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Abstract

In this research article, by making use of Salagean differential operator, we introduce and investigate a new subclass of analytic and bi-univalent functions using the Horadam polynomial. We derive the coefficient estimate and obtain Fekete-szegö inequality for functions in this subclass.

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1 Introduction

Let $\mathcal{A}$ denote the class of all analytic functions f defined on the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, which is normalized under the condition $f(0) = f'(0) = 1$ having the Taylor series expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \ z \in \Delta. \tag{1}$$

and $\mathcal{S}$, the class of functions in $\mathcal{A}$ which are univalent in Δ. Let the function f and g be analytic in Δ. Then we say that the function f is subordinate to g, if there exist a schwarz function $w(z)$ which is analytic in Δ with

$$w(0) = 0 \ , |w(z)| < 1, (z \in \Delta)$$

satisfying

$$f(z) = g(w(z)).$$

It is known that,

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \text{ and } f(\Delta) \subset g(\Delta).$$

By, the Koebe one-quarter theorem [10] every function $f \in \mathcal{S}$ has an inverse f^{-1} defined by

$$f^{-1}(f(z)) = z, (z \in \Delta)$$

and

$$f(f^{-1}(w)) = w, \left(|w| < r_0(f),\ r_0(f) \geq \frac{1}{4}\right)$$

where

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - ... \tag{2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in Δ if both f and f^{-1} are univalent in Δ. Denote by Σ the class of bi-univalent functions in Δ. Examples of bi-univalent functions are

$$\tfrac{z}{1-z}, -log(1-z), \tfrac{1}{2}log(\tfrac{1+z}{1-z}), ...$$

The familiar Koebe function is not a member of Σ.

Lewin [16] investigated the class of bi-univalent function Σ and showed $|a_2| < 1.51$ and motivated by the work of Lewin, Brannan and Clunie [8] conjectured that $|a_2| \leq \sqrt{2}$. The best known estimate for functions in Σ is obtained by Tan [21] in 1984, that is $|a_2| < 1.485$. The coefficient estimate problem for $|a_n| (n \in \mathbb{N}, n \geq 3)$ is still open [18]. The study of bi-univalent functions gained interest mainly due to the work of Srivastava et al [18]. Several researchers got motivated by this, (see[1,2,3,4,5,6,7,9,10,11,12,18,19,20,22,23]) and investigated interesting subclasses of the class Σ and found non-sharp estimates for the first two Taylor-Maclaurin coefficients.

Definition 1.1. *(see [13,14]) The Horadam polynomials $h_n(r)$ are given by the following recurrence relation:*

$$h_n(r) = prh_{n-1}(r) + qh_{n-2}(r)\ (r \in \mathbb{R};\ n \in \mathbb{N} = \{1, 2, 3...\}) \tag{3}$$

with

$$h_1(r) = a \text{ and } h_2(r) = br,$$

for some real constants a, b, p and q. Moreover, the characteristic equation of the recurrence relation (3) is given by

$$t^2 - prt - q = 0,$$

which has the following two real roots:

$$\alpha = \frac{pr+\sqrt{p^2r^2+4q}}{2} \text{ and } \beta = \frac{pr-\sqrt{p^2r^2+4q}}{2}.$$

By choosing appropriately the parameters a, b, p and q, we get some special cases of the Horadam polynomials $h_n(r)$.

- Taking a = b = p = q = 1, we obtain the Fibonacci polynomials $F_n(r)$.
- Taking a = 2 and b = p = q = 1, we get the Lucas polynomials $L_n(r)$.
- Taking a = q = 1 and b = p = 2, we have the Pell polynomials $P_n(r)$.
- Taking a = b = p = 2 and q = 1, we find the Pell-Lucas polynomials $Q_n(r)$.
- Taking a = b = 1, p = 2 and q = -1, we obtain the Chebyshev polynomials $T_n(r)$ of the first kind.
- Taking a = 1, b = p = 2 and q = -1, we have the Chebyshev polynomials $U_n(r)$ of the second kind.

The generating function of the Horadam polynomials $h_n(r)$ (see [14]) are given by

$$\Omega(r,z) = \Sigma_{n=1}^{\infty} h_n(r) z^{n-1} = \frac{a+(b-ap)rz}{1-prz-qz^2}. \tag{4}$$

We now define and discuss (p,q)-analogue of Salagean differential operator:

$$\begin{aligned}
&\mathfrak{T}^0_{p,q} f(z) = f(z),\\
&\mathfrak{T}^1_{p,q} f(z) = z(\mathfrak{T}_{p,q} f(z)),\\
&\quad\quad .\\
&\quad\quad .\\
&\quad\quad .\\
&\mathfrak{T}^k_{p,q} f(z) = z\mathfrak{T}_{p,q}(\mathfrak{T}^{k-1}_{p,q} f(z)),\\
&\mathfrak{T}^k_{p,q} f(z) = z + \sum_{n=2}^{\infty} [n]^k_{p,q} a_n z^n \ (k \in \mathbb{N}_0 = \mathbb{N}\cup\{0\},\ z \in \Delta).
\end{aligned}$$

If we let p=1 and $q \to 1^-$, then $\mathfrak{T}^k_{p,q} f(z)$ reduces to the well-known Salagean differential operator [17].

Definition 1.2. *For $\zeta \geq 1$, $\varrho \geq 0$ and $\delta \geq 0$, a function $f \in \mathcal{A}$ given by (1) is said to be in the class $\mathfrak{M}_\zeta(p,q,k,\varrho)$ if the following subordinations are satisfied:*

$$(1-\zeta)\left(\frac{\mathfrak{T}^k_{p,q} f(z)}{z}\right)^{\varrho} + \zeta(\mathfrak{T}^k_{p,q} f(z))'\left(\frac{\mathfrak{T}^k_{p,q} f(z)}{z}\right)^{\varrho-1} + \delta z(\mathfrak{T}^k_{p,q} f(z))'' \prec \Omega(r,z)+1-a \tag{5}$$

Definition 1.3. *For $\zeta \geq 1$, $\varrho \geq 0$ and $\delta \geq 0$, a function $f \in \Sigma$ given by* (1) *is said to be in the class $\mathfrak{B}_\zeta(p,q,k,\varrho)$ if the following subordinations are satisfied:*

$$(1-\zeta)\left(\frac{\mathfrak{T}^k_{p,q}f(z)}{z}\right)^{\varrho}+\zeta(\mathfrak{T}^k_{p,q}f(z))'\left(\frac{\mathfrak{T}^k_{p,q}f(z)}{z}\right)^{\varrho-1}+\delta z(\mathfrak{T}^k_{p,q}f(z))''\prec\Omega(r,z)+1-a \tag{6}$$

and

$$(1-\zeta)\left(\frac{\mathfrak{T}^k_{p,q}g(w)}{w}\right)^{\varrho}+\zeta(\mathfrak{T}^k_{p,q}g(w))'\left(\frac{\mathfrak{T}^k_{p,q}g(w)}{w}\right)^{\varrho-1}+\delta w(\mathfrak{T}^k_{p,q}g(w))''\prec\Omega(r,w)+1-a \tag{7}$$

where $g(w)=f^{-1}(w)$ is defined by (2)

2 Coefficient bounds for $f \in \mathfrak{M}_\zeta(p,q,k,\varrho)$

Let $\mathcal{B}=\{\omega\in\mathcal{H}:|\omega(z)|\leq 1, z\in\Delta\}$ and $\mathcal{B}_0$ be the subclass of $\mathcal{B}$ of all ω such that $\omega(0)=0$. The elements of $\mathcal{B}_0$ are known as Schwarz functions.

We will apply a lemma below to prove the main theorem of this section.

Lemma 2.1. ([15]) *If $\omega\in\mathcal{B}_0$ is of the form*

$$\omega(z)=\sum_{n=1}^{\infty}\omega_n z^n,\ z\in\Delta, \tag{8}$$

then for $\nu\in\mathbb{C}$,

$$|\omega_2-\nu\omega_1^2|\leq max\{1,|\nu|\}. \tag{9}$$

Theorem 2.1. *Let f given by* (1) *be in the class $\mathfrak{M}_\zeta(p,q,k,\varrho)$. Then*

$$|a_2|\leq\frac{|br|}{[2]^k_{p,q}(\varrho+\zeta+2\delta)},$$

$$|a_3|\leq\frac{|br|}{(\varrho+2\zeta+6\delta)[3]^k_{p,q}}\ max\left\{1,\left|\left(\frac{(\varrho+2\zeta)(\varrho-1)br}{2(\varrho+\zeta+2\delta)^2}\right)-\frac{pbr^2+aq}{br}\right|\right\}$$

and

$$|a_3-\varrho a_2^2|\leq\frac{|br|}{(\varrho+2\zeta+6\delta)[3]^k_{p,q}}max\left\{1,\left|\frac{(\varrho+2\zeta)br}{2(\varrho+\zeta+2\delta)^2}\left(\frac{2\varrho(\varrho+2\zeta+6\delta)[3]^k_{p,q}}{(\varrho+2\zeta)([2]^k_{p,q})^2}+\varrho-1\right)-\frac{pbr^2+aq}{br}\right|\right\}.$$

Proof. Let f is in the class $\mathfrak{M}_\zeta(p,q,k,\varrho)$ then from Definition 1.2, for some analytic functions u and v such that $u(0)=v(0)=0$,

$$| u(z) |=| u_1 z + u_2 z^2 + u_3 z^3 + ... |< 1, \ (z \in \Delta)$$

then

$$| u_t |\leq 1 \ for \ t \in \mathbb{N}. \tag{10}$$

$$(1-\zeta)\left(\frac{\mathfrak{T}^k_{p,q}f(z)}{z}\right)^{\varrho} + \ \zeta(\mathfrak{T}^k_{p,q}f(z))'\left(\frac{\mathfrak{T}^k_{p,q}f(z)}{z}\right)^{\varrho-1} + \\ \delta z(\mathfrak{T}^k_{p,q}f(z))'' = \Omega(r, u(z)) + 1 - a$$

or equivalently,

$$(1-\zeta)\left(\frac{\mathfrak{T}^k_{p,q}f(z)}{z}\right)^{\varrho} + \zeta(\mathfrak{T}^k_{p,q}f(z))'\left(\frac{\mathfrak{T}^k_{p,q}f(z)}{z}\right)^{\varrho-1} + \delta z(\mathfrak{T}^k_{p,q}f(z))'' = \\ 1 + h_1(r) + h_2(r)u(z) + h_3(r)(u(z))^2 + ... - a \tag{11}$$

From the equality (11)

$$(1-\zeta)\left(\frac{\mathfrak{T}^k_{p,q}f(z)}{z}\right)^{\varrho} + \zeta(\mathfrak{T}^k_{p,q}f(z))'\left(\frac{\mathfrak{T}^k_{p,q}f(z)}{z}\right)^{\varrho-1} + \delta z(\mathfrak{T}^k_{p,q}f(z))'' = \\ 1 + h_2(r)u_1(z) + [h_2(r)u_2 + h_3(r)u_1^2]z^2 + ... \tag{12}$$

Comparing the coefficients of equation (12), we get

$$[2]^k_{p,q}(\varrho + \zeta + 2\delta)a_2 = h_2(r)u_1 \tag{13}$$

$$(\varrho + 2\zeta)\left\{\left(\frac{\varrho - 1}{2}\right)([2]^k_{p,q})^2 a_2^2 + \left(1 + \frac{6\delta}{\varrho + 2\zeta}\right)[3]^k_{p,q}a_3\right\} = h_2(r)u_2 + h_3(r)u_1^2 \tag{14}$$

From (13) we get,

$$a_2 = \frac{h_2(r)u_1}{[2]^k_{p,q}(\varrho + \zeta + 2\delta)}$$

$$|a_2| \leq \frac{|br|}{[2]^k_{p,q}(\varrho + \zeta + 2\delta)}. \tag{15}$$

Now we get,

$$\begin{aligned}(\varrho + 2\zeta)(1 + \frac{6\delta}{\varrho + 2\zeta})[3]^k_{p,q}a_3 &= h_2(r)u_2 + h_3(r)u_1^2 - (\varrho + 2\zeta)\left(\frac{\varrho - 1}{2}\right)([2]^k_{p,q})^2 a_2^2 \\ &= h_2(r)u_2 + h_3(r)u_1^2 - (\varrho + 2\zeta)\left(\frac{\varrho - 1}{2}\right)\left(\frac{h_2(r)u_1}{\varrho + \zeta + 2\delta}\right)^2 \\ &= h_2(r)u_2 - \frac{u_1^2}{2}\left[\left(\frac{h_2(r)(\varrho + 2\zeta)(\varrho - 1)}{(\varrho + \zeta + 2\delta)^2}\right) - 2h_3(r)\right].\end{aligned}$$

Thus we have

$$a_3 = \frac{h_2(r)}{(\varrho+2\zeta+6\delta)[3]^k_{p,q}}\left\{u_2 - u_1^2\left[\left(\frac{(\varrho+2\zeta)(\varrho-1)h_2(r)}{2(\varrho+\zeta+2\delta)^2}\right) - \frac{h_3(r)}{h_2(r)}\right]\right\}$$

$$= \frac{h_2(r)}{(\varrho+2\zeta+6\delta)[3]^k_{p,q}}\{u_2 - \aleph u_1^2\} \tag{16}$$

where

$$\aleph = \left[\left(\frac{(\varrho+2\zeta)(\varrho-1)h_2(r)}{2(\varrho+\zeta+2\delta)^2}\right) - \frac{h_3(r)}{h_2(r)}\right].$$

By applying Lemma 2.1, we get

$$|a_3| = \frac{|h_2(r)|}{(\varrho+2\zeta+6\delta)[3]^k_{p,q}}|u_2 - \aleph u_1^2|$$

$$\leq \frac{|br|}{(\varrho+2\zeta+6\delta)[3]^k_{p,q}}\ max\left\{1, \left|\left(\frac{(\varrho+2\zeta)(\varrho-1)br}{2(\varrho+\zeta+2\delta)^2}\right) - \frac{pbr^2+aq}{br}\right|\right\}.$$

For any $\varrho \in \mathbb{C}$, we get

$$a_3 - \varrho a_2^2 = \frac{h_2(r)}{(\varrho+2\zeta+6\delta)[3]^k_{p,q}}\left\{u_2 - u_1^2\left[\left(\frac{(\varrho+2\zeta)(\varrho-1)h_2(r)}{2(\varrho+\zeta+2\delta)^2}\right) - \frac{h_3(r)}{h_2(r)}\right]\right\} - \varrho\left(\frac{h_2(r)u_1}{[2]^k_{p,q}(\varrho+\zeta+2\delta)}\right)^2$$

$$= \frac{h_2(r)}{(\varrho+2\zeta+6\delta)[3]^k_{p,q}}\{u_2 - \eta u_1^2\}$$

where

$$\eta = \frac{(\varrho+2\zeta)h_2(r)}{2(\varrho+\zeta+2\delta)^2}\left(\frac{2\varrho(\varrho+2\zeta+6\delta)[3]^k_{p,q}}{(\varrho+2\zeta)([2]^k_{p,q})^2} + \varrho - 1\right) - \frac{h_3(r)}{h_2(r)}.$$

By applying Lemma 2.1, we get

$$|a_3 - \varrho a_2^2| = \frac{|h_2(r)|}{(\varrho+2\zeta+6\delta)[3]^k_{p,q}}|u_2 - \eta u_1^2|$$

$$|a_3 - \varrho a_2^2| \leq \frac{|br|}{(\varrho+2\zeta+6\delta)[3]^k_{p,q}} max\left\{1, \left|\frac{(\varrho+2\zeta)br}{2(\varrho+\zeta+2\delta)^2}\left(\frac{2\varrho(\varrho+2\zeta+6\delta)[3]^k_{p,q}}{(\varrho+2\zeta)([2]^k_{p,q})^2} + \varrho - 1\right) - \frac{pbr^2+aq}{br}\right|\right\}.$$

□

Theorem 2.2. *Let f given by (1) be in the class $\mathfrak{B}_\zeta(p,q,k,\varrho)$. Then*

$$| a_2 | \leq \frac{| br | \sqrt{2 | br |}}{\sqrt{|\Theta(\varrho,\zeta,p,q,k)|}}$$

and

$$|a_3| \leq \frac{b^2r^2}{([2]^k_{p,q})^2(\varrho+\zeta+2\delta)^2} + \frac{|br|}{(\varrho+2\zeta)(1+\frac{6\delta}{2\zeta+1})[3]^k_{p,q}}$$

where

$$\Theta(\varrho,\zeta,p,q,k) = \{(\varrho+2\zeta)[(\varrho-1)([2]^k_{p,q})^2 + 2(1+\frac{6\delta}{\varrho+2\zeta})[3]^k_{p,q}]b \\ - 2([2]^k_{p,q}(\varrho+\zeta+2\delta))^2p\}br^2 - 2[[2]^k_{p,q}(\varrho+\zeta+2\delta)^2]aq. \quad (17)$$

Proof. Let f is in the class $\mathfrak{B}_\zeta(p,q,k,\varrho)$ then from Definition 1.3, for some analytic functions u and v such that $u(0)=v(0)=0$,

$$| u(z) |=| u_1z + u_2z^2 + u_3z^3 + ... |< 1, \ (z \in \Delta)$$

and

$$| v(w) |=| v_1w + v_2w^2 + v_3w^3 + ... |< 1, \ (w \in \Delta)$$

then

$$| u_t | \leq 1 \ and \ | v_t | \leq 1 \ for \ t \in \mathbb{N}. \quad (18)$$

$$(1-\zeta)\left(\frac{\mathfrak{T}^k_{p,q}f(z)}{z}\right)^\varrho + \zeta(\mathfrak{T}^k_{p,q}f(z))'\left(\frac{\mathfrak{T}^k_{p,q}f(z)}{z}\right)^{\varrho-1} + \delta z(\mathfrak{T}^k_{p,q}f(z))'' = \Omega(r,u(z))+1-a$$

$$(1-\zeta)\left(\frac{\mathfrak{T}^k_{p,q}g(w)}{w}\right)^\varrho + \zeta(\mathfrak{T}^k_{p,q}g(w))'\left(\frac{\mathfrak{T}^k_{p,q}g(w)}{w}\right)^{\varrho-1} + \delta w(\mathfrak{T}^k_{p,q}g(w))'' = \Omega(r,v(w))+1-a$$

or equivalently,

$$(1-\zeta)\left(\frac{\mathfrak{T}^k_{p,q}f(z)}{z}\right)^\varrho + \zeta(\mathfrak{T}^k_{p,q}f(z))'\left(\frac{\mathfrak{T}^k_{p,q}f(z)}{z}\right)^{\varrho-1} + \delta z(\mathfrak{T}^k_{p,q}f(z))'' = \\ 1 + h_1(r) + h_2(r)u(z) + h_3(r)(u(z))^2 + ... - a \quad (19)$$

$$(1-\zeta)\left(\frac{\mathfrak{T}^k_{p,q}g(w)}{w}\right)^\varrho + \zeta(\mathfrak{T}^k_{p,q}g(w))'\left(\frac{\mathfrak{T}^k_{p,q}g(w)}{w}\right)^{\varrho-1} + \delta w(\mathfrak{T}^k_{p,q}g(w))'' = \\ 1 + h_1(r) + h_2(r)v(w) + h_3(r)(v(w))^2 + ... - a \quad (20)$$

From the equalities (19) and (20),

$$(1-\zeta)\left(\frac{\mathfrak{T}_{p,q}^{k}f(z)}{z}\right)^{\varrho}+\zeta(\mathfrak{T}_{p,q}^{k}f(z))'\left(\frac{\mathfrak{T}_{p,q}^{k}f(z)}{z}\right)^{\varrho-1}+\delta z(\mathfrak{T}_{p,q}^{k}f(z))''=$$
$$1+h_2(r)u_1(z)+[h_2(r)u_2+h_3(r)u_1^2]z^2+... \quad (21)$$

$$(1-\zeta)\left(\frac{\mathfrak{T}_{p,q}^{k}g(w)}{w}\right)^{\varrho}+\zeta(\mathfrak{T}_{p,q}^{k}g(w))'\left(\frac{\mathfrak{T}_{p,q}^{k}g(w)}{w}\right)^{\varrho-1}+\delta w(\mathfrak{T}_{p,q}^{k}g(w))''=$$
$$1+h_2(r)v_1(w)+[h_2(r)v_2+h_3(r)v_1^2]w^2+... \quad (22)$$

Comparing the coefficients of equation (21) and (22), we get

$$[2]_{p,q}^{k}(\varrho+\zeta+2\delta)a_2=h_2(r)u_1 \quad (23)$$

$$(\varrho+2\zeta)\left\{\left(\frac{\varrho-1}{2}\right)([2]_{p,q}^{k})^2a_2^2+\left(1+\frac{6\delta}{\varrho+2\zeta}\right)[3]_{p,q}^{k}a_3\right\}=h_2(r)u_2+h_3(r)u_1^2 \quad (24)$$

$$-[2]_{p,q}^{k}(\varrho+\zeta+2\delta)a_2=h_2(r)v_1 \quad (25)$$

$$(\varrho+2\zeta)\left\{\left(\frac{\varrho-1}{2}\right)([2]_{p,q}^{k})^2a_2^2+\left(1+\frac{6\delta}{\varrho+2\zeta}\right)[3]_{p,q}^{k}(2a_2^2-a_3)\right\}$$
$$=h_2(r)v_2+h_3(r)v_1^2 \quad (26)$$

From (23) and (25) we get,

$$u_1=-v_1 \quad (27)$$

$$2\{[2]_{p,q}^{k}(\varrho+\zeta+2\delta)\}^2a_2^2=h_2^2(r)(u_1^2+v_1^2) \quad (28)$$

Adding (24) and (26) we get,

$$2(\varrho+2\zeta)\left\{\frac{\varrho-1}{2}([2]_{p,q}^{k})^2+\left(1+\frac{6\delta}{\varrho+2\zeta}\right)[3]_{p,q}^{k}\right\}a_2^2=h_2(r)(u_2+v_2)+h_3(r)(u_1^2+v_1^2) \quad (29)$$

Substituting the value of $(u_1^2+v_1^2)$ from (28) in the right hand side of (29) we get,

$$a_2^2=\frac{h_2^3(r)(u_2+v_2)}{(\varrho+2\zeta)\left\{(\varrho-1)([2]_{p,q}^{k})^2+2\left(1+\frac{6\delta}{\varrho+2\zeta}\right)[3]_{p,q}^{k}\right\}h_2^2(r)-2h_3(r)([2]_{p,q}^{k}(\varrho+\zeta+2\delta))^2} \quad (30)$$

Compute using (3), (17), (18) and (30),

$$\mid a_2\mid\leq\frac{\mid br\mid\sqrt{2\mid br\mid}}{\sqrt{|\Theta(\varrho,\zeta,p,q,k)|}}.$$

Subtracting (26) from (24) we obtain,

$$2(\varrho+2\zeta)\left(1+\frac{6\delta}{\varrho+2\zeta}\right)[3]^k_{p,q}(a_3-a_2^2)=h_2(r)(u_2-v_2). \tag{31}$$

In view of (28) and (30), Equation (31) becomes

$$a_3=\frac{h_2^2(r)(u_1^2+v_1^2)}{2([2]^k_{p,q}(\varrho+\zeta+2\delta))^2}+\frac{h_2(r)(u_2-v_2)}{2(\varrho+2\zeta)\left(1+\frac{6\delta}{\varrho+2\zeta}\right)[3]^k_{p,q}}$$

By applying (3), we get,

$$|a_3|\le\frac{b^2r^2}{([2]^k_{p,q})^2(\varrho+\zeta+2\delta)^2}+\frac{|br|}{(\varrho+2\zeta)(1+\frac{6\delta}{\varrho+2\zeta})[3]^k_{p,q}}.$$

□

By setting $\varrho=\delta=0$ and $\zeta=1$ in Theorem 2.2, we obtain the following consequence.

Corollary 2.1. *If f of the form (1) is in the class $\mathfrak{B}_1(p,q,k)$ then*

$$|\ a_2\ |\le\frac{|\ br\ |\sqrt{|\ br\ |}}{\sqrt{|\{(2[3]^k_{p,q}-([2]^k_{p,q})^2)b-[2]^k_{p,q}p\}br^2-[2]^k_{p,q}aq|}}$$

and

$$|\ a_3\ |\le\frac{b^2r^2}{([2]^k_{p,q})^2}+\frac{|\ br\ |}{2[3]^k_{p,q}}.$$

setting $\varrho=\delta=0$, $\zeta=1$ and $k=0$ in Theorem 2.2, we obtain

Corollary 2.2. *If f of the form (1) is in the class $\mathfrak{B}_1(r)$ then*

$$|\ a_2\ |\le\frac{|\ br\ |\sqrt{|\ br\ |}}{\sqrt{|\{b-p\}br^2-aq|}}$$

and

$$|\ a_3\ |\le b^2r^2+\frac{|\ br\ |}{2}.$$

3 Fekete-Szegö inequality for the class $\mathfrak{B}_\zeta(p, q, k, \varrho)$:

In this section, we prove Fekete-Szegö inequalities for functions in the class $\mathfrak{B}_\zeta(p, q, k, \varrho)$. These inequalities are given in the following theorem.

Theorem 3.1. *Let f given by (1) be in the class $\mathfrak{B}_\zeta(p, q, k, \varrho)$ and $\mu \in \mathcal{R}$ Then*

$$| a_3 - \varrho a_2^2 | \leq \begin{cases} \frac{2|br|}{2(\varrho+2\zeta)\left(1+\frac{6\delta}{\varrho+2\zeta}\right)[3]_{p,q}^k}, & 0 \leq| \phi(\varrho, r) | \leq \frac{1}{2(\varrho+2\zeta)\left(1+\frac{6\delta}{\varrho+2\zeta}\right)[3]_{p,q}^k} \\ 2|br||\phi(\varrho, r)|, & |\phi(\varrho, r)| \geq \frac{1}{2(\varrho+2\zeta)\left(1+\frac{6\delta}{\varrho+2\zeta}\right)[3]_{p,q}^k} \end{cases}$$

where

$$\phi(\varrho, r) = \frac{h_2^2(r)(1-\varrho)}{\Upsilon(p, q, k, \varrho)}$$

and

$$\Upsilon(p, q, k, \varrho) = (\varrho + 2\zeta) \left\{ (\varrho - 1)([2]_{p,q}^k)^2 + 2\left(1 + \frac{6\delta}{\varrho + 2\zeta}\right)[3]_{p,q}^k \right\} h_2^2(r) - 2h_3(r)([2]_{p,q}^k(\varrho + \zeta + 2\delta))^2. \quad (32)$$

Proof. From (30) and (31)

$$a_3 - \varrho a_2^2 = \frac{(1-\varrho)h_2^3(r)(u_2 + v_2)}{\Upsilon(p, q, k, \varrho)} + \frac{h_2(r)(u_2 - v_2)}{2(\varrho + 2\zeta)\left(1 + \frac{6\delta}{\varrho+2\zeta}\right)[3]_{p,q}^k}$$

$$= h_2(r) \left\{ \left[\phi(\varrho, r) + \frac{1}{2(\varrho + 2\zeta)\left(1 + \frac{6\delta}{\varrho+2\zeta}\right)[3]_{p,q}^k} \right] u_2 + \left[\phi(\varrho, r) - \frac{1}{2(\varrho + 2\zeta)\left(1 + \frac{6\delta}{\varrho+2\zeta}\right)[3]_{p,q}^k} \right] v_2 \right\}$$

where

$$\phi(\varrho, r) = \frac{h_2^2(r)(1-\varrho)}{\Upsilon(p, q, k, \varrho)}$$

and $\Upsilon(p, q, k, \varrho)$ is given in (32).

$$| a_3 - \varrho a_2^2 | \leq \begin{cases} \frac{2|br|}{2(\varrho+2\zeta)\left(1+\frac{6\delta}{\varrho+2\zeta}\right)[3]_{p,q}^k}, & 0 \leq| \phi(\varrho, r) | \leq \frac{1}{2(\varrho+2\zeta)\left(1+\frac{6\delta}{\varrho+2\zeta}\right)[3]_{p,q}^k} \\ 2|br||\phi(\varrho, r)|, & |\phi(\varrho, r)| \geq \frac{1}{2(\varrho+2\zeta)\left(1+\frac{6\delta}{\varrho+2\zeta}\right)[3]_{p,q}^k}. \end{cases}$$

□

4 Conclusion

In the present work, by making use of Salagean differential operator, we define a new subclass of analytic and bi-univalent functions using the Horadam polynomial. Coefficient estimate $|a_2|$, $|a_3|$ and Fekete szegö inequality of the functions has been studied.

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