Heap Paradox in Fish Schools

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Abstract: The heap paradox is an enduring problem in complex systems. One variation of the heap paradox considers how many parts of a whole are required to generate group behaviour. The original heap paradox indicates that the recursive adding grain process finally makes a heap exceeding at a certain point; however, in this study, we insist that this recursive operation must involve ontological judgement (i.e. anticipation for the forthcoming heap from non-heap in every procedure). Applying integrated information theory (IIT) 3.0 provides two solutions to this paradox. Here, the group behaviour of P. altivelis is analysed with group sizes ranging from 2–5 fish. Resultingly, the subgroup embedded in the whole group has qualitatively different group integrity compared with a whole group of the same size: 2-fish subschools lose the followership property, but gain the leadership property, which is never observed in a 2-fish whole school. Thus, our result suggests that each fish tentatively solves the heap paradox according to its own ontological judgements concerning its group size.

Keywords: The paradox of the heap, Collective behaviour, Integrated information theory 3.0

1. INTRODUCTION

The heap paradox is a representative argument regarding the relation between parts and the whole. The heap paradox considers the following: If one picks a grain out of a heap of wheat, successively, when does this heap become a non-heap? Alternatively, if one puts a grain on the ground successively, when do the collective grains become a heap? Here, the latter version is considered; the objective of this paradox investigates how many individuals are needed to make a whole. Accordingly, the heap paradox implies that execution of the same recursive operation qualitatively changes the object.

The heap paradox is inevitable in collective behaviour studies. Conflicts between the parts and the whole observed in this paradox are confirmed in several disciplines, including decision making and information transfer [1-5]. Such studies have attempted to identify the optimal regions in which both the parts and the whole are satisfied (e.g. phase transition in birds flocks [2, 3]). However, the following fundamental question remains unanswered in collective behaviour studies: how many individuals are needed to produce genuine group behaviour? Accordingly, the number of parts required to comprise a whole is investigated here.

Typically, the quantity of components that generates a whole is either addressed intuitively or relies upon implicit assumptions. Some studies imply that two or three individuals are sufficient to generate collective (qualitatively different) behaviour [6-8]. Other studies solely examine the group behaviours from the list of groups from two to ten and above individuals [9]. Those studies never explicitly mention what the criterions of group behaviour are, but they implicitly assume that group behaviour emerges in these numbers. Although some studies [6, 8] suggest that the shift in qualitative difference occurs between two and three fish, this result is difficult to generalize. Accordingly, a thorough and generalizable solution to the heap paradox remains unachieved.

In establishing a method to develop a solution, the details of the logical structure of the heap paradox are considered as follows. The paradox posits that adding grains produces the heap; however, the location at which grains are added before the heap exists must be determined. Adding the grain randomly never makes the heap under the unbounded area condition because uniform distribution of grains does not constitute a heap. Generally, the operation of the heap paradox assumes that the addition of grains occurs at a specific fixed position [10, 11]. By fixing this grain addition point, the heap eventually emerges through the recursive operation.

However, careful reflection leads to the consideration that this fixed drop point is invalid because it is chosen based on the assumption that the characteristics of the heap are known before it is built. In other words, constructing a solution to the paradox requires the following process: first, the characteristics of the heap is known in advance; then, it is essentially forgotten; finally, it is reinvestigated. This observation provides essential insight into the heap paradox; the recursive operation inevitably involves the ontological judgement regarding the location at which the parts accumulate in every step in the process of making the heap. The operator must anticipate the forthcoming heap from non-heap in each procedure. This is what we call "ontological judgement" hereafter. Without such judgement, the heap will never be constructed.

To address this, the manipulationist view [12] in integrated information theory (IIT) 3.0 bridges the heap paradox and each agent's ontological judgement. Although IIT originally proposed measuring the degree of consciousness (Φ), some studies have succeeded in demonstrating that this theory can also apply to general complex systems [13-16]. Particularly, in collective

behaviour, Φ represents "group integrity" as autonomous causal structure for a given system. The manipulationist view in IIT 3.0 posits that Φ (i.e. the group integrity) is based on the computation of all possible virtual interventions [12] (or all counterfactual states [15]) in the causal system. These virtual commitments to the system suggest how each fish virtually effects the others with regard to maintaining an aggregation.

In our previous study [15], IIT 3.0 revealed the discontinuity between 3- and 4-fish schools in terms of the existence of leadership. Additionally, it was suggested that IIT 3.0 can discriminate 2- to 5-fish schools as fundamentally different autonomous systems [16]. Accordingly, the causal structure is investigated here in greater detail. This paper investigates the behavioural differences in the fish of a 2-fish group compared with 2 fish in a group of 3 or more in terms of causal structure. Differences therein indirectly suggests that each fish performs different ontological judgements with respect to its group size, as different ontological judgement contain different virtual interventions.

2. MATERIALS AND METHODS

2.1 Experimental Settings

We studied the behaviour of *P. altivelis*. Juveniles (7–14 cm) display typical schooling behaviour. We purchased juveniles from Tarumi Aquafarming (Kasumigaura, Ibaraki, Japan), which were then were housed in controlled laboratory conditions. Approximately 150 fish were kept in an 8 m^3 tank with continuously filtered and recycled fresh water maintained at 16.4 °C. Just before each experiment was conducted, randomly chosen fish were separated into each school size and moved to the experimental arena without pre-training. The experimental arena consisted of a 3×3 m shallow white tank with a water depth of about 8 cm. This experimental setup was used in previous studies [7]. The temperature of the water in this experiment tank was also 16.4 °C. The fish could only move in two dimensions such that they seldom overlapped with each other vertically. The behaviour in the tank was recorded with an overhead grayscale video camera (Library GE 60, Library Co. Ltd., Tokyo, Japan) with a spatial resolution of 640×480 px and a temporal resolution of 120 fps. The recording was started after 5 min of free swimming.

2.2 Tracking

From the grayscale images, the time series of the individuals' positions were tracked with image processing software (Library Move-tr/2D ver. 8.31, Library Co. Ltd.), which identified the dark shape of each fish against the white background and the geometric centre of the fish. The fish trajectories were constructed by tracking individuals from one frame to the next. When the fish overlapped or came into contact with each other, they were separated with the manual tracking mode of the software. Accordingly, for each observed time duration, each individual's position was obtained in (x, y) coordinates as a single pixel with a side length of 4.76

mm. The time interval between two consecutive reconstructions of the individuals' coordinates was fixed at $dt = 0.2$ s (24 fps). The position of each fish was represented by (x, y) coordinates.
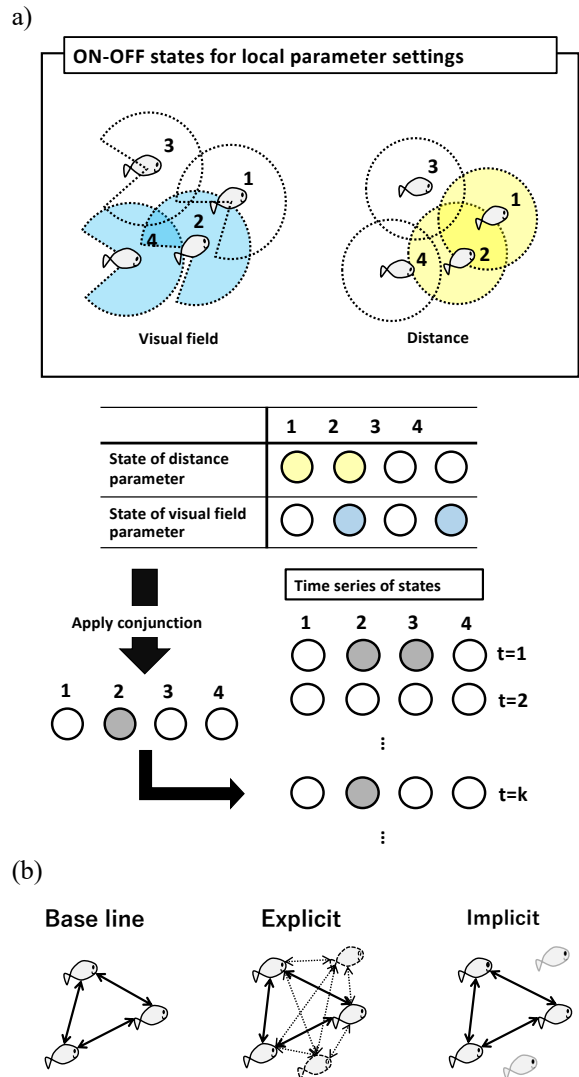
a)

(b)

Fig. 1 (a) The definition of ON/OFF states for each fish and the procedure for constructing a collective bit states sequence. The time step is fixed at $dt = 0.2$ s (24 fps). (b) Differences between a base line group (3-fish school), the subgroup as explicit interaction, and the subgroup as implicit interaction (for 3 fish in 5-fish schools). The network is fully connected without a self-loop.

2.3 Parameter settings and computing Φ

Here, all computations were performed using the PyPhi software package with the CUT_ONE_APPROXIMATION to Φ [14]. To compute Φ values for each network state, the transition probability matrix (TPM), which satisfies conditional independence, is needed. For readers who are unfamiliar with IIT, it suffices to know that the degree of integrity, Φ, is the loss of information by a certain cut on the system, called the minimal information partition (MIP) [13-15]. If the loss of information is low, that system has low integrity since

the division of the system results in minimal effects on that system. Throughout this paper, it is assumed that high group integrity (high Φ values) represents the group behaviour.

To apply IIT 3.0, a binary state among the fish schools must be defined. The interaction radius with the visual field can determine the ON/OFF state for each fish. This definition is based on the classical flocking model. In our previous studies [15, 16], the ON/OFF state was adopted as the turning rate with an interaction radius; here, only the interaction radius without the turning rate was used for simplicity. If the fish of interest has any neighbours within its interaction radius, its state is ON, otherwise it is OFF (Fig. 1(a)).

After determining each state for each step, a collective state is obtained, i.e. a n-bit sequence for a n-fish school, e.g. $b_t=x_1x_2...x_n$, where x is 0:OFF or 1:ON). The TMP is uniquely determined from this collective state sequence $(b_1, b_2, ..., b_T)$, where T is the maximal time. Then, the Φ values can be computed for each collective state. Note that there are 2^n Φ values since each collective state has its own Φ value. For instance, for a 2-fish school, there are four collective states: 00, 01, 10, and 11. Therefore, there are four values: Φ_{00}, Φ_{01}, Φ_{10}, and Φ_{11}, respectively. Here, we only focus on the mean Φ values, denoted $<\Phi>$, to compare the average degree of the group integrity.

Additionally, the group integrity, Φ, is considered for subgroups of the whole group. When computing the subsystem's Φ value in the same TPM, the rest of the system is treated as the noise to that subsystem (Fig. 1(b): explicit interaction). These noises give different Φ values for each subsystem. IIT sometimes devotes much attention to the subsystems of a whole system because the main complex (a set of elements that have the highest Φ value) is often a subsystem; however, this reversal trend (that is, the subsystem's Φ value is larger than the whole system's Φ value) was never confirmed to occur in a fully connected network.

Another computational method was also attempted, that is, the subgroup is not a subsystem of IIT, but the subgroup is selected individuals independently from the whole group. In other words, different TMPs were reconstructed from the selected fish as if the rest of the members never existed. The effects from the rest is then not treated as noise in the TPM, but as the restriction of the subgroup's trajectory (Fig. 1(b): implicit interaction). However, the effect of this interaction has been confirmed to be much weaker than that of explicit interaction. Accordingly, the graph of implicit interactions is not included. Instead, the reference value to the explicit interaction is included.

2.4 Experimental data

The experimental data used for IIT 3.0 is listed in Table 1. Three data sequence are obtained for the 2-, 4- and 5-fish schools and four data sequences for the 3-fish schools. All data for the minimum distance (body contact with each other) is nearly zero. Accordingly, an all connected network was chosen, as it is reasonable to assume that physical contact generates some information transfer.

2.5 Ethics statement

This study was carried out in strict accordance with the recommendations in the Guide for the Care and Use of Laboratory Animals of the National Institutes of Health. The protocol was approved by the Committee on the Ethics of Animal Experiments of the University of Tsukuba (Permit Number: 14-386). All efforts were made to minimise suffering.

Table 1. Data summary for fish schools of varying group sizes.

Group size	Average distance (mm)	Average velocity (mm/s)	Minimum distance (mm)	Total time (s)
2	166.3	268.8	1.90	861
	90.67	271.7	0.10	828
	122.0	256.1	1.60	893
3	170.8	301.2	1.80	750
	159.1	343.2	1.83	697
	173.1	300.0	2.82	812
	132.0	240.0	1.67	782
4	164.3	270.7	1.18	886
	141.5	190.8	1.38	860
	114.9	148.6	1.83	817
5	143.8	260.0	0.79	857
	146.0	213.1	1.16	767
	143.7	259.2	1.44	811

3. RESULTS

3.1 $<\Phi>$ as base line

As noted in the section 2.3, only mean Φ values ($<\Phi>$) are used to evaluate the group integrity. It is reasonable to assume that high $<\Phi>$ values are meaningful for fish schools since low group integrity signifies an unordered individual collective.

Figure 2 shows the heat map distribution of $<\Phi>$ for every pair of parameters. The different parameter settings generate different $<\Phi>$ corresponding to different TPMs. In previous studies [15, 16], the discontinuity between 3- and 4-fish schools was confirmed. The meaning of discontinuity is in the interrelation between heat map distribution where the peak of $<\Phi>$ shifted from the full visual field (Fig. 2(a)) to a blind spot (Fig. 2(b)). The peak shift in heat map can discriminate in terms of the different kinds of interaction (i.e. pair of parameters) to generate high group integrity with respect to each size.

Notably, the MIP, or the partitioning where most information decreases, in the blind spot condition always cuts between the group leader and the rest of its members. We confirmed that this IIT-induced leader corresponds with the positional leadership, i.e., the top fish along with group moving direction; however, in the full visual field condition (Fig. 2(a)-(b)), the MIP cut never ensures the leadership partition. Therefore, the peak $<\Phi>$ around the second row of the heat map (Fig. 2(c)-(d)) represent the existence of leadership.

Contrastingly, there are other peaks around the bottom left of the heatmap in Fig. 2 as (a)-(d). Notably, the peak value of $<\Phi>$ exceeds all other cells for the 2-fish school, with the highest $<\Phi>$ at (200 mm, 36 degrees). This is considered followership, to distinguish from the

leadership discussed above. The narrow visual fields from 36 to 72 (deg) in this condition emphasize the chasing behaviour for moving fish ahead of the current fish. This chasing behaviour is predominant in the 2-fish school.
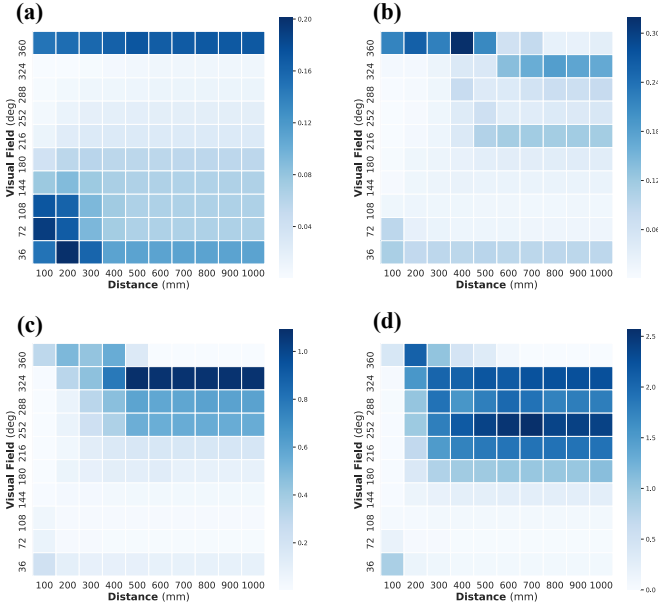

Fig.2 eat map distribution of $<\Phi>$ …with respect to school sizes for (a) 2-, (b) 3-, (c) 4-, and (d) 5- fish schools.

The property of baseline conditions considered in the following sections is summarized as follows: (I) a genuine 2-fish school: has followership; (II) a genuine 3-fish school: has no leader and followership and the full visual field condition is the matter for group integrity; (III) genuine 4-fish school: leadership emerges; (IV) genuine 5-fish school: leadership also emerges, however, the visual field parameter becomes narrower to generate the peak $<\Phi>$. In [16], more details of the difference among these systems are discussed.

3.2 $<\Phi>$ for 2-fish subschool in 3-fish schools and larger

In the previous section, $<\Phi>$ heatmap distributions were estimated according to parameter settings for each group size. In this section, changes to $<\Phi>$ heatmap distributions are considered when the group is embedded in larger systems. Additionally, the difference in terms of group integrity between a 2-fish school and 2 fish in 3-fish and larger schools are considered. Evaluating these differences informs the different ontological judgements that occur amongst large groups.

In this section, the genuine 2-fish school and the 2-fish school as a subgroup of larger systems are compared. Computing all combinations of data ($_nC_2$ data for n-fish school of each data) and averaging them, the heat map distribution can be obtained as in Fig. 2; however, to clarify the differences among them, three regions were selected where the most remarkable differences are observed: Region 1, 2, and 3 in Fig. 3(a).

In Fig. 3(b), the boxplot for each subsystem and baseline (genuine 2-fish school) shows their $<\Phi>$,

respectively. Note that the baseline uses the same data as in Fig. 2(a). The denotation m/n in Fig. 3(b) represents an m-fish subschool in an n-fish school.

The difference between the baseline and the other subsystems is clear. First, in region 1, the followership observed in the baseline diminished. The high $<\Phi>$ values observed in Fig. 2(a) drops almost to zero (Mann–Whitney U-test: $<10^{-10}$ for each data compared with the baseline). This tendency was also observed in the implicit interaction ($<\Phi>$ for 2/3-fish subschool: 0.07, 2/4-fish subschool: 0.06, 2/5-fish subschool: 0.05). Furthermore, the other high $<\Phi>$ concentrated area (Region 2) also reduces its values dramatically (Mann–Whitney U-test: $<10^{-10}$ for each data compared with baseline). This indicates that the pair of parameters needed to generate high group integrity in the genuine 2-fish school never generates high group integrity in the subsystems.
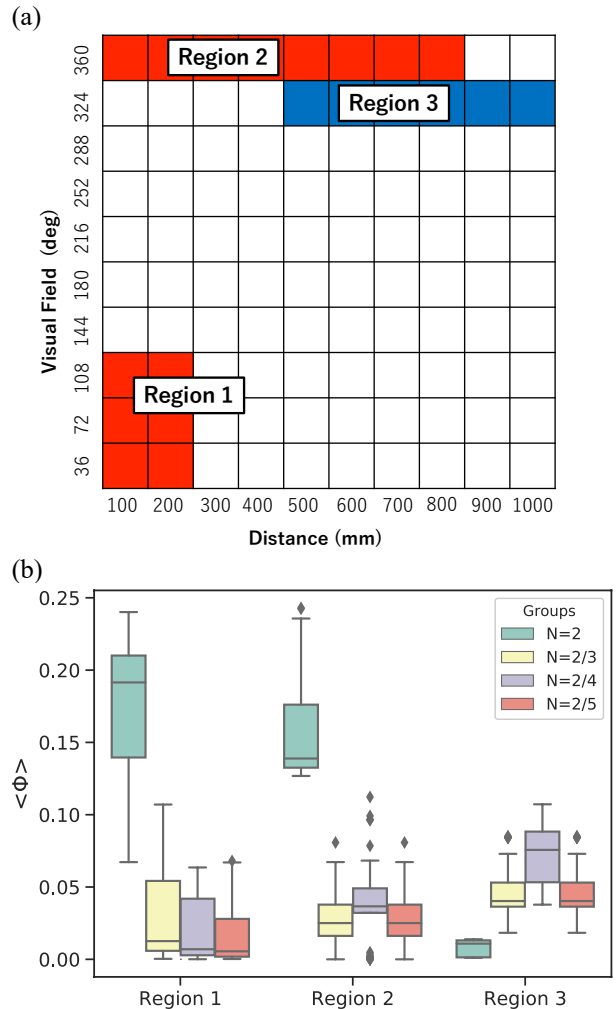

Fig.3 (a) Comparison regions of subsystems of the same size group and the baseline (genuine 2-fish school). The red regions represent the decreasing zone and the blue region represents the increasing zone. (b) The box plot of $<\Phi>$ for the baseline and the subgroup in the large group, across the defined regions. For group sizes N, m/n an m-fish subschool in an n-fish school.

In contrast, $<\Phi>$ increase in Region 3. This is the only

region in which <Φ> increase. As discussed above, this region corresponds to the existence of leadership. Raising <Φ> in Region 3 instead of Region 2 signifies that the subschool in the group uses a different kind of interaction to generate high group integrity. Interestingly, the <Φ> distribution of implicit interaction settings shows little difference compared with the 2-fish baseline.
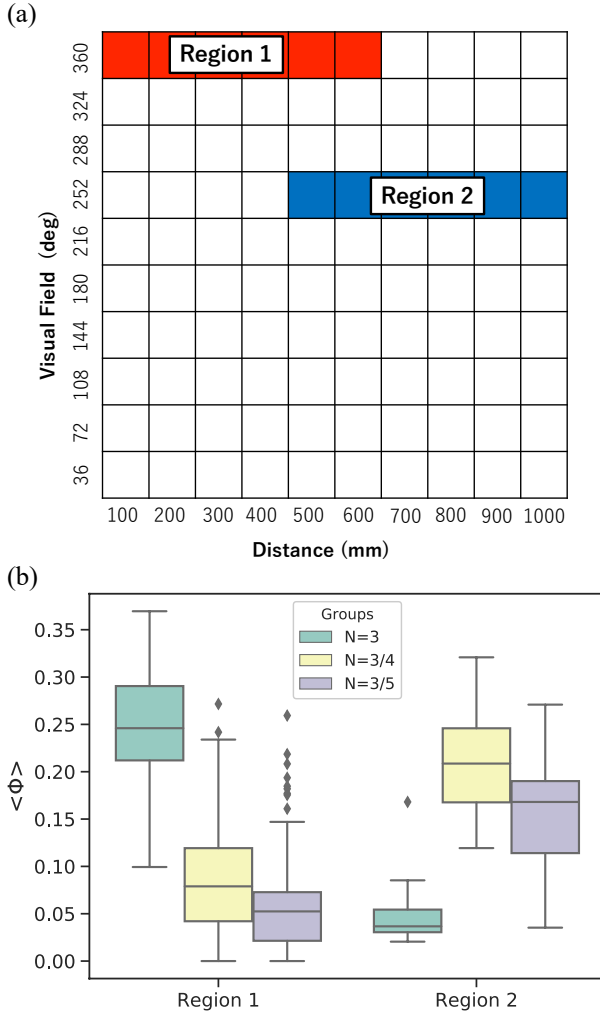
(a)

(b)

Fig. 4 (a) Comparison regions of the subsystems of the same size group and the baseline. The red region represents the decreasing zone and the blue region represents the increasing zone. (b) Box plot of <Φ> of for the baseline and subgroups in the large group, across the defined regions. For group sizes N, *m/n* an *m*-fish subschool in an *n*-fish school. Each data is significantly different compared with the baseline (Mann–Whitney U-test: $<10^{-10}$).

3.3 <Φ> for a 3-fish subschool in 4- and 5-fish schools

In this section, the group integrity of a 3-fish subschool is examined (3 fish in 4- and 5-fish schools). As seen in the 2-fish subschools, we confirmed the same tendency (i.e. <Φ> reduced with full visual field condition and <Φ> increased with blind spots conditions) except for followership. The main differences are seen in two regions (Fig. 4(a)).

In Region 1, <Φ> in the full visual field conditions decreases nearly to zero. Instead of a decreasing <Φ> in

Region 1, <Φ> in Region 2 increases the most. This peak shift signifies that 3 fish as a subgroup achieve high group integrity with the existence of leadership (the MIP cut divides between the leader and the rest). Notably, the blind spot (outside of the visual field) is larger than the 2-fish subgroup, generating the group integrity.

Additionally, <Φ> of the implicit interaction has a minimal impact on its baseline hereafter. The difference between them shows some difference significantly, but they are almost the same values (For instance, in Region 1, <Φ> for 3/4-fish subschool is 0.20, and for 3/5-fish subschool is 0.19; in contrast, the <Φ> of baseline is 0.24. The comparison of both values shows significant difference, but their difference is not so larger than explicit interaction in Fig. 4(b)).
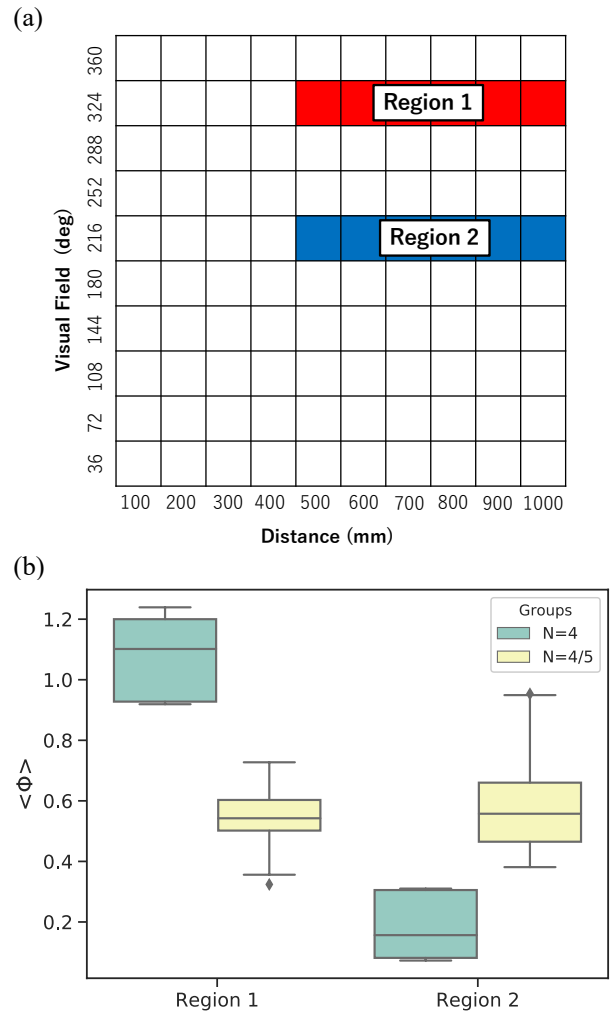
(a)

(b)

Fig. 5 (a) Comparison regions of subsystems of the same size group and the baseline. The red region represents the decreasing zone and the blue region represents the increasing zone. (b) The box plot of <Φ> for the baseline and subgroup in large group, among the defined regions. For group sizes N, *m/n* an *m*-fish subschool in an *n*-fish school. Each data is significantly different compared with the baseline (Mann–Whitney U-test: $<10^{-10}$).

3.3 <Φ> for a 4-fish subschool in a 5-fish school

Finally, the 4-fish subschool in a 5-fish school is

examined. The difference is less drastic than in sections 3.1 and 3.2. Two regions were selected for comparison (Fig 5(a)). The difference between these two regions appears in the width of the blind spots. The peak $<\Phi>$ are around the second row from the top in the genuine 4-fish school (Fig. 2(c)); in contrast, these peaks shift to lower regions (Fig. 5(b)) as if the 4-fish subschools show resemblance in terms of group integrities in certain parameter settings. Note that this is a selective case for explicit interaction in subgroups. The subschool in implicit interactions never shows contrast as in that of explicit ones (In Region 1, $<\Phi>$ for 4/5-fish subschool is 0.85 but the baseline is 1.08.)

4. DISCUSSION

Here, our analysis for subschools embedded in whole schools is summarized. The subgroup applies a different pair of parameters to achieve high group integrity. Notably, the applied parameter settings in a subgroup never emerge in a whole group. For instance, the 2-fish subschool, which showed the most radical shift in the parametric space, has a leadership instead of a followership. The peak of integrity in the leadership is not observed in the 2-fish school for any parameter setting [16]. The two subgroups in the whole school obtain qualitatively different group integrities. Similarly, the 3-fish school also shows the leadership only as a subgroup, and the 4-fish school with its peak of $<\Phi>$ has more blind spots to generate a high group integrity

As discussed in the Introduction, the heap paradox contains a hidden assumption regarding the ontological judgement for the recursive procedure. The grain dropper, or operation which recursively adds parts to the whole, must select the location at which the parts are added by anticipating the forthcoming heap at each addition. This assumption posits that the grain dropper must construct an image of the heap from the non-heap grain components. This discussion also applies to the fish school: constructing a group forces each fish to determine what the group behaviour should be.

Accordingly, the solution to this paradox considers the point at which a collection of fish constitutes a group. Two tentative solutions are proposed. The first suggests that 2-fish schools have different causal structures compared with 3-fish and larger fish schools. Likewise, 3-fish schools also have different causal structures compared with other numbers of fish schools, and so on (recall that computation of Φ by definition is based on all possible virtual intervention to the given system. See the discussion in [16]). There are always qualitative discontinuities in terms of group integrity according to the group size. Therefore, the characteristics of the group are different for each size. The assumption of the paradox, where there is one qualitative discontinuity among the group size, collapses. Hence, the paradox is solved by invaliding its assumption.

The second solution applies the findings from the experiments conducted herein. Examining the subsystem details elucidates various causal structures compared to the whole school of the same size. As shown above, a 2-fish subschool is qualitatively different than a 2-fish

whole school, with regard to the emergence of leadership. This difference emerges because the virtual interventions to the system are different in each system.

The interventions to the system can be rephrased as the ontological judgements to that system. Accordingly, the group integrity, Φ, represents the total possible commitments of the fish. The intervention to each fish estimates the irreducible components as subgroups (this irreducible component in IIT and finally gives Φ [13-15]). Some fish may estimate the irreducible group constructed from 2 fish and other fish may estimate the irreducible group constructed from other numbers. These different ontological judgements make the whole group unique. This argument also applies to the subgroups. This consideration suggests that each fish inevitably confronts the paradox of the group's characteristics and provides a temporary solution in a given condition. Accordingly, the second solution posits that each fish tentatively solves the heap paradox of its given condition in different ways.

It is not insisted that IIT 3.0 alone can provide a solution to the heap paradox. There may be other ways to identify qualitative discontinuity among the different group sizes; however, we believe that IIT 3.0 provides an essential insight to understanding animal collective behaviour. These results may apply to only *P. altivelis* schools in these particular conditions. It is likely that different animals may exhibit different causal structures according to IIT 3.0. Different ontological judgements to the group by other animals could explain diverse natural group sizes.

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