Weight Distributions of a Class of Linear Codes

Xina Zhang, Xiaoni Du, Rong Wang and Fujun Zhang
weight distributions by character sums over finite fields. Namely, let $D = \{d_1, d_2, \ldots, d_n\} \subseteq F_q$. The linear code $C_D$ of length $n$ over $F_p$ is defined by

$$C_D = \{(Tr(xd_1), Tr(xd_2), \ldots, Tr(xd_n)) : x \in F_q\},$$

here the set $D$ is called the defining set of linear code $C_D$. From then on, many works have been done in this study by scholars (See Refs. [1], [6], [7], [9], [10], [11], [13], [14], [16] for example). In particular, there are optimal codes and almost optimal codes in the literature [7], [10], [11], [13], [14], [16].

In this literature, with the construction method as mentioned above, we obtain a class of linear codes with a few weights. Meanwhile, we determine their weight distributions by character sums over finite fields.

II. Preliminaries

An additive character of $F_q$ is a non-zero function $\chi$ from $F_q$ to the set of complex numbers of absolute value 1 such that $\chi(x+y) = \chi(x)\chi(y)$ for any pairs $(x, y) \in F^2_q$. For each $u \in F_q$, the function

$$\chi_u(v) = \zeta^{Tr(uv)}, \quad v \in F_q$$

is an additive character of $F_q$. The absolute trace function [12] from $F_q$ onto $F_p$ is defined by

$$Tr(x^{q^n}) = x + x^q + \cdots + x^{q^{n-1}},$$

for any $x \in F_q$.
denotes an additive character of $\mathbb{F}_q$, where $\zeta = e^{2\pi i/p}$ is a primitive $p$-th root of unity and $i = \sqrt{-1}$. Since $\chi_0(v) = 1$ for all $v \in \mathbb{F}_q$, we call it the trivial additive character of $\mathbb{F}_q$. We call $\chi_1$ the canonical additive character of $\mathbb{F}_q$ and $\chi_u(x) = \chi_1(ax)$ for all $u \in \mathbb{F}_q$. The additive character satisfies the orthogonal property [12], that is

$$\sum_{v \in \mathbb{F}_q} \chi_u(v) = \begin{cases} q, & \text{if } u = 0, \\ 0, & \text{if } u \neq 0. \end{cases}$$

Let $g$ be a fixed primitive element of $\mathbb{F}_q$. For each $j = 0, 1, \ldots, q - 2$, the function $\lambda_j(g^k) = e^{2\pi i jk/(q-1)}$ for $k = 0, 1, \ldots, q - 2$ defines a multiplicative character of $\mathbb{F}_q$, we extend these characters by setting $\lambda_j(0) = 0$. It satisfies

$$\sum_{v \in \mathbb{F}_q} \lambda_j(v) = \begin{cases} q - 1, & \text{if } j = 0, \\ 0, & \text{if } j \neq 0. \end{cases}$$

The following text we refer to $\eta = \lambda_{(q-1)/2}$ and $\eta' = \lambda_{(p-1)/2}$ the quadratic characters over $\mathbb{F}_q$ and $\mathbb{F}_p$, respectively. The Gauss sums over $\mathbb{F}_q$ and $\mathbb{F}_p$ are defined respectively by

$$G(\lambda) = \sum_{v \in \mathbb{F}_q^*} \lambda(v) \chi_1(v)$$

and

$$G'(\lambda') = \sum_{v \in \mathbb{F}_p^*} \lambda'(v) \chi'_1(v),$$

where $\lambda'$ and $\chi'_1$ are the canonical multiplicative and additive characters of $\mathbb{F}_p$, respectively.

III. MAIN RESULTS

In this section, we present the main results, including the construction, the parameters and the weight distributions of the linear codes $C_D$. The proofs will be given in section IV.

We firstly select the defining set

$$D = \{(x_1, x_2) \in \mathbb{F}_q^2 : Tr(x_1^{p^s+1}) = 1, Tr(x_2) = 1\} \quad (2)$$

to construct linear code

$$C_D = \{(Tr(ax_1 + bx_2)(x_1, x_2) \in D) : a, b \in \mathbb{F}_q\}, \quad (3)$$

where $m = 2s$ and $s$ is a positive integer.

Define

$$Sp = \{x \in \mathbb{F}_p^*\} \quad \text{and} \quad NSp = \mathbb{F}_p^* \setminus Sp. \quad (4)$$

**Theorem 1:** With the notation above, the weight distributions of the code $C_D$ with the parameter $|p^{2m+2} + p^m + s, 2m|$ is shown in Table I. Obviously, the code is 5-weight.

**Example 1:** If $(p, m) = (5, 2)$, then the code $C_D$ has parameters $[30, 4, 20]$ and weight enumerator $1 + 60z^{20} + 500z^{24} + 242z^{25} + 40z^{30}$, which from Magma program confirmed the result in Theorem 1.

**Example 2:** If $(p, m) = (3, 2)$, then the code $C_D$ has parameters $[12, 4, 6]$ and weight enumerator $1 + 12z^{6} + 54z^{8} + 8z^{9} + 6z^{12}$, which from Magma program confirmed the result in Theorem 1.

**Example 3:** If $(p, m) = (3, 4)$, then the code $C_D$ has parameters $[810, 8, 486]$ and weight enumerator $1 + 110z^{486} + 6318z^{540} + 100z^{567} + 30z^{648} + 2z^{810}$, which from Magma program confirmed the result in Theorem 1.

IV. AUXILIARY LEMMAS AND PROOFS OF THE MAIN RESULTS

We begin this section by introducing some known facts on exponential sums.

**Lemma 1:** [4] Let $m = 2s$, where $s$ is a positive integer. For any $a \in \mathbb{F}_p^*$ and $b \in \mathbb{F}_p^m$, we have

$$\sum_{x \in \mathbb{F}_q} \zeta^{Tr(a(x^{p^s+1}) + \alpha x)} = -p^s \zeta^{-Tr(a(\frac{x^{p^s+1}}{a}))}.$$  

**Lemma 2:** [12] If $f(x) = a_2x^2 + a_1x + a_0 \in \mathbb{F}_q[x]$, where $a_2 \neq 0$, then

$$\sum_{x \in \mathbb{F}_q} \zeta^{Tr(f(x))} = \zeta^{Tr(a_0 - a_1^2(4a_2)^{-1})} \eta(a_2)G(\eta).$$

**Lemma 3:** [12] With the notation above, we have

$$G(\eta) = (-1)^{m-1} \sqrt{(p^s)^m}, \quad G'(\eta') = \sqrt{p^s},$$
where \( p^* = (-1)^{\frac{p-1}{2}} \).

The following Lemmas 4-5 are essential to compute the length and the weight distributions of linear code \( C_D \).

**Lemma 4:** The length of the \( C_D \) is

\[
n = \left| \{(x_1, x_2) \in \mathbb{F}_q^2 : \text{Tr}(x_1^{p^*+1}) = 1, \text{Tr}(x_2) = 1 \} \right| = p^{2m-2} + p^{m+s-2}.
\] (5)

**Proof:** Since

\[
n = \left| \{(x_1, x_2) \in \mathbb{F}_q^2 : \text{Tr}(x_2) = 1, \text{Tr}(x_1^{p^*+1}) = 1 \} \right|
= \sum_{x_1, x_2 \in \mathbb{F}_q} \left( \frac{1}{p} \sum_{y_1 \in \mathbb{F}_p} \zeta y_1 (\text{Tr}(x_1^{p^*+1})) \right)
= \left( \frac{1}{p} \sum_{y_2 \in \mathbb{F}_p} \zeta y_2 (\text{Tr}(x_2)) \right)
= p^{2m-2} + \frac{1}{p^2} (\Omega_1 + \Omega_2 + \Omega_3),
\]

where

\[
\Omega_1 = \sum_{x_1, x_2 \in \mathbb{F}_q} \sum_{y_2 \in \mathbb{F}_p} \zeta y_2 \text{Tr}(x_2) - y_2
= \sum_{y_2 \in \mathbb{F}_p} \zeta y_2 \sum_{x_1, x_2 \in \mathbb{F}_q} \text{Tr}(x_2)
= 0.
\]

\[
\Omega_2 = \sum_{x_1, x_2 \in \mathbb{F}_q} \sum_{y_1 \in \mathbb{F}_p} \zeta y_1 \text{Tr}(x_1^{p^*+1}) - y_1
= \sum_{y_1 \in \mathbb{F}_p} \zeta y_1 \sum_{x_1, x_2 \in \mathbb{F}_q} \text{Tr}(x_1^{p^*+1})
= p^{m+s},
\]

\[
\Omega_3 = \sum_{x_1, x_2 \in \mathbb{F}_q} \zeta y_1 \text{Tr}(x_1^{p^*+1}) - y_1
= \sum_{y_2 \in \mathbb{F}_p} \zeta y_2 \sum_{x_1, x_2 \in \mathbb{F}_q} \text{Tr}(x_2)
= 0.
\]

Then \( n = p^{2m-2} + p^{m+s-2} \). \( \square \)

Let

\[
T = \left| \{(x_1, x_2) \in \mathbb{F}_q^2 : \text{Tr}(x_1^{p^*+1}) = 1, \text{Tr}(x_2) = 1, \text{Tr}(ax_1 + bx_2) = 0 \} \right|.
\]

For any \( a, b \in \mathbb{F}_q \) and \( c(a, b) \in C_D \), the weight of \( c(a, b) \) is as follows

\[
w(c(a, b)) = n - T.
\] (6)

Then

\[
T = \frac{1}{p} \sum_{x_1, x_2 \in \mathbb{F}_q} \zeta \text{Tr}(x_1^{p^*+1} - z_1) \left( \sum_{z_2 \in \mathbb{F}_p} \zeta \text{Tr}(ax_1 + bx_2) \right)
= \frac{n}{p} + \frac{1}{p^3} (\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4),
\] (8)

where \( \varphi_1 \), \( \varphi_2 \) and \( \varphi_3 \) are showed as follows, and \( \varphi_4 \) is showed at the top of next page.
Lemma 5: With the notation above, we have  
I. if $a = 0, b = 0$, then $T = n$.  
II. if $a = 0, b \neq 0$,  
(i) $b \in \mathbb{F}_p^*$, then $T = 0$.  
(ii) $b \notin \mathbb{F}_q \setminus \mathbb{F}_p^*$, then $T = p^{2m-3} + p^{m+s-3}$.  
III. if $a \neq 0, b = 0$,  
(i) $Tr(a^{\rho+1}) = 0$, then $T = p^{2m-3} + p^{m+s-2}$.  
(ii) $Tr(a^{\rho+1}) \notin Sp$, then $T = (-1)^{\frac{m-1}{2}}p^{m+s-2} + p^{2m-3}$.  
(iii) $Tr(a^{\rho+1}) \in NSp$, then $T = (-1)^{\frac{m-1}{2}}p^{m+s-2} + p^{2m-3}$.  
IV. if $a \neq 0, b \neq 0$,  
(i) $b \in \mathbb{F}_p^*$ and $Tr(a^{\rho+1}) \in \{0, b^2\}$, then $T = p^{2m-3}$.  
(ii) $b \in \mathbb{F}_p^*$, $Tr(a^{\rho+1}) \neq 0$ and $Tr(a^{\rho+1}) - b^2 \in NSp$, then $T = (-1)^{\frac{m-1}{2}}p^{m+s-2} + p^{2m-3}$.  
(iii) $b \in \mathbb{F}_p^*$, $Tr(a^{\rho+1}) \neq 0$ and $Tr(a^{\rho+1}) - b^2 \in Sp$, then $T = (-1)^{\frac{m-1}{2}}p^{m+s-2} + p^{2m-3}$.  
(iv) $b \notin \mathbb{F}_q \setminus \mathbb{F}_p^*$, then $T = p^{2m-3} + p^{m+s-3}$.

Proof: The desired conclusion then follows from $\varphi_1, \varphi_2, \varphi_3, \varphi_4$, Lemmas 1-3 and Eq.(7). Thus we complete the proof of the lemma. □

Proof of Theorem 1: we denote the non-zero weights of the lines $1 - 5$ in Table I by $w_i$, and the corresponding multiplicity by $A_{w_i}(1 \leq i \leq 5)$.

Then from Lemmas 4, 5 and MacWilliams equations [8] we have

$$
\begin{align*}
\sum_{i=1}^{5} A_{w_i} &= p^{2m-1} - 1, \\
\sum_{i=1}^{5} w_i A_{w_i} &= p^{2m-1}(p^{2m-2} + p^{m+s+2})(p-1), \\
\sum_{i=1}^{5} w_i^2 A_{w_i} &= p^{2m-2}(p-1)(p^{2m-2} + p^{m+s-2}) \\
& \cdot (p^{2m-1} + p^{m+s-1} - p^{2m-2} - p^{m+s-2} + 1),
\end{align*}
$$

where $A_1 = p - 1, A_2 = p^m(p^m - p)$. Thus we complete the proof of Theorem 1. □

V. CONCLUDING REMARKS

In this paper, we firstly constructed a class of linear codes with a few weights by a proper selection of the defining set, then we determined the parameters and the weight distributions by the character sums over finite fields. We also wrote magma programs to verify the correctness of the results.

REFERENCES

\[ \varphi_4 = \sum_{x_1, x_2 \in \mathbb{F}_p} \sum_{z_1 \in \mathbb{F}_p^*} \zeta z_1 (Tr(x_1^{p^s + 1}x_1^{p^t + 1})) - 1) \sum_{z_2 \in \mathbb{F}_p} \zeta z_2 (Tr(x_2^{p^s + 1}x_2^{p^t + 1})) - 1) \sum_{z_3 \in \mathbb{F}_p} \zeta z_3 (Tr(ax_1 + bx_2)) \]

\[ = \sum_{z_1 \in \mathbb{F}_p} \zeta z_1 \sum_{z_2 \in \mathbb{F}_p} \zeta z_2 \sum_{z_3 \in \mathbb{F}_p} \zeta \left( Tr((2z_1x_1^{p^s + 1})x_1^{p^t + 1}) + Tr((2z_2x_2^{p^s + 1})x_2^{p^t + 1}) \right) \sum_{x_2 \in \mathbb{F}_q} \zeta \left( Tr((2z_3x_3^{p^s + 1})x_3^{p^t + 1}) \right) \]

\[ = \begin{cases} 
- p^s \sum_{z_1 \in \mathbb{F}_p} \zeta z_1 \sum_{z_2 \in \mathbb{F}_p} \zeta z_2 \left( p^m \zeta \left( \frac{Tr(a^{p^s + 1})}{4x_1} \right) z_2^2 \right) + \sum_{z_3 \neq -b^{-1}z_2} \zeta \left( \frac{Tr(a^{p^s + 1})}{4x_1} \right) z_3, & b \in \mathbb{F}_p^s, \\
\sum_{x_2 \in \mathbb{F}_q} \zeta Tr((2z_2x_2^{p^s + 1})x_2^{p^t + 1}), & b \in \mathbb{F}_p^s \setminus \mathbb{F}_p^s, \\
-p^m + s, & b \in \mathbb{F}_p^s \text{ and } Tr(a^{p^s + 1}) = 0, \\
-p^m + s G'(\eta') \eta' \left( \frac{Tr(a^{p^s + 1})}{4x_1} \right) \sum_{z_1 \in \mathbb{F}_p} \zeta z_1 (1 - \frac{b^2}{Tr(a^{p^s + 1})}) \eta'(-z_1) - p^m + s, & b \in \mathbb{F}_p^s \text{ and } Tr(a^{p^s + 1}) \neq 0, b \in \mathbb{F}_q \setminus \mathbb{F}_p^s, \\
0, & b \in \mathbb{F}_p^s \text{ and } Tr(a^{p^s + 1}) \in \{0, b^2\}, \\
(-1)^{\frac{p^s}{2}} p^m + s - p^m + s, & b \in \mathbb{F}_p^s, Tr(a^{p^s + 1}) \neq 0 \text{ and } Tr(a^{p^s + 1}) - b^2 \in NSp, \\
(-1)^{\frac{p^t}{2}} p^m + s - p^m + s, & b \in \mathbb{F}_p^s, Tr(a^{p^s + 1}) \neq 0 \text{ and } Tr(a^{p^s + 1}) - b^2 \in Sp, \\
0, & b \in \mathbb{F}_q \setminus \mathbb{F}_p^s. 
\end{cases} \]
