A Generalized Method to Find the Square Root of Matrix Whose Characteristic Equation Is Quadratic

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Abstract
In this paper, we generalized the method to calculating the square root of matrix whose characteristic is quadratic and how to Cayley-Hamilton theorem may be used to determine the formula for all square root of matrix whose order is 2×2.

Keywords : Eigen values, Matrix equation, Square root of matrix.

1. Introduction
Let \( M_n(C) \) be the set of all complex matrices whose order is \( n \times n \). Matrix \( Q \) is said to be a square root of matrix \( P \), if the matrix product \( Q^2 = P \).

Now, what is the square root of matrix such as \[ \begin{bmatrix} p & q \\ r & s \end{bmatrix} \]. It is not, in general \[ \begin{bmatrix} \sqrt{p} & \sqrt{q} \\ \sqrt{r} & \sqrt{s} \end{bmatrix} \]

It is easy to see that the upper left entry of its square is \( p + \sqrt{q} \) and not \( p \).

In recent years, several article have been written about the root of a matrix, and one can refer to [4-6]. A number of method have been proposed to computing the square root of matrix and these are usually based on Newton’s method, either directly or the sign function.(see e.g., [1-3]).

2. Generalized Method
The set of all matrices which their square is \( P \), denoted by \( \sqrt{P} \), i.e., \[ \sqrt{P} = \{ Y : Y \in M_n(C), Y^2 = P \} \]

This set can be very large. For example, we will see that \( \sqrt{I} \) has infinite members. We can define the \( n \)-th root of a matrix \( P \) as follows.

\[ \sqrt{P} = \{ Y : Y \in M_n(C), Y^n = P \} \]

It is well known to all, if \( P = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \), then characteristic equation is

\[ \lambda^2 - (\text{Trace } P) \lambda + \text{det } P = 0 \] (1)

Apply cayley-Hamilton theorem, putting \( \lambda = P \), then equation (1) is

\[ P^2 - (\text{Trace } P)P + (\text{det } P)I = 0 \]

Thus, we have

\[ P^2 = (\text{Trace } P)P - (\text{det } P)I \] (2)

Putting \( , P^2 = Q \), then equation (2) is

\[ Q = (\text{Trace } P)P - (\text{det } P)I \]

\[ Q + (\text{det } P)I = (\text{Trace } P)P \]

\[ \frac{1}{(\text{Trace } P)}(Q + (\text{det } P)I) = P \] (3)

Lemma 2.1. Let \( P \) be a 2×2 matrix. Then \( \text{trace } P^2 = (\text{trace } P)^2 - 2 \text{ det } P \).

Proof: Suppose \( \lambda_1 \) and \( \lambda_2 \) are the two eigen values of the matrix \( P \). Then we can easy to see that \( \lambda_1^2 \) and \( \lambda_2^2 \) are the eigen values of \( P^2 \). We know that , \( \text{trace } P = \lambda_1 + \lambda_2 \) and \( \text{det } P = \lambda_1 \lambda_2 \).

Then, \( \text{trace } P^2 = \lambda_1^2 + \lambda_2^2 \)

\[ = (\lambda_1 + \lambda_2)^2 - 2\lambda_1 \lambda_2 \]

\[ = (\text{trace } P)^2 - 2 \text{ det } P \]

Second proof: In other words, let \( P = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \)

Then,

\[ P^2 = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} \]

\[ = \begin{bmatrix} p^2 + rq & pq + qs \\ pr + rs & s^2 + rq \end{bmatrix} \]

Therefore,

\[ \text{Trace } P^2 = (p^2 + rq) + (s^2 + rq) \]

\[ \text{Trace } P^2 = p^2 + s^2 + 2rq \]

\[ \text{Trace } P^2 = p^2 + s^2 + 2ps - 2ps + 2rq \]

\[ \text{Trace } P^2 = (p + s)^2 - 2(ps - qr) \] (1)

But, \( \text{trace } P = p + s \) and \( \text{det } P = ps - qr \), then equation (1),

\[ \text{Trace } P \text{Let } P,Q \in M_n(C) \]

\[ = (\text{trace } P)^2 - 2 \text{ det } P \]
Remark 1. Let $P, Q \in M_2(C)$ and $P^2 = Q$. Then the following statements are holds:

(1) $\det P = \sqrt{\det Q}$

(2) $\text{trace } P = \sqrt{\text{trace } Q + 2\sqrt{\det Q}}$

Example 2.1 Let $Q = \begin{bmatrix} 8 & 5 \\ 3 & 8 \end{bmatrix}$. So $\det Q = 64-15 = 49$, and $\text{trace } Q = 8 + 16 = 24$, therefore if $P^2 = Q$ then,

$\det P = \sqrt{\det Q} = \sqrt{49} = \pm 7$, and

$\text{trace } P = \sqrt{\text{trace } Q + 2\sqrt{\det Q}} = \sqrt{16 + 2\sqrt{49}} = \sqrt{16 \pm 14}$, taking positive and negative sign then,

$\text{trace } P = \pm \sqrt{30}$ or $\text{trace } P = \pm \sqrt{2}$, thus, from equation (3),

$P = \frac{1}{\sqrt{\text{trace } P}} \left[ Q + (\text{det } P)I \right],

P = \frac{1}{\sqrt{30}} \begin{bmatrix} 8 & 5 \\ 3 & 8 \end{bmatrix} + \left( \pm 1 \right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Therefore,

$P = \frac{1}{\sqrt{30}} \begin{bmatrix} 8 & 5 \\ 3 & 8 \end{bmatrix} + (7) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $P = \frac{1}{\sqrt{30}} \begin{bmatrix} 8 & 5 \\ 3 & 8 \end{bmatrix} + (-7) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and

$P = \frac{1}{\sqrt{2}} \begin{bmatrix} 15 & 5 \\ 3 & 15 \end{bmatrix}$ or $P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $P = \pm \frac{1}{\sqrt{2}} \begin{bmatrix} 15 & 5 \\ 3 & 15 \end{bmatrix}$ or $P = \pm \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Lemma 2.2 Let $P \in M_2(C)$. If trace $P = 0$, then $P^2 \in \{I\}$. Proof. We will prove this lemma in two ways. In general, we have $P^2 - (\text{Trace } P)P + (\text{det } P)I = 0$ .................(1)

Therefore, if trace $P = 0$, then from (1) we obtain.

$P^2 + (\text{det } P)I = 0$

$P^2 = -(\text{det } P)I$ and $P^2 \in \{I\}$

Second proof: Let $P = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$, and $p + s = 0$

Then,

$p^2 = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} p^2 + qr & pq + qs \\ pr + rs & s^2 + rq \end{bmatrix}$

Putting $p = -s$, then

$p^2 = \begin{bmatrix} p^2 + qr & 0 \\ 0 & s^2 + rq \end{bmatrix}$

Hence, when $p^2 = s^2$ then $P^2 = (p^2 + qr)$.

Example 2.2 Let $Q = \begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix}$. Then $\det Q = 2 - 6 = -4$, and trace $Q = 1 + 2 = 3$. If $P^2 = Q$ then

$\det P = \sqrt{\det Q} = \sqrt{-4} = 2i$, and

$\text{trace } P = \sqrt{\text{trace } Q + 2\sqrt{\det Q}}

= \sqrt{3 + 2\sqrt{4}}

= \sqrt{3 + 4i}$.

Now,

$P = \frac{1}{\sqrt{\text{trace } P}} \left[ Q + (\text{det } P)I \right],

P = \frac{1}{\sqrt{2\sqrt{13} + 4i}} \begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix} + 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}

P = \frac{1}{\sqrt{2\sqrt{13} + 4i}} \left[ \begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix} + 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right]$.

Lemma 2.3 For each $\beta \in C$ and any matrix $P$, $\sqrt{\beta P} = \sqrt{\beta} \sqrt{P}$. Proof: Suppose that $\beta \neq 0$ and $Y \in \sqrt{\beta P}$. So $Y^2 \in \beta P$, hence $\frac{1}{\beta} Y \in \sqrt{P}$ which implies that $Y \in \sqrt{\beta} \sqrt{P}$.

Conversely, if $Y \in \sqrt{\beta P}$, then $\frac{1}{\sqrt{\beta}} Y = P$. Hence $Y^2 = \beta P$ and $Y \in \sqrt{\beta P}$.

Now, we try to compute $\sqrt{I}$. Suppose that $P \in M_2(C)$ and $P^2 = I$. Let $P = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$.

Then,

$p^2 = \begin{bmatrix} p^2 + qr & pq + qs \\ pr + rs & s^2 + rq \end{bmatrix}$, but $P^2 = I$, then

$I = \begin{bmatrix} p^2 + qr & pq + qs \\ pr + rs & s^2 + rq \end{bmatrix}$

Hence we have,

$p^2 + qr = 1$ .................(1)

$pq + qs = 0$ .................(2)

$pr + rs = 0$ .................(3)

$s^2 + rq = 1$ .................(4)
From (2) and (3), \( q = 0 \) or \( p + s = 0 \) and \( r = 0 \) or \( p + s = 0 \). We consider two cases:

1. If \( p + s = 0 \), then equation (2) and (3) hold. We have \( p^2 + r = 1 \) or \( p = \sqrt{1 - \frac{r}{q}} \). Therefore
   \[
   p = \left\{ \begin{array}{cc}
   \sqrt{1 - \frac{r}{q}} & 0 \\
   0 & -\sqrt{1 - \frac{r}{q}}
   \end{array} \right\}; b, c \in C.
   \]

2. If \( p + s \neq 0 \) we must have \( q = 0 \) and \( r = 0 \). Hence \( p = \pm \sqrt{1} \) and \( s = \pm \sqrt{1} \). Therefore there are two solutions \( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \) and \( \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \). Hence we can write
   \[
   \sqrt{I} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\} \cup \left\{ \begin{array}{cc}
   \sqrt{1 - \frac{r}{q}} & 0 \\
   0 & -\sqrt{1 - \frac{r}{q}}
   \end{array} \right\}; b, c \in C
   \]

Example 2.3 Let \( Q = \begin{bmatrix} 16 & 0 \\ 0 & 16 \end{bmatrix} \). Therefore \( Q = 16 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 16 I \). Then \( \sqrt{Q} = 4\sqrt{I} \), hence we have
   \[
   \sqrt{I} = \left\{ \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \right\} \cup \left\{ \begin{array}{cc}
   4\sqrt{1 - \frac{r}{q}} & 2q \\
   2r & -4\sqrt{1 - \frac{r}{q}}
   \end{array} \right\}; b, c \in C
   \]

References


