On Epi-Artinian Rings and Modules

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Abstract

An $R$ module $M$ is said to be epi-Artinian if for every descending chain $M_1 \geq M_2 \geq \ldots$ of submodules of $M$, there exists an index $n$ such that $M_{i+1}$ is homomorphic image of $M_i$, $\forall i \geq n$. In this paper, we discuss some properties of epi-Artinian rings and modules. We characterize epi-Artinian modules with iso-Artinian modules. We also discuss some properties of iso-Artinian rings and modules.

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1 Introduction and Preliminaries

In this paper, all rings are associative with unit element and all modules are unitary right modules.

In [7], Facchini et. al. defined the notion of iso-Noetherian and iso-Artinian modules. According to them, a module $M$ iso-Noetherian (iso-Artinian ) if for every ascending (descending) chain $M_1 \leq M_2 \leq M_3 \leq \ldots$ ($M_1 \geq M_2 \geq M_3 \geq \ldots$) of submodules of $M$, there exists an index $n$ such that $M_n$ is isomorphic to $M_i$ for every $i \geq n$. A ring $R$ is said to be right iso-Noetherian (right iso-Artinian) if the right module $R_R$ is iso-Noetherian (iso-Artinian).

In [12], we define epi-Artinian module. We say that a modules $M$ is epi-Artinian if for every descending chain $M_1 \geq M_2 \geq M_3 \geq \ldots$ of submodules of $M$, there exists an index $n$ such that $M_{i+1}$ is a homomorphic image of $M_i$ for every $i \geq n$. In [5], the authors call epi-dcc this chain condition. We say that a ring $R$ is right epi-Artinian if the right module $R_R$ is epi-Artinian.

Every iso-Artinian module is epi-Artinian, but epi-Artinian modules need not be iso-Artinian. We give examples of an epi-Artinian module which is not iso-Artinian. We provide sufficient conditions for epi-Artinian modules to be iso-Artinian. If $R$ is an integral domain, then $R_R$ is iso-Artinian if and only if $R_R$ is epi-Artinian.
2 Epi-Artinian and Iso-Artinian Rings and Modules

We begin this section with the following lemma.

Lemma 2.1. Let $R$ be an iso-Artinian ring. Then $R$ contains a uniform ideal.

Proof. If $R$ is uniform, then we are done. If not, $R$ contains a direct sum of nonzero ideals, say $R = I_0 \supset I_1 \oplus I_1'$. If either of $I_1, I_1'$ is uniform, then we are done. If not, repeating this argument for $I_1$, we get $I_1, I_2, I_2'$ such that $I_1 \supset I_2 \oplus I_2'$. If either of $I_2, I_2'$ is uniform, then we are done. If not, repeating the process for $I_3, I_4, \ldots$, we get a direct sum $I_1' \oplus I_2' \oplus I_3' \oplus \ldots$. The finite uniform dimension shows that this process must stop at a finite step $k$. At this stage the ideal $I_k$ is uniform. \hfill \Box

In the following lemma, we we discuss structure of essential right ideal of a right iso-Artinian ring in terms of uniform right ideals.

Proposition 2.2. If $R$ is a right iso-Artinian ring. Then $R$ contains an essential right ideal which is a finite direct sum of uniform right ideal.

Proof. Let $I' = \oplus_{i=1}^n U_i$ be a direct sum of uniform right ideals $U_i$ of $R$. Suppose that $I'$ is not essential in $R$. Then there exists a nonzero right ideal $J$ of $R$ such that $I' \cap J = 0$. By lemma 2.1, $J$ contains a uniform right ideal, say $U_{n+1}$ and $R \supset I' \oplus U_{n+1} = \oplus_{i=1}^{n+1} U_i$. If $\oplus_{i=1}^{n+1} U_i$ is not essential in $R$, then repeating this process, either we get an essential submodule or else an infinite direct sum, which is not possible because $R$ has finite uniform dimension. \hfill \Box

Remark 2.3. [12, Remark 3.8] Every iso-Artinian module is epi-Artinian. But, in general, epi-Artinian modules need not be iso-Artinian. For example, let $M = \oplus_{p \in \mathbb{P}} \mathbb{Z}_p$, where $\mathbb{P}$ be the set of all prime integers. Then $M$ is epi-Artinian, but not iso-Artinian. In the following, we provide sufficient conditions for epi-Artinian modules to be iso-Artinian.

Proposition 2.4. Let $M$ be a uniform torsion free $R$-module. Then $M$ is epi-Artinian if and only if $M$ is iso-Artinian.

Proof. Let $M_1 \geq M_2 \geq M_3 \geq \ldots$ be a descending chain of submodules of $M$. Since $M$ is epi-Artinian, there exists an index $n$ such that $M_{i+1}$ is homomorphic image of $M_i$, for all $i \geq n$. Let $f : M_i \to M_{i+1}$ be epimorphism then $f$ is an endomorphism of $M_n$. Since $M_n$ is uniform and torsion free hence satisfies (*)-property. Thus $f$ is monomorphism. Therefore $f$ is isomorphism. \hfill \Box

Proposition 2.5. Let $R$ be a right epi-artinian ring. If every nonzero right ideal of $R$ contains a right regular element, then $R$ is right Noetherian.
**Proof.** It is sufficient to show that every right ideal of $R$ is finitely generated. On contrary, let $I$ be a non finitely generated right ideal of $R$. Let $x \in I$ be a right regular element. Then $R \cong xR$ and $xR$ contains a right ideal $xI$, which is isomorphic to $I$ as a right $R$-module. Thus we construct a descending chain $R \supseteq I \supseteq xR \supseteq x^2R \supseteq x^3I \supseteq \ldots$, where $x^nI \cong I$ is not finitely generated and $x^nR \cong R$ is finitely generated. Since $R$ is epi-artinian, there exists an index $m$ such that $x^mI$ is epimorphic image of $x^iR$, for all $i \geq m$. This shows that $x^mI \cong I$ is finitely generated, a contradiction. Thus $I$ is finitely generated. Therefore $R$ is right Noetherian.

Recall by [2] that an $R$-module $M$ is virtually semisimple if every submodule of $M$ is isomorphic to a direct summand of $M$. If every submodule of $M$ is virtually semisimple then $M$ is said to be completely virtually semisimple.

**Lemma 2.6.** Let $R$ be a semiprime iso-Artinian ring. Then every projective $R$-module $M$ is completely virtually semisimple.

**Proof.** Since $R$ is semiprime iso-Artinian ring. Therefore $R$ is direct sum of iso-retractable ideals, by [12, Proposition 2.7]. Now by [2, Theorem 3.11], $R$ is a left completely virtually semisimple ring. Thus by [2, Proposition 3.3], every projective left $R$-module is completely virtually semisimple. 

We know that every right Artinian ring is right Noetherian. In the following, we show that under semiprimeness condition iso-Artinian ring becomes right Noetherian.

**Corollary 2.7.** Let $R$ be a semiprime iso-Artinian ring. Then $R$ is a right Noetherian ring.

**Proof.** By Lemma 2.6, $R$ is completely virtually semisimple and $u.dim(R) < \infty$. Thus [2, Proposition 2.8] implies that $R$ is right Noetherian ring.

**Proposition 2.8.** Let $R$ be a semiprime iso-Artinian hereditary ring. Then every finitely generated projective $R$-module is iso-Artinian.

**Proof.** By Lemma 2.6.

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References


