Application of Path Integration to Three-Dimensional Harmonic Oscillator Using General Canonical Transformations

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Theme: Application of Path Integration to Three-dimensional Harmonic Oscillator using general canonical transformations

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Abstract

Usually general canonical transformations are applied to a system whose hamiltonien is time dependent, In our work, these transformations are used to derive the propagator of harmonic osillator. This is achieved after knowing of free particle propagator.
keywords: Path intergation - Harmonic oscillator - Free propagator - Canonical transformations

1. Introduction

It is well known that an exact and analytical solution of the Shrödinger equation can only be found for limited number of potentials. If these are, in addition, time- dependent, it is very rare to be able to find exact solution, as the Shrödinger equation becomes then a partial-derivative with two variables. For example, for the variable-frequency harmonic potential oscillator, an analytical solution is found through different methods, among them the invariant approach [1–3]. The variable-frequency or variable mass oscillator, as well as an infinite rectangular potential well of variable width, belong to the class of potentials which can be resolved via the generalized canonical transformations GCT method. These canonical transformations, followed by time transformation [4], are defined by:

$$\begin{aligned} \overrightarrow{X} &= \overrightarrow{Q}\rho(t/t_0) \\ \overrightarrow{P} &= \overrightarrow{p}/\rho(t/t_0) \end{aligned} \tag{1}$$

$$\frac{ds}{dt} = \frac{1}{\rho^2(t/t_0)}, \tag{2}$$

where $\rho(t/t_0)$ is an arbitrary dimensionless function, and t_0 is the time unit which is subsequently taken to be 1, Q and p are the new variables.

The subject of this work, in the first stage, is the study of certain time-dependent physical system [5] which admit, in general, neither invariant nor auxilliary equations. When the coordinates of such system are submitted to the space-time transformations [1] and [2], the system becomes equivalent to a variable-frequency harmonic oscillator.

Following the standard procedure by using the Hamiltonian formalism, that is known to be well adapted to canonical transformation, we establish the general relation which exists between the propagators when one changes the coordinate system via the GCT. This relation is valid for all time-dependent potentials.

In the second stage, we apply the result to find the propagator of harmonic oscillator with constant frequency. This preliminary and fundamental result can be used in other more complicated situations where the propagator was not tractable by usually methods [5–7].

2. Propagator for time-dependent potential

In the canonical formulation of the path integrals, the propagator is written formally, in standard notation, as follows:

$$K(\overrightarrow{X}_f, t_f/\overrightarrow{X}_i, t_i) = \int D[\overrightarrow{X}(t)]D[\overrightarrow{P}(t)] \exp\left\{ i\int_{t_i}^{t_f} (\overrightarrow{P}\dot{\overrightarrow{X}} - H(\overrightarrow{P}, \overrightarrow{X}, t))dt \right\} \tag{3}$$

where we have posed $\hbar = 1$. Via transformations (1) and (2), the Hamiltonian

$$H(\overrightarrow{P}, \overrightarrow{X}, t) = \frac{\overrightarrow{P}^2}{2m} + V(\overrightarrow{X}, t) \tag{4}$$

transforms to

$$H(\overrightarrow{p}, \overrightarrow{X}, t) = \frac{\overrightarrow{p}^2}{2m\rho^2(t)} - \frac{\overrightarrow{p}\,\overrightarrow{Q}\dot{\rho}(t)}{\rho(t)} + V(\overrightarrow{Q}\rho(t), t), \tag{5}$$

where the generating function responsible for this transformation is:

$$F_2(\overrightarrow{p}, \overrightarrow{X}, t) = \overrightarrow{p}\frac{\vec{X}}{\rho(t)}.$$

Moreover, the measure is transformed as:

$$D[\overrightarrow{X}(t)]D[\overrightarrow{P}(t)] = (\rho_i\rho_f)^{-3/2}D[\overrightarrow{Q}(t)]D[\overrightarrow{p}(t)]. \tag{6}$$

The evolution of the physical system is then described in the coordinates $(\overrightarrow{p}, \overrightarrow{Q}, t)$ by the propagator

$$\begin{aligned} K(\overrightarrow{Q}_f, t_f/\overrightarrow{Q}_i, t_i) &= (\rho_i\rho_f)^{-3/2}D[\overrightarrow{Q}(t)]D[\overrightarrow{p}(t)] \\ \exp\left\{ i\int_{t_i}^{t_f} \left[\overrightarrow{p}\dot{\overrightarrow{Q}} - \left(\frac{\overrightarrow{p}^2}{2m\rho^2(t)} - \frac{\overrightarrow{p}\,\overrightarrow{Q}\dot{\rho}(t)}{\rho(t)} + V(\overrightarrow{Q}\rho(t), t) \right) \right] dt \right\} \end{aligned} \tag{7}$$

and the use of (2) brings to a constant mass appearing in kinetic term of (7); the propagator can written as follows:

$$\begin{aligned} K(\overrightarrow{Q}_f, t_f/\overrightarrow{Q}_i, t_i) &= (\rho_i\rho_f)^{-3/2}D[\overrightarrow{Q}(s)]D[\overrightarrow{p}(s)] \\ \exp\left\{ i\int_{t_i}^{t_f} \left[\overrightarrow{p}\dot{\overrightarrow{Q}} - \left(\frac{\overrightarrow{p}^2}{2m} - \frac{\overrightarrow{p}\,\overrightarrow{Q}\dot{\nu}(s)}{\nu(s)} + \nu^2(s)V(\overrightarrow{Q}\nu(s), \int_s \nu(\sigma)d\sigma) \right) \right] ds \right\} \end{aligned} \tag{8}$$

where the derivative is relative to s and:

$$s_i = \int^{t_i} \frac{d\sigma}{\rho^2(\sigma)} \quad , \quad s_f = \int^{t_f} \frac{d\sigma}{\rho^2(\sigma)} \quad \text{and} \quad \nu(s) = \rho(t(s)). \tag{9}$$

The additional term in Eq.(9) can be eliminated by the following canonical transformation:

$$\overrightarrow{B} = \overrightarrow{p} - m\overrightarrow{Q}\frac{\dot{\nu}(s)}{\nu(s)} \quad \text{and} \quad \overrightarrow{Q} = \overrightarrow{Q} \tag{10}$$

with the associated generating function $F_2(\overrightarrow{B},\overrightarrow{Q},s)=\overrightarrow{B}\overrightarrow{Q}+m\overrightarrow{Q}^2\frac{\dot{\nu}}{2\nu}$ leads to a new Hamiltonian

$$h(\overrightarrow{B},\overrightarrow{Q},s)=\frac{\overrightarrow{B}^2}{2m}+\frac{m\bar{\omega}^2(s)\overrightarrow{Q}^2}{2}+\nu^2(s)V(\overrightarrow{Q}\nu(s),\int_s\nu(\sigma)d\sigma) \tag{11}$$

where

$$\bar{\omega}^2(s)=\frac{\bar{\nu}(s)}{\nu(s)}-2\left(\frac{\dot{\nu}(s)}{\nu}\right)^2=\rho^3(t)\ddot{\rho}(t). \tag{12}$$

Taking into account the invariance in this last canonical transformation (10), of the measure, the propagator (8) is written as:

$$\begin{aligned}K(\overrightarrow{Q}_f,t_f/\overrightarrow{Q}_i,t_i)=(\rho_f\rho_i)^{-3/2}\exp\left[\frac{im}{2}\left\{\frac{\dot{\nu}(s_f)}{\nu(s_f)}\overrightarrow{Q}_f^2-\frac{\dot{\nu}(s_i)}{\nu(s_i)}\overrightarrow{Q}_i^2\right\}\right]\int D[\overrightarrow{Q}(s)]D[\overrightarrow{B}(s)]\\ \exp\left\{i\int_{s_i}^{s_f}\left[\overrightarrow{B}\dot{\overrightarrow{Q}}-(\frac{\overrightarrow{B}^2}{2m}+\frac{m\bar{\omega}^2(s)\overrightarrow{Q}^2}{2}+\nu^2(s)V(\overrightarrow{Q}\nu(s),\int_s\nu^2(\sigma)d\sigma))\right]ds\right\}\end{aligned} \tag{13}$$

or, equivalently, by use of (1) and the relation: $\frac{\dot{\nu}(s)}{\nu(s)}=\rho(t)\dot{\rho}(t)$

$$\begin{aligned}K(\overrightarrow{X}_f,t_f/\overrightarrow{X}_i,t_i)=(\rho_f\rho_i)^{-3/2}\exp\left[\frac{im}{2}\left\{\frac{\dot{\rho}(T)}{\rho(T)}\overrightarrow{X}_f^2-\frac{\dot{\rho}(0)}{\rho(0)}\overrightarrow{X}_i^2\right\}\right]\int D[\overrightarrow{Q}(s)]D[\overrightarrow{B}(s)]\\ \exp\left\{i\int_{s_i}^{s_f}\left[\overrightarrow{B}\dot{\overrightarrow{Q}}-(\frac{\overrightarrow{B}^2}{2m}+\frac{m\bar{\omega}^2(s)\overrightarrow{Q}^2}{2}+\nu^2(s)V(\overrightarrow{Q}\nu(s),\int_s\nu^2(\sigma)d\sigma))\right]ds\right\}\end{aligned} \tag{14}$$

3. Application to harmonic oscillator

In the case of harmonic oscillator the potential has an independent time form:

$$V(\overrightarrow{X},t)=\frac{m}{2}\dot{\overrightarrow{X}}^2.$$

After substituting this potential in (14) and putting zero the factor beside a quadratic term using (12), we obtain

$$\rho^3(t)\ddot{\rho}(t)+\rho^4(t)=0$$

whose general solutions is: $\rho(t)=a\cos(t)+b\sin(t)$.

Finallay, we are dealing with propagator of free particle:

$$\begin{aligned}K(\overrightarrow{X}_f,t_f/\overrightarrow{X}_i,t_i)=(\rho_f\rho_i)^{-3/2}\exp\left[\frac{im}{2}\left\{\frac{\dot{\rho}(T)}{\rho(T)}\overrightarrow{X}_f^2-\frac{\dot{\rho}(0)}{\rho(0)}\overrightarrow{X}_i^2\right\}\right]\int D[\overrightarrow{Q}(s)]D[\overrightarrow{B}(s)]\\ \exp\left\{i\int_{s_i}^{s_f}\left[\overrightarrow{B}\dot{\overrightarrow{Q}}-\frac{\overrightarrow{B}^2}{2m}\right]ds\right\}=(\rho_f\rho_i)^{-3/2}\exp\left[\frac{im}{2}\left\{\frac{\dot{\rho}(T)}{\rho(T)}\overrightarrow{X}_f^2-\frac{\dot{\rho}(0)}{\rho(0)}\overrightarrow{X}_i^2\right\}\right]K_{\text{free}}(\overrightarrow{X}_f,t_f/\overrightarrow{X}_i,t_i)\end{aligned} \tag{15}$$

Knowing that:

$$s_f-s_i=\Delta s=\int_0^T\frac{d\sigma}{\rho^2(\sigma)}=\int_0^T\frac{d\sigma}{(a\cos(\sigma)+b\sin(\sigma))^2}=\frac{\sin(T)}{a\rho(T)} \tag{16}$$

and the propagator of free particle:

$$K_{\text{free}}(\overrightarrow{Q}_f,s_f/\overrightarrow{Q}_i,s_i)=\left(\frac{m}{2\pi i(\Delta s)}\right)^{3/2}\exp\frac{im}{2(\Delta s)}(\overrightarrow{Q}_f-\overrightarrow{Q}_i)^2, \tag{17}$$

we find immediately after subtitution of (16) and (17) in (15) the propagator of the harmonic oscillator [6]:

$$K(\overrightarrow{X}_f, t_f/\overrightarrow{X}_i, t_i) = \left(\frac{m}{2\pi i \sin(T)}\right)^{3/2} \exp \frac{im}{2\sin(T)} \left\{(\overrightarrow{X}_f^2 + \overrightarrow{X}_i^2)\cos(T) - 2\overrightarrow{X}_f \overrightarrow{X}_i\right\} \tag{18}$$

4. Conclusion

In this work, we have presented the method of general canonical transformations (GCT) applied to harmonic oscillator moves. Thanks to this method, we have calculated the propagator of three-dimensional harmonic oscillator via the one of free particle. Of course, the generalization to D-dimensional case follows the same scheme. This method can, undoubtedly, be applied to other problems in a large area of physics.

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