Stable matching with an entrance criterion for teams

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Abstract

In this paper we generalize the results of former papers that dealt with the assignment of students to dormitories at the Technion–Israel Institute of Technology, under an entrance criterion. Here, we consider the case where students may apply in groups, thereafter called teams, each consisting of one or more students. A team-application means that students from any given team, want to be assigned together to the same dormitory-group. More specifically, students of the same team prefer living off-campus rather than living in different dormitory-groups.

The underlying assumption in our model is that the dormitory-groups share a common preference over the teams, which is given by a strictly increasing ranking of the teams' credit scores. We adjust the definition of a quasi-stable outcome to incorporate team applications, and show that a quasi-stable outcome always exists. Furthermore, an algorithm that finds all the quasi-stable outcomes, is presented. Apparently, some of the properties of the model for teams of a single student, continue to hold also under the model of team applications. In addition, we consider the incentive compatibility property of the outcomes generated by the proposed algorithm, and show, in particular, that the algorithm that produce a specific quasi-stable outcome, is manipulation-proof, i.e., no subset of teams can gain by misrepresenting their preferences over the dormitory-groups.

1 Introduction

Matching problems form an important topic in game theory. The original formal model was developed by David Gale and Lloyd Shapley (GS) in 1962 (see [7]). Their paper defines a notion of stability for two-sided matching in populations where individuals have preferences over being matched with individuals of the other side. Further, [7] provides an algorithm that finds a stable match. There are many variants and extensions of the original model of GS that have many useful applications, see, for example, [11, 18, 10, 27, 22, 3, 1, 2, 5, 28, 24], and references therein. In what follows, we refer to the model of GS as the classic matching model.

A case study of assignment of students to dormitory-groups at the Technion - institute of technology, led to a development of a matching model which incorporates an "entrance criterion" (see [16]). Under this model, students are considered for an assignment only if they meet a competitive eligibility condition, characterized by a merit-score. The assignment of eligible students is based on the students' preferences over the dormitory-groups as well as a common and complete preference-list of the dormitory-groups over the students, which is characterized by a credit-score of each student. The paper describes and analyzes an algorithm that generates a corresponding stable outcome with some desirable properties.

The framework of [16] is generalized in [17], by relaxing the assumption that preference-lists of the dormitory-groups are common and complete. In particular, dormitory-groups are allowed to use different evaluation criteria for stating their preferences over the students. The proposed algorithm for the generalized model preserves most of the properties of the original algorithm. Moreover, [17] demonstrates that the new algorithm satisfies the incentive compatibility property for a single student, which means that a single student cannot improve his assignment by misrepresenting his preference-list while all other students state their true preference-lists.

In the models presented above, the players in at least one of the populations, compete as individuals. However, some applications require team assignment, such as in the dormitory-groups assignment, where students want to be assigned to the same room as their friends, and otherwise, they prefer living off campus. Similarly, in the process of students' assignment to schools, there may be educational needs that impose constraints for assigning some students to the same school as their mates. Team assignment in the classic matching model is discussed, for example, in [13, 15, 12, 18, 4]. The problem of team assignment under capacity constraints on the population of the other side, is more complicated than the case of single students, and a stable assignment does not necessarily exist. See, for example, [18].

In this paper we extend the matching model with entrance criterion of [16, 17] to incorporate team assignment. More specifically, we assume that the population of the first side consists of different sized groups of students, later referred to as teams, where the population of the second side consists of dormitory-groups, each is characterized by its capacity of beds. A team is formed if all its members prefer living off campus rather than being separated and assigned to different dormitory-groups. Further, we assume that in any team, the preference-lists of all its students over the dormitory-groups, are common. In addition, the dormitory-groups are assumed to rate teams rather than individual students. There are many ways for a dormitory-group to decide how to rate a team, as for example, to rate the least/most preferred student in that team, etc. In this paper we do not discuss the methodology of forming the preference-lists for the agents, but assume that they are provided to us as an input.

In this paper, as in [16], we refer to the specific case where the preferences of the dormitory-groups are common, and are determined by the credit scores of the teams. The paper is organized as follows: In section 2 we present the team assignment model, while adapting the definitions of [16, 17] to include team assignments. In section 3 we present the properties of team assignment in the classic matching model, where dormitory-groups' preferences are common. Section 4 describes the properties of team assignment in the general model which incorporates an entrance criterion: a stable assignment for any instance is shown to exist and an algorithm that returns all possible stable assignments, is proposed. Some of the properties discussed in [16, 17], are shown to continue to hold in this model. Finally, in section 5 we discuss the existence of the incentive compatibility property for any set which contains one or more teams, i.e., the property that there is no possibility of gaining by misrepresenting the preference-lists of all the teams in the set.

2 Preliminaries and notations

In this section, a modification of the stable matching model with an entrance criterion (as presented in [16, 17]), which, incorporates "team applications", is presented.

Decisions about the assignment of students to dormitories at the Technion - institute of technology, and possibly in many other universities and colleges, are based on a three-step process (see section 2 in [16]). The first step determines the eligibility of applicants for on-campus housing. The second step allocates the students that were found eligible, to dormitory-groups. Finally, the third step assigns students to specific rooms\apartments. Different criteria are used for these steps: socio-economic and personal data are the main factors for determining the merit score of each student for the first step, where academic seniority and academic excellence determine the credit score for the second step, and finally, student preferences over rooms and room-mates are the main factors in the actual assignment of students to rooms/apartments.

Here we consider the first two steps, under the assumption that students may apply in groups rather than applying one by one. Throughout, we refer to a group of students as a team. Any team that applies for a dormitory is assigned both a merit-score and a credit-score that are functions of the individual characteristics of the students in the team. The underlying assumption for a team that fills an application is that the students in the team prefer living outside the campus than being separated and assigned to different dormitories/apartments.[1] Throughout this paper we assume that both merit scores and credit scores for each team are given to us as an input by the dormitory management of the university/college.

The data for our model includes two disjoint finite sets G and D, referred to as the set of teams and the set of dormitory-groups, respectively. Let $|G| = n$, $|D| = k$, $G = \{1, ..., n\}$ (the indices of the teams) and $D = \{1, ..., k\}$ (the indices of the dormitory-groups). Note that any student, which applies, belongs to a

[1] Note that if we just referred to the preferences of the members of a team over room-mates, the three-step process described here could be reduced to a two-step process by using the model presented in this paper, while eliminating the third step, and assigning teams directly to rooms rather than to dormitory-groups. In reality, the step of assigning students to rooms is based also on some other characteristics such as smoking, religions, gender, etc.

unique team $g \in G$, which is possibly a singleton. Each team $g \in G$ is associated with three non-negative numbers q_g, m_g and c_g: q_g is the number of students in team g, m_g is the merit score of team g, and c_g is the credit score of team g. The special case where $\sum_{g \in G} q_g = n$ refers to singleton teams, which is considered in [16]. Throughout we assume that: (1) each of the sequences $m_1, ..., m_n$ and $c_1, ..., c_n$ consists of distinct elements, i.e., for $g1 \neq g2$: $m_{g1} \neq m_{g2}$ and $c_{g1} \neq c_{g2}$, and (2) w.l.o.g, teams in G are indexed according to their credit scores from the highest to the lowest (i.e. $\forall g1, g2 \in G : g1 > g2 \Leftrightarrow c_{g1} < c_{g2}$). In addition, each team $g \in G$ is associated with a non-empty set $\emptyset \subset D_g \subseteq D$, and a ranking $\succ_g$ of $D_g \cup \{g\}$, where g is the least preferred element by team g. We refer to $\succ_g$ as the preference of team g over the set of dormitory-groups in D_g and over being unassigned to any dormitory-group in its preference-list, where the latter case is represented by g. We say that dormitory-group $d \in D$ is *acceptable* by team $g \in G$ if and only if $d \in D_g$. Similarly, $d \notin D_g$ means that team $g \in G$ finds dormitory-group $d \in D$ *unacceptable* and it prefers to be assigned to g, which means living off-campus, rather than living in dormitory-group d. Each dormitory-group is associated with a capacity, which is the number of beds that it offers. However, in order to avoid the case of having beds that no team is interested in, we consider the "effective capacity" b_d of each dormitory-group d, to be the minimum between the number of beds in d and $\sum_{\{g \in G | d \in D_g\}} q_g$, where the second term stands for the total number of students that accept dormitory-group d. For simplicity, in the sequel we assume that the effective capacity of a dormitory-group equals its given capacity. Thus, the total number of students that are interested in living in dormitories, under some personal preferences, is $\sum_{g \in G} q_g$, and the total number of beds in the dormitories is $\sum_{d \in D} b_d$, where according to our assumption, $k * \sum_{g \in G} q_g \geq \sum_{d \in D} b_d$. In addition, we assume that for any team $g \in G$ and any dormitory-group $d \in D$, $q_g > b_d$ implies that $d \notin D_g$. Therefore, for each team $g \in G$, since $D_g \neq \emptyset$, $q_g \leq \max_{d \in D} b_d$.

In this study we assume that the credit scores of the teams are common and complete, implying a common and complete preference-list of the dormitory-groups over the teams. The more general case where each dormitory-group has its own preference-list over the teams, is open for further research. The case with non-common preference-lists of the dormitory-groups where teams are singletons, is discussed in [17].

In what follows, the pair (G', D), where $G' \subseteq G$, is referred to as a *market*. Let $n(G')$ be the number of teams in G', i.e., $n(G') \leq n(G) = n$.

An *assignment* for a set $A \subseteq G$ over the dormitory-groups in D is a set of pairs, $\mu = \{(g, d) | g \in A, d \in D\}$, where for any $(g, d), (g', d') \in \mu : g \neq g'$, i.e., the teams in different pairs are distinct, $\sum_{g:(g,d) \in \mu} q_g \leq b_d$ for each $d \in D$, and $(g, d) \in \mu$ implies $d \in D_g$. μ can also be interpreted as a function $\mu(\cdot) : A \to D$ having $\mu(g) = d$ for $g \in A$. Under this interpretation, μ represents the assignment of teams in A to dormitory-groups, while the other teams are not assigned to any dormitory-group.

An *outcome* is a triplet (μ_A, W, R) where μ is the current assignment for a set $A \subseteq G$, and the pair

(W, R) partitions the teams that have not yet been assigned a dormitory-group, namely $G \backslash A$, into two disjoint sets: the teams in W are in a *waiting list*, while those in R are called *refugees*, as they are excluded from any further consideration. The difference between the sets W and R is that W contains the teams that are still waiting to be considered for an appropriate assignment, while R contains the teams that have already been considered but, unfortunately, no suitable dormitory-group has been found to host them. For simplicity, we omit the subscript A from the outcome triplet since clearly $A = G \backslash (W \cup R)$.

Next, we present a definition of plausibility that generalizes the corresponding definition in [16, 17], where all teams are singletons.

Definition 1 *An outcome (μ, W, R) is said to be plausible if:*

(a) $m_g < m_{g'}$ for each $g \in W$ and $g' \in G \backslash W$, and

(b) Either $W = \emptyset$ or $\sum_{d \in D} b_d - \sum_{(g,d) \in \mu} q_g < q_{\hat{g}}$, where $\hat{g}$ is the team with the highest merit score in W.

Condition (a) of Definition 1 asserts that the merit-score of each team in the waiting list is lower than the merit score of any team, which is not in the waiting-list (namely, a team which has either been assigned to a dormitory-group or has been excluded from further consideration). It follows immediately that if (μ, W, R) and (μ', W', R') are two outcomes that satisfy condition (a) in Definition 1, then W and W' are ordered by set-inclusion.

Condition (b) of Definition 1 asserts that either all teams are processed (meaning assigned to a dormitory-group or determined to be refugees), or the total number of empty beds is less than the size of the next-to-be-processed team in W, according to the merit score. In particular, by condition (b), if all teams are singletons, all dormitory-groups are at full capacity.

Recall that Definition 1 has not made any use of the credit scores, which next plays a central role in the determination of the final outcome.

Definition 2 *A pair $(g, d) \in G \times D$ is a blocking pair of an outcome (μ, W, R) if:*

(a) $g \in G \backslash W$ and

(b) $d \in D_g$ and

(c) $(g, d) \notin \mu$ and

(d) If $g \notin R$: $d >_g \mu(g)$ and

(e) $q_g + \sum_{(i,d)\in\mu} q_i - \sum_{i\in G'_g} q_i \le b_d$, *where* $G'_g \equiv \{i|(i,d) \in \mu, and, c_g > c_i\}$.

In other words, according to Definition 2, a blocking pair in a specific outcome consists of a team, which is not in the waiting list, and a dormitory-group, where the two are not assigned one to the other, though each of them prefers to be assigned to the other rather than their current state in the outcome.

Definition 3 *An outcome* (μ, W, R) *is called* *internally stable* *if it has no blocking pairs, and is called* *quasi-stable* *if it is plausible, see Definition 1, and internally stable.*

Note: If all teams consisted of the same number of students q, then we could scale down the the size of the teams and the capacity of each dormitory-group by q, to generate a model of single students, which is discussed in [16, 17]. Thus, in this research, we assume that the teams in G are of non-identical size.

3 A stable assignment for an exogenously given waiting list

This section presents some properties of internally stable outcomes, see Definition 3. Given a certain waiting list W, where $\emptyset \subseteq W \subset G$, we restrict ourselves to a market (G', D) for a subset $G' \subseteq G$ where $G' = G \backslash W$.

In the following, we claim that for any given G', an internally stable outcome, for market (G', D) exists, and it is unique. The proof starts by presenting a constructive algorithm. The algorithm scans the teams of G' according to their indices, and assigns each team to its most preferred dormitory-group that still has enough beds to accommodate it. If such a dormitory-group doesn't exist, the team is classified as a refugee.

The algorithm is using the following data-structure:

- g - the current team scanned by the algorithm.
- $\hat{\succ}$ - the preference-list of the current team g, which contains the dormitory-groups in D_g that have not yet rejected it.
- d - the current most preferred dormitory-group in $\hat{\succ}$.
- R - the current set of unassigned teams from G', that will eventually remain unassigned to any dormitory-group.
- μ - the current assignment of teams from $G' \backslash R$ to dormitory-groups.

Team Internally Stable Assignment (TISA) algorithm :

Input: Market (G', D) (assuming $G' \neq \emptyset$).

Initialization: Let μ be the empty assignment, $R = \emptyset$, and $g = 1$. All the beds in the dormitory-groups are free. In particular, the set of temporarily assigned teams to each dormitory-group is empty.

Begin:

- While $g < n + 1$ do the following steps:

  1. If $g \in G'$
     (a) $\hat{\succ} = \succ_g$.
     (b) Let d be the most preferred dormitory-group in $\hat{\succ}$.
     (c) If such d does not exist, then add team g to R.
     (d) Otherwise,
        i. If $b_d - \sum_{\{g' | (g', d) \in \mu\}} q_j < q_g$:
           A. Delete d from $\hat{\succ}$.
           B. Go to (1b).
        ii. Otherwise, insert (g, d) to μ.
  2. $g \leftarrow g + 1$.

  Endwhile.

- Output (μ, R).

End $\circ$

The output (μ, R) is returned where μ is the set of pairs $(g, d) \in G' \times D$ that are matched and $R \subseteq G'$ is the set of refugees that could not be assigned by the algorithm. Note that all the students in $G' \backslash R$ are assigned a dormitory-group under μ.

The following theorem provides a characterization of the outcome generated by the TISA algorithm:

Theorem 1 *For any market (G', D), $G' \subseteq G$, there exists a single internally stable outcome of the form $(\mu, G \backslash G', R)$.*

Proof: We start by showing that the TISA algorithm terminates, and produces an internally stable outcome of the form $(\mu, G\backslash G', R)$.

Termination is immediate as each team is scanned exactly once, and all preference-lists of the teams are finite. Let $(\mu, G\backslash G', R)$ be the outcome generated by the TISA algorithm on market (G', D). Acceptability of μ is also immediate since a team is assigned to a dormitory-group only if the dormitory-group is acceptable by the team and there are enough beds in the dormitory-group to accommodate the team. Now, assume by contradiction that (g, d) is a blocking pair in $(\mu, G\backslash G', R)$. Consider the time of addressing team g during the algorithm. At this time, dormitory-group d contains only teams with a higher credit-score than c_g. If at this time, there were enough beds to accommodate g in d, g would be assigned to d or to a dormitory-group that g prefers better than d. The contradiction follows since no team leaves a dormitory-group once it was assigned to it by the algorithm.

Next, we show that $(\mu, G\backslash G', R)$ is a unique internally stable outcome for market (G', D). This part is also proved by contradiction.

Given a market (G', D), assume that there were two different internally stable outcomes $(\mu_1, G\backslash G', R_1)$ and $(\mu_2, G\backslash G', R_2)$. Let $g^* \in G'$ be the team with the highest credit-score for which $\mu_1(g^*) \neq \mu_2(g^*)$. W.l.o.g., assume that g^* prefers the outcome under $(\mu_1, G\backslash G', R_1)$ (which implies $g^* \notin R_1$) than under $(\mu_2, G\backslash G', R_2)$, i.e.:

$$d_1 = \mu_1(g^*) \succ_{g^*} \mu_2(g^*) \quad \textit{or} \quad g^* \in R_2 \tag{1}$$

Consider the outcome $(\mu_2, G\backslash G', R_2)$: Let $\tilde{G}_{g^*} = \{i | \mu_2(i) = d_1 \text{ and } c_{g^*} < c_i\}$ be the set of teams that were assigned by μ_2 to d_1 and their credit scores are higher than the credit score of team g^*. As g^* is the highest credit-scored team whose assignment differs between $(\mu_1, G\backslash G', R_1)$ and $(\mu_2, G\backslash G', R_2)$, and since g^* is assigned to d_1 under μ_1, clearly,

$$q_{g^*} + \sum_{i \in \tilde{G}_{g^*}} q_i \leq b_{d_1}. \tag{2}$$

From (1) and (2) we conclude that (g^*, d_1) blocks $(\mu_2, G\backslash G', R_2)$, in contradiction to the assumption on the internal stability of $(\mu_2, G\backslash G', R_2)$. ∎

Comment 1: The complexity of the TISA algorithm is of order $o(n(G') * k)$ since all teams in G' are scanned exactly once, and for any team, its preference-list, which, in the worst case, might contain all dormitory-groups in D, are scanned.

4 Quasi-stability

In the previous section, the unique internally stable assignment under a certain market was determined. In this section, we consider the general model with merit scores, as discussed in section 2, and elaborate on some of its properties. Our first observation is that in the model considered here, unlike the model with singleton teams (see section 3 in [16] and [17]), there may exist two outcomes, $(\mu, W, R), (\mu', W', R')$, that fulfil condition (a) of plausibility, where (μ, W, R) is plausible, $W' \subset W$, but (μ', W', R') is not.

Example 1 *Let* $G = \{1, 2, 3\}$*,* $D = \{d\}$*,* $q_1 = q_2 = 1, q_3 = 2$ *and* $b_d = 2$*. Assume* d *is acceptable by* $1, 2$ *and* 3*,* $m_1 < m_2 < m_3$*, and recall that* $c_1 > c_2 > c_3$*. Consider the following outcomes:*

- (μ, W, R)*, where* $\mu = \{(3, d)\}$*,* $W = \{1, 2\}$ *and* $R = \emptyset$*.*
- (μ', W', R')*, where* $\mu' = \{(2, d)\}$*,* $W' = \{1\}$ *and* $R' = \{3\}$*.*

Clearly, (μ, W, R) *is a quasi-stable outcome, but* (μ', W', R') *is not, as it does not satisfy condition (b) of Definition 1, even though* $W' \subset W$*.* $\circ$

In the next sub-section, we propose an algorithm that finds all quasi-stable outcomes when the set of teams and the set of dormitory-groups are given. Then, in sub-sections 4.2 and 4.3, we concentrate on a specific quasi-stable outcome and discuss some of its properties.

4.1 Finding all quasi-stable outcomes

For the classic matching model of men and women (see [7]), a polynomial time algorithm that finds all stable matchings, is presented in [10]. The following algorithm finds all quasi-stable outcomes for a given set of teams G and a given set of dormitory-groups D. More specifically, the algorithm finds an internally stable outcome for any possible waiting-list that satisfies condition (a) of plausibility, by running the TISA algorithm, while filtering out outcomes that don't satisfy condition (b) of plausibility.

The algorithm uses the following data structure:

- $\hat{W}$ - a stack of teams ordered in a decreasing order of their merit scores. In each iteration, this stack holds the current teams in the waiting-list.[2]
- K - the set of quasi-stable outcomes that have already been found.

[2] Note that the teams in the stack $\hat{W}$, are ordered according to the merit-scores, while we indexed them according to their credit-scores.

- k - the index of the current quasi-stable outcome inserted to K.
- G' - is $G \backslash \hat{W}$.
- itr - iteration counter (which is equal to $|G \backslash \hat{W}|$).

Finding all Quasi-Stable Outcomes (QSO) algorithm

Input: A non-empty set of teams G, and a non-empty set of dormitory-groups D.

1. Let $g* \in G$ be the team with highest merit score.
2. Push to $\hat{W}$ the teams in $G \backslash \{g^*\}$ in a descending order of their merit scores.
3. Let $K = \emptyset$, $k = 1$, $G' = \{g^*\}$, and $itr = 1$.
4. While $itr \leq n$:
   (a) Apply the TISA algorithm on market (G', D). Denote the output of this run by (μ', R').
   (b) If condition (b) of plausibility holds for $(\mu', G \backslash G', R')$:
      i. Let $(\mu_k, W_k, R_k) = (\mu', G \backslash G', R')$, and add it to K.
      ii. $k \leftarrow k + 1$.
   (c) Pop the first team from $\hat{W}$ and insert it to G'.
   (d) $itr \leftarrow itr + 1$.

   Endwhile.
5. Return K. $\circ$

Comment 2: According to Comment 1, each run of the TISA algorithm on market (G', D) is of complexity $o(n(G') * k)$. The QSO algorithm on G and D, considers all possible waiting-lists that consist of a consecutive set of the lowest possible merit scores, and for which condition (a) of plausibility holds. There are $n(G)+1$ such possibilities. For each such waiting-list, the QSO algorithm generates a run of the TISA algorithm. Therefore, the complexity of the QSO algorithm is of order $o((n(G))^2 * k) = o(n^2 * k)$.

Comment 3: In order to simplify the presentation, we start the QSO algorithm when the set G' consists of the highest merit-scored team. However, the same output can be obtained by a more efficient algorithm, if we initialize G' to include the set of teams $\bar{G} \subseteq G$ with the highest merit scores such that $\sum_{g \in \bar{G}} q_g \leq \sum_{d \in D} b_d$, and $\hat{W} = G \backslash \bar{G}$.

We conclude this subsection by the following theorem:

Theorem 2 *The output K of the QSO algorithm is equal to the set of all quasi-stable outcomes for the set of teams G and the set of dormitory-groups D.*

Proof: Any element $(\mu_k, W_k, R_k) \in K$, produced by the QSO algorithm, is an outcome by definition. Any such outcome, is internally stable in view of Theorem 1. The plausibility of (μ_k, W_k, R_k) is due to the way of determining W_k by the QSO algorithm (see step 4b). The proof that K consists of all quasi-stable outcomes for the set of teams G and the set of dormitory-groups D, follows immediately from Theorem 1, and from the fact that the QSO algorithm considers all possible waiting-lists for this data. ∎

Note that in the last iteration of the QSO algorithm, the specific outcome, which is the result of running the TISA algorithm on market (G, D), whose waiting-list is empty, is added to K. Theorem 1 asserts that such an outcome exists, and is unique, and it satisfies both conditions of plausibility as its waiting-list is empty. The following observation is, thus, straightforward:

Observation 1 *For any set of teams G, and any set of dormitory-groups D, there exists at least one quasi-stable outcome, i.e., $|K| \geq 1$.*

4.2 Properties of quasi-stable outcomes

There are several optimality criteria that one can use in the evaluation of quasi-stable outcomes. For example, the one that has the least number of refugee teams, the one that has the least number of refugee students as individuals, the one where most teams (or students as individuals) get their most preferred assignment, the one with the least number of empty beds, etc. In this sub-section we consider some possible optimality criteria. We start by saying that scanning the list K that the QSO algorithm generates can serve in finding a quasi-stable outcome under any possible criterion.

In this sub-section we refer to the first outcome that is generated by the QSO algorithm, namely (μ_1, W_1, R_1), and elaborate on some of its properties. Recall that (μ_1, W_1, R_1) is the lowest indexed outcome in K.[3]

First, the following theorem shows that W_1 contains the largest number of teams (and, therefore, the largest number of students as individuals), and that R_1 contains the least number of students as individuals, among all quasi-stable outcomes. For this sake we introduce the function $s : 2^G \to N$, which returns the number of students in any subset G' of G.

Theorem 3 *Let K be the output of the QSO algorithm on the set of teams G and the set of dormitory-groups D, and suppose that $|K| > 1$. Consider two different outcomes in K, (μ_1, W_1, R_1) and (μ_*, W_*, R_*). Then*

1. $W_* \subset W_1$.
2. $s(W_*) < s(W_1)$.
3. $s(R_*) > s(R_1)$.

Proof: The proof of item 1 follows immediately from the way of constructing the elements of K by the QSO algorithm, implying also the proof of item 2. In order to prove item 3, let e_1 and e_* be the total number of unassigned beds in (μ_1, W_1, R_1) and (μ_*, W_*, R_*), respectively, i.e., $e_1 = \sum_{d\in D} b_d - \sum_{(g,d)\in\mu_1} q_g$ and $e_* = \sum_{d\in D} b_d - \sum_{(g,d)\in\mu_*} q_g$. Therefore, $s(R_1) = \sum_{g\in G\setminus W_1} q_g - (\sum_{d\in D} b_d - e_1)$, and $s(R_*) = \sum_{g\in G\setminus W_*} q_g - (\sum_{d\in D} b_d - e_*)$. By item 1, $W_* \subset W_1$, and therefore, $G\setminus W_1 \subset G\setminus W_*$. Let g^* be the team with the highest merit score in W_1. Clearly, $g^* \in G\setminus W_*$. Condition (b) of plausibility for (μ_1, W_1, R_1) asserts that $q_{g^*} > e_1$. Thus,

$$s(R_*) - s(R_1) = \sum_{g\in G\setminus W_*} q_g - \sum_{g\in G\setminus W_1} q_g + e_* - e_1 \geq q_{g^*} + e_* - e_1 \geq q_{g^*} - e_1 > 0 \tag{3}$$

concluding the proof. ■

Unfortunately, the following example shows that in terms of the number of teams, there may be markets where R_1 is not necessarily minimal among all quasi-stable outcomes.

Example 2 *Let $G = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $D = \{d1, d2, d3, d4\}$, $q_1 = q_3 = q_4 = q_5 = q_8 = q_9 = 1$, $q_2 = q_6 = q_7 = 2$ and $b_{d1} = b_{d2} = b_{d3} = b_{d4} = 2$. Recall that $c_1 > c_2 > c_3 > c_4 > c_5 > c_6 > c_7 > c_8 > c_9$, and let $m_9 > m_8 > m_7 > m_6 > m_3 > m_2 > m_4 > m_5 > m_1$. The following table presents the preference-lists of the teams:*

[3] In the case where all teams are of the same size, the outcome (μ_1, W_1, R_1) coincides with the outcome of Dor-AA in [16]. Therefore, in such a case, this outcome satisfies all optimality criteria stated above.

1	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
d1	*d1*	*d1*	*d2*	*d2*	*d3*	*d4*	*d3*	*d4*
	d2		*d3*	*d4*				

For this data there are two quasi-stable outcomes. The first outcome of the QSO algorithm, namely, (μ_1, W_1, R_1) *where* $\mu_1 = \{(2, d1), (4, d2), (5, d2), (6, d3), (7, d4)\}$*,* $W_1 = \{1\}$*, and* $R_1 = \{3, 8, 9\}$ *and* (μ_2, W_2, R_2)*, where* $\mu_2 = \{(1, d1), (2, d2), (3, d1), (4, d3), (5, d4), (8, d3), (9, d4)\}$*,* $W_2 = \emptyset$ *and* $R_2 = \{6, 7\}$*.*

In this example, in (μ_1, W_1, R_1) *there are three teams which are determined to be refugees, and in* (μ_2, W_2, R_2) *only two teams are determined to be refugees.* ∘

4.3 Optimality criteria

4.3.1 Optimality for teams and students

In this sub-section we propose the following definition of optimality for individual student and for teams:

Definition 4 *A quasi-stable outcome is said to be optimal for a certain student (team) if there is no other quasi-stable outcome in which the student (team) is assigned to a dormitory-group that he(it) better prefers.*[4]

The following example shows that there may exist a team set G and a dormitory-group set D, where any quasi-stable outcome is not optimal for at least one team in G, and particularly for all its students. Therefore, unlike the case where the teams are singletons (see [16, 17]), no algorithm can find an optimal outcome for all teams or students in G.

Example 3 *Let* $G = \{1, 2, 3\}$*,* $D = \{d1, d2\}$*,* $q_1 = 1, q_2 = 2, q_3 = 1$ *and* $b_{d1} = 1, b_{d2} = 2$*. Recall that* $c_1 > c_2 > c_3$*, and let* $m_3 > m_2 > m_1$*. The following table presents the preference-lists of the teams:*

1	*2*	*3*
d2	*d2*	*d2*
d1		*d1*

For this data there are two quasi-stable outcomes. The first outcome of the QSO algorithm, namely, (μ_1, W_1, R_1)*, where* $\mu_1 = \{(2, d2), (3, d1)\}$*,* $W_1 = \{1\}$ *and* $R_1 = \emptyset$*, and* (μ_2, W_2, R_2)*, where* $\mu_2 = \{(1, d2), (3, d2)\}$*,* $W_2 = \emptyset$ *and* $R_2 = \{2\}$*.*

In this example, optimality for team 2 *is reached under* (μ_1, W_1, R_1) *and optimality for team* 3 *is reached under* (μ_2, W_2, R_2)*.* ∘

[4] Definition 4 includes the preference of being assigned to a dormitory-group over being a refugee, as being a refugee is the least preferred option for any student or team.

Unfortunately, unlike the case where all teams are singletons (discussed in [16, 17]), the following example shows that there may exist a quasi stable outcome $(\mu_*, W_*, R_*) \neq (\mu_1, W_1, R_1)$, where most students, and also most teams in $G\backslash W_1$, prefer over outcome (μ_1, W_1, R_1).

Example 4 *Let* $G = \{1,2,3,4,5\}$, $D = \{d1, d2, d3, d4\}$, $q_2 = 2, q_1 = q_3 = q_4 = q_5 = 1$ *and* $b_{d1} = 2, b_{d2} = b_{d3} = b_{d4} = 1$. *Recall that* $c_1 > c_2 > c_3 > c_4 > c_5$, *and let* $m_2 > m_3 > m_4 > m_5 > m_1$. *The following table presents the preference-lists of the teams:*

1	*2*	*3*	*4*	*5*
d1	*d1*	*d1*	*d1*	*d1*
d2		*d2*	*d2*	*d2*
d3		*d3*	*d3*	*d3*
d4		*d4*	*d4*	*d4*

The first outcome is the outcome that the QSO algorithm generates, namely, (μ_1, W_1, R_1), *where* $\mu_1 = \{(2, d1), (3, d2), (4, d3), (5, d4)\}$, $W_1 = \{1\}$ *and* $R_1 = \emptyset$. *However, if the waiting-list is empty, the following quasi-stable outcome is generated:* (μ_2, W_2, R_2), *where* $\mu_2 = \{(1, d1), (3, d1), (4, d2), (5, d3)\}$, $W_2 = \emptyset$ *and* $R_2 = \{2\}$.

Here, the outcome (μ_2, W_2, R_2) *is strictly better than* (μ_1, W_1, R_1) *for teams* $3, 4, 5 \in G\backslash W_1$ *(and also for* $1 \in W_1$*), where for team* 2 *the opposite holds. As* $3 = q_3 + q_4 + q_5 > q_2 = 2$, *we conclude that the outcome* (μ_2, W_2, R_2) *is optimal for more students and more teams in* $G\backslash W_1$ *than is the case under* (μ_1, W_1, R_1). $\circ$

4.3.2 Optimality for a team with a complete preference-list

In the case of singletons teams, see [16], it is shown that a team who listed all dormitory-groups as acceptable cannot end up as a refugee team in (μ_1, W_1, R_1). However, this claim does not hold for the general case, as the following two examples show. The first example refers to a team of size 2, and the second refers to a singleton team.

Example 5 *Let* $G = \{1,2,3\}$, $D = \{d1, d2\}$, $q_1 = q_2 = 1, q_3 = 2$ *and* $b_{d1} = b_{d2} = 2$. *Recall that* $c_1 > c_2 > c_3$, *and let* $m_1 < m_2 < m_3$. *The following table presents the preference-lists of the teams:*

1	*2*	*3*
d1	*d2*	*d1*
d2	*d1*	*d2*

Consider the only quasi-stable outcome for this data, namely, (μ_1, W_1, R_1)*, where* $\mu_1 = \{(1, d1), (2, d2)\}$*,* $W_1 = \emptyset$*, and* $R_1 = \{3\}$*. Here, all dormitory-groups are acceptable by team* 3*, but* $3 \in R_1$*.* $\circ$

Example 6 *Let* $G = \{1, 2, 3, 4\}$*,* $D = \{d1, d2\}$*,* $q_2 = 2, q_1 = q_3 = q_4 = 1$ *and* $b_{d1} = b_{d2} = 2$*. Recall that* $c_1 > c_2 > c_3 > c_4$*, and let* $m_4 > m_3 > m_2 > m_1$*. The following table presents the preference-lists of the teams:*

1	*2*	*3*	*4*
d1	*d1*	*d1*	*d1*
d2	*d2*		*d2*

Consider the only quasi-stable outcome for this data, namely, (μ_1, W_1, R_1)*, where* $\mu_1 = \{(1, d1), (3, d1), (2, d2)\}$*,* $W_1 = \emptyset$ *and* $R_1 = \{4\}$*. Thus, all dormitory-groups are acceptable by team* 4*, but* $4 \in R_1$*.* $\circ$

Comment 4: Note that all the examples presented in this paper, refer to markets where the teams consist of two students and singletons only. Thus, the results of this paper continue to hold also for markets with teams of size larger than two.

5 Incentive compatibility

Incentive compatibility addresses the question of whether each member of a group of one or more agents can gain by misrepresenting his preference-list, while all agents outside this group state their true preference-lists. It is well known that in the classic stable matching model (see [7]), in the implementation of the men courting version of the GS algorithm, such a manipulation is not profitable, i.e., any subset of men, where each of its members presents a false preference-list, while all other men state their true preference-lists, contains at least one man, who prefers less his match when running the algorithm under this misrepresentation of preference-lists than his match when all agents state their true preference-lists (see [6, 27]). An immediate conclusion is that such a manipulation is not profitable when applying the GS algorithm in the one to many matching models, such as the assignment of students to schools, as any school with k positions can be presented as k schools of one position.

In this section we refer to the aforementioned results, and verify if a set of one or more teams can all gain by misrepresenting their preference-lists, while all other teams state their true preference-lists.

We first refer to the TISA algorithm for market (G, D). The following theorem demonstrates that under the TISA algorithm, a gain for all the teams in any subset of G is impossible.

Theorem 4 *Let $\succ_G = (\succ_1, ..., \succ_n)$ be the true preference-lists vector of the teams in market (G, D). Let $\tilde{\succ_G} = (\tilde{\succ_1}, ..., \tilde{\succ_n})$ be a vector of preference-lists that differs from $\succ_G$ in a minimal non-empty subset of teams $\emptyset \subset \hat{G} \subseteq G$. Finally, let (μ, R) and $(\tilde{\mu}, \tilde{R})$ be the internally stable assignments generated by application of the TISA algorithm on market (G, D) under $\succ_G$ and under $\tilde{\succ_G}$, respectively. Then, there exists a team $g \in \hat{G}$, for which $\mu(g) \succeq_g \tilde{\mu}(g)$, or, alternatively, $g \in R \cap \tilde{R}$.*

Proof: Let g be the team with the highest credit-score in $\hat{G}$, i.e., g is the lowest indexed team in $\hat{G}$. The order of scanning the teams while running the TISA algorithm on market (G, D), is independent of the preference-lists. Consider the point of time when g is considered by the TISA algorithm. The temporary assignments at this specific point of time, when running the TISA algorithm with $\succ_G$ and with $\tilde{\succ_G}$, coincide, as for all $g' < g$, $\succ_{g'} = \tilde{\succ_{g'}}$ holds. Thus, at this point of time, g is assigned to the best dormitory-group, according to its stated preference-list, as long as it has a sufficient number of beds to host it, implying that g cannot get a better assignment by misrepresenting its preference-list. Finally, since no team is being removed from a dormitory-group during the run of the TISA algorithm, $\mu(g) \succeq_g \tilde{\mu}(g)$, or, alternatively, g is a member of both R and $\tilde{R}$. ■

Theorem 4 refers to internally stable outcomes in market (G, D). However, an immediate conclusion from Theorem 4 and the definition of the QSO algorithm, is that manipulation is impossible for the last quasi-stable outcome of G and D, to be added to the set K. In fact, one can define an algorithm, which we denote by $|K|$-QSO, that generates this specific quasi-stable outcome. The algorithm consists of the final iteration of the QSO algorithm, when applied on the set of teams G and the set of dormitory-groups D, where the waiting list is empty, and all the teams in G are either assigned to a dormitory-group or are determined to be refugees. The $|K|$-QSO algorithm boils down to running the TISA algorithm on market (G, D), and therefore, under the $|K|$-QSO algorithm, a gain for all the teams in any subset of G, is impossible.

Next, we refer to the first outcome in the set K generated by the QSO algorithm, namely (μ_1, W_1, R_1). Consider the algorithm, which we denote by 1-QSO algorithm, that consists of the first iteration of the QSO algorithm.[5] For the singleton teams case, it is proved in [17], that when running the 1-QSO algorithm, no student can be assigned to a better dormitory-group, according to his preference-list, by misrepresenting his preferences, while all other students state their true preference-lists.[6] However, the following example shows that in the general case, a single team can be assigned by the 1-QSO algorithm to a more preferred dormitory-group if it misrepresents its preference-list.

[5]The 1-QSO algorithm can be efficiently implemented as a run of the QSO algorithm, which stops when $|K| = 1$.

[6]For the singleton teams case, the 1-QSO algorithm coincides with G-DorAA, which is presented in [17].

Example 7 *Consider example 4. The first outcome of the QSO algorithm is* (μ_1, W_1, R_1)*, where* $\mu_1 = \{(2, d1), (3, d2), (4, d3), (5, d4)\}$*,* $W_1 = \{1\}$*, and* $R_1 = \emptyset$*.*

Consider the modified preference-list of the teams which differs only for team 5:

1	*2*	*3*	*4*	*5*
d1	*d1*	*d1*	*d1*	*d1*
d2		*d2*	*d2*	*d2*
d3		*d3*	*d3*	*d3*
d4		*d4*	*d4*	

Now, the first outcome of the QSO algorithm is $(\tilde{\mu_1}, \tilde{W_1}, \tilde{R_1})$*, where* $\tilde{\mu_1} = \{(1, d1), (3, d1), (4, d2), (5, d3)\}$*,* $\tilde{W_1} = \emptyset$ *and* $\tilde{R_1} = \{2\}$*. Here, team 5 is assigned to* $d3$ *under* $(\tilde{\mu_1}, \tilde{W_1}, \tilde{R_1})$ *and to* $d4$*, which is less preferred than* $d3$*, by team 5, under* (μ_1, W_1, R_1)*.* $\circ$

Example 7 demonstrates that not only a team can get a better assignment for itself by misrepresenting its preference-list, but also the assignment may be better for other teams. Moreover, in this example, all students, except students in team 2, gain from this "lie", implying that most of the students in G gain from this "lie". Example 7 also shows that a "lie" of a single team can influence the size of the waiting-list of outcome (μ_1, W_1, R_1).

6 Conclusions and comments

In this paper we extend the stable matching model with entrance criterion to include applications of teams rather than individual students. The meaning of a team-application is that students in a team prefer living off-campus than being assigned separately to different dormitory-groups. We adjust the definition of quasi-stable outcomes, and present some properties of the new model along with algorithms that generate outcomes with certain desirable properties. In addition, we show that some of the properties that exist in the model for singleton teams, continue to hold for the generalized model. Finally, the existence of the incentive compatibility property in the model, is discussed.

The model presented here refers to a specific situation where the dormitory-groups' preferences, that are represented by the credit scores of the teams, are common and complete. Nevertheless, all results and their proofs continue to hold when removing the completeness requirement, i.e., negative credit-scores for some of the teams, is allowed. A negative credit score means that the respective team is not acceptable by any dormitory-group, as for example, if the team consists of both genders. In such a case, in all quasi-stable outcomes, the unacceptable teams will either be in the waiting-list or will end up as refugees.

In our following work, we intend to generalize the model by letting the dormitory-groups to rate single students rather than teams, and verify if a quasi-stable outcome, where all students from the same team are assigned together, exists.

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