distinct from one another.

**Proof.** First, the left\(\rightarrow\)right side of any premise of any rule of \(\text{LBWC}\) is empty, whenever that of its conclusion is so,\(^8\) in which case no Right\(\rightarrow\)Left \(\sim\)-rule with empty \(\Delta|\Gamma\) is derivable in \(\text{LBWC}\), and so its consequence is distinct from the case of \(\text{LBWC}_{\sim}\). Next, let \(h \in \text{hom}(\mathfrak{m}_{\Sigma(-\sim)}, \mathbb{B}_{[\sim]}_{4,01(-\sim)})\) extend \(\{\{p_0, b\}, \{p_1, f\}\} \cup \{(\{\wedge \setminus \{p_0, p_1\} \times \{n\}\), in which case \((\{p_0, p_2\} \rightarrow p_1) \rightarrow (h^{-1}[\mathbb{F}_{\text{LBWC}_{/[\sim]_01}}]) \not\subseteq \ridor\{p_0, p_2\} \rightarrow \sim p_0\), and so Lemma 6.1 and Theorem 6.4 complete the argument.

Thus, it is \(\text{LBWC}_{\sim}\), that appears the right deductive fragment of \(\text{LBWC}_{/[\sim]_01}\). And what is more, it is these instances of sequent calculi that justify the necessity of fuzziness of universal semantics of sequent calculi.

The non-optional version of the above completeness theorem collectively with \([26]\) provides a much more immediate and transparent insight into the algebraic completeness theorem for the non-empty-both-of-sequent-sides fragment of \(\text{LBWC}\) originally obtained in \([24, 25]\) with using rather esoteric advanced algebraically-logical tools, not at all applicable to proving the optional left-side version of the above theorem just because of absence of interpretation of propositional \(\Sigma\)-sequents with empty either side by means of \(\{\text{sets of}\} \Sigma\)-equations. And what is more, the optional right-side version of the above theorem (collectively with \([29]\)) provides a new as well as much more immediate and transparent insight into the algebraic completeness theorem for \(\text{LBWC}_{/[\sim]_01}\) originally obtained in \([25]\) with using rather esoteric advanced algebraically-logical tools. These points once more demonstrate the power and usefulness of the generic elaboration presented here.

\(^8\)This is why the non-empty-either-/both-of-sequent-sides fragments of \(\text{LBWC} /\text{"explicitly studied in [24, 25]\} is well-defined.\)
6.3. Two multiple-conclusion Gentzen-style axiomatizations of the implication-less fragment of Gödel’s three-valued logic.

6.3.1. A multiplicative calculus. Let $\mathbb{LS}$ be the two-side propositional $\Sigma$-sequent calculus constituted by basic structural rules, Sharings, Cuts, $(\sim(\wedge|\vee))$ and Left ($\sim$) as well as Right ($\sim$) with $\phi \in \sim \text{Fm}_S$. This is actually what is originally introduced in [21] and then factually studied in [35].

**Lemma 6.7.** $\mathbb{LS}$ is multiplicative.

*Proof.* Note that $G \triangleq (\mathbb{LS} \setminus (\mathcal{B}_S^2 \cup \mathcal{S}_S^2 \cup \mathcal{C}_S^2))$ is schematic. Moreover, for any $(\Gamma \rightarrow \Phi) \in G$ and all $\Psi \in \text{Seq}_S$, it holds that $((\Gamma \triangleright \Psi) \rightarrow (\Phi \triangleright \Psi)) \in G$. Hence, $G$ is multiplicative. Then, Proposition 3.1 completes the argument. ■

Now, let $S_3$ be the $\Sigma$-algebra with carrier 3 and operations defined by $b \triangleq (\min|\max)(a,b)$ and $\sim^{S_3} a \triangleq (2 \cdot (1 - \min(1,a)))$, for all $a, b \in 3$. Then, let $F \triangleq \{2\}$ and $S_3 \triangleq \langle S_3, (F, F) \rangle$.

**Theorem 6.8** (Completeness Theorem; cf. [35]). The consequence of $\mathbb{LS}$ is left-dual defined by $S_3$.

*Proof.* First, $D \ni | \not \ni \{2\}0$ is a prime filter of the bounded lattice $S_3 \vdash \{\wedge, \vee\}$ with zero/unit 0/2, while $(\sim^{S_3})^{-1}[F] = \{0\}$ is a prime ideal of it, whereas the following Stone identities

$$((\sim x_0(\wedge|\vee) \sim x_0)(\vee|\wedge)x_1) \approx x_1$$

are true in $S_3$. Hence, all rules in $\mathbb{LS}$ other than basic structural ones as well as both Cuts and Sharings are true in $S_3(S_3)$. Thus, by Corollary 3.12, we get $\vdash_{\mathbb{LS}} \subseteq \vdash_{\mathbb{S}_3}$.

Conversely, by Lemma 6.7 and Corollary 3.11, $\vdash_{\mathbb{LS}}$ is left-dual defined by a left-dual reflexive transitive crisp-decomposable 2-side propositional $\Sigma$-sequent semantics $S$. Consider any $v \in S$, in which case there is some $C_2 \subseteq C_0 \triangleq \text{Fm}_S$ such that $v = (\{C_2, C_2\})$, and so, by Left ($\sim$), $C_2 \subseteq C_1 \triangleq (\text{Fm}_S \setminus \sim^{-1}[C_0]) \subseteq C_0$. Likewise, by ($\sim\wedge$) and ($\sim\vee$), $(C_2, C_1)$ is both $\wedge$-conjunctive and $\vee$-disjunctive. Therefore, by those instances of ($\sim$), which are in $\mathbb{LS}$, $h \triangleq \{\varphi \cdot \max\{i \in n \mid \varphi \in C_i\}\} \varphi \in \text{Fm}_S \}$ is hom(\text{Fm}_S, S_3) is such that $C_2 = h^{-1}[F]$. Thus, $v \in S_3(S_3)$, in which case $S \subseteq S_3(S_3)$, and so $\vdash_{\mathbb{LS}} \supseteq \vdash_{\mathbb{S}_3}$, as required. ■

This provides a much more immediate and transparent insight into the Completeness Theorem for $\mathbb{LS}$, originally being due to [35], thus once more demonstrating the power and usefulness of the generic elaboration presented here.

6.3.2. A non-multiplicative calculus. Let $\mathbb{LLL}$ be the 2-side propositional $\Sigma$-sequent calculus constituted by basic structural rules, Sharings, Cuts, $(\wedge|\vee)$, Left ($\sim$) and:

$$((\Gamma \cup \Delta) \rightarrow \Gamma \rightarrow \sim \Delta,$$

where $\Gamma, \Delta \in \varphi_{\omega}(\text{Fm}_S)$, in which case (20) are derivable in it, and so are both ($\sim (\wedge|\vee)$) and Right ($\sim$) with $\phi \in \sim \text{Fm}_S$. This was introduced and proved to have Cut Elimination Property in [32].

Next, let $S_3^F$ be the 2-side fuzzy $\Sigma$-matrix with underlying algebra $S_3$, grading lattice $S_3 \vdash \{\wedge, \vee\}$ and diagonal membership functions.

**Theorem 6.9** (Completeness Theorem; cf. [32]). $\vdash_{\mathbb{LLL}}$ is left-dual defined by $S_3^F$. 
De Morgan (19) and Stone (21) identities in 
Proof. The inclusion $\mathbb{L}\mathbb{L}\mathbb{L} \subseteq \vdash_{\mathbb{S}^3}$ is immediate by the truth of distributive lattice, 
(a ∈ \rightarrow Corollary 6.11. (precisely, we have: G"odel’s three-valued logic. these calculi are Gentzen-style axiomatizations of the implication-less fragment of 
(G邝\ldots \Pi) \subseteq (h^{-1}[\mathbb{F}\mathbb{S}^3_+]) \not\equiv (\Pi \nets {\sim p_0, p_1}), and so Lemma 6.7 and Theorem 6.9 complete 
Thus, $\mathbb{L}\mathbb{L}\mathbb{L}$ equally justifies the necessity of fuzziness of universal semantics of 
In this connection, we should like to highlight that, in view of 
This provides a new as well as much more immediate and transparent insight into the Completeness Theorem of [32], thus once again highlighting the usefulness and power of the universal elaboration presented here. 
Since $F$ is a prime filter of the lattice $\mathbb{S}^3 \upharpoonright \{\land, \lor\}$, in which case axioms of $\vdash_{\mathbb{S}^3}$ are those of $\vdash_{\mathbb{S}^3}$, while the implication-less fragment of G"odel’s three-valued logic is defined by the logical matrix $(\mathbb{S}^3, F)$ in the conventional sense (cf., e.g., [17]), by Theorems 6.8 and 6.9, we, first, get: 

**Corollary 6.10.** Derivable axioms of $\mathbb{L}\mathbb{S}$ are exactly those of $\mathbb{L}\mathbb{L}\mathbb{L}$. In particular, these calculi are Gentzen-style axiomatizations of the implication-less fragment of G"odel’s three-valued logic.

Nevertheless, such is not, generally speaking, the case for proper rules. More precisely, we have:

**Corollary 6.11.** $(p_0 \rightarrow p_1) \rightarrow (\neg \neg (p_0, p_1)) \not\equiv \vdash_{\mathbb{L}\mathbb{L}\mathbb{L}}$, in which case $(p_0 \rightarrow )$ \rightarrow (\neg \neg (p_0)) \in $\mathbb{L}\mathbb{L}\mathbb{L}$ is not derivable in $\mathbb{L}\mathbb{S}$, and so $\vdash_{\mathbb{L}\mathbb{L}\mathbb{L}} \not\supseteq \vdash_{\mathbb{L}\mathbb{S}}$. In particular, $\mathbb{L}\mathbb{L}\mathbb{L}$, being non-multiplicative, is (left-dual) defined by no crisp-decomposable 2-side propositional $\Sigma$-sequent semantics.

**Proof.** Let $h \in \text{hom}(\mathfrak{S}^3, \mathbb{S}^3)$ extend $V_\omega \times \{1\}$, in which case $(p_0 \rightarrow p_1) \in (h^{-1}[\mathbb{F}\mathbb{S}^3_+]) \not\equiv (\neg \neg (p_0, p_1))$, and so Lemma 6.7 and Theorem 6.9 complete the argument.

Thus, $\mathbb{L}\mathbb{L}\mathbb{L}$ equally justifies the necessity of fuzziness of universal semantics of sequent calculi. In this connection, we should like to highlight that, in view of Theorem 6.9, it is this sequent calculus that corresponds to the logic involved according to the generic algebraic approach developed in [24] and then advanced in [47, 50].

7. Conclusions

Thus, we have completely implemented the research program announced in Section 1. We should like to highlight that, as for two-side sequent calculi, the principal advance of the present study with regard to [28] consists in not merely extending the scopes of fuzzy semantics of derivable rules of sequent calculi by eliminating Sharings and/or Cuts but mainly clarifying how (more specifically, where from) fuzziness arises when dealing with non-multiplicative calculi. Namely, this is just because direct products of non-one-element families of crisp sets are always fuzzy.

Perhaps, it is equally noteworthy that a one more substantial advance of the present study with regard to [28] consists in Booleanity and complete distributivity (i.e., atomic Booleanity; cf. [57]) of complete grading lattices of fuzzy valuations/matrices constituting fuzzy /matrix semantics of /propositional sequent calculus.
calculi with merely basic structural rules, the lattice completeness, not highlighted therein explicitly, being due to argumentation of main results obtained therein.

As for possible directions of further work, it would be interesting and valuable to develop an algebraic study of fuzzy matrices, advanced applications of which have been demonstrated by a one more algebraic proof of the Cut Elimination Theorem for Gentzen’s calculus [10] (cf. Corollary 5.5). In addition, it would be especially valuable to explore single-conclusion calculi as to such a fuzzy semantics. The problem is that, as opposed to multiple-conclusion calculi, single-conclusion ones do not possess universal disjunction (like ⊎ for former ones), so this point is far from being straightforward. These advanced issues go far beyond the scopes of the present paper constituting merely foundations of the topic involved and are going to be discussed elsewhere.

References


9Recall that the first algebraic argumentation of this theorem, following a quite different and much more complicated method, has been due to [56]. In this way, the present paper provides a new and much more transparent insight into the issue. It appears that Cut Elimination (at least, in the propositional case and under presence of inverse logical rules) is just due to certain homomorphisms from the four-element Kleene lattice onto the three- and two-element ones.


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