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THE DISPERSIONLESS INTEGRABLE SYSTEMS AND RELATED CONFORMAL STRUCTURE GENERATING EQUATIONS OF MATHEMATICAL PHYSICS

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Abstract. Based on the vector fields on the complexified torus and the related Lie-algebraic structures, we devise an approach to constructing multidimensional dispersionless integrable systems, describing conformal structure generating equations of mathematical physics. As examples, we have analyzed Einstein–Weyl metric equation, the modified Einstein–Weyl metric equation, the Dunajski heavenly equations, first and second conformal structure generating equations, inverse first Shabat reduction heavenly equation, first Plebański heavenly equation, modified Plebański equation and

Husain heavenly equation.

1. Vector fields on the complexified torus $\mathbb{T}^n_{\mathbb{C}}$ and the related Lie-algebraic properties

It is well known [13] that the loop Lie algebra $\tilde{\mathcal{G}} := \widetilde{diff}(\mathbb{T}^n)$, consisting of the set of smooth mappings $\{\mathbb{C}^1 \supset \mathbb{S}^1 \longrightarrow \mathcal{G} = diff(\mathbb{T}^n\}$, extended, respectively, holomorphically from the circle $\mathbb{S}^1 \subset \mathbb{C}^1$ on the disc $\mathbb{D}^1_+$ of the internal points $\lambda \in \mathbb{D}^1$ and on the disc $\mathbb{D}^1_-$ of the external points $\lambda \in \mathbb{C}\backslash\mathbb{D}^1$, can be centrally extended as $\widehat{\mathcal{G}} := (\widetilde{diff}(\mathbb{T}^n); \mathbb{R}^1)$, where for elements $(\tilde{a};\alpha)$ and $(\tilde{b};\beta) \in \widehat{\mathcal{G}}$ the commutator

$$[(\tilde{a};\alpha),(\tilde{b};\beta)] = ([\tilde{a},\tilde{b}];\omega_2(\tilde{a},\tilde{b})) \in \tilde{\mathcal{G}} \tag{1.1}$$

and the 2-cocycle $\omega_2 : \tilde{\mathcal{G}} \times \tilde{\mathcal{G}} \to \mathbb{R}^1$ satisfies the condition

$$\omega_2([\tilde{a},\tilde{b}],\tilde{c}) + \omega_2([\tilde{b},\tilde{c}],\tilde{a}) + \omega_2([\tilde{c},\tilde{a}],\tilde{b}) = 0 \tag{1.2}$$

for any $\tilde{a},\tilde{b}$ and $\tilde{c} \in \tilde{\mathcal{G}}$. For arbitrary $n \in \mathbb{Z}_+$ the cocycle $\omega_2 : \tilde{\mathcal{G}} \times \tilde{\mathcal{G}} \to \mathbb{R}^1$ can be taken in the unique Cartan-Maurer form

$$\omega_2(\tilde{a},\tilde{b}) = \underset{\lambda\in\mathbb{C}}{res} \int_{\mathbb{T}^n\times\mathbb{S}^1} (< a(x,y;\lambda), \partial b(x,y;\lambda) > dxdy, \tag{1.3}$$

where have denoted by $< \cdot, >$ the standard scalar product in the Euclidean space $\mathbb{E}^n$ and parametrized the Lie algebra $\tilde{\mathcal{G}} = \widetilde{diff}(\mathbb{T}^n)$ by means of an additional spatial parameter $y \in \mathbb{S}^1$. For the case $n = 1$ the cocycle (1.3) above can be extended by means the Gelfand–Fuchs 2-cocycle [6]

$$\tilde{\omega}_2(\tilde{a},\tilde{b}) = \underset{\lambda\in\mathbb{C}}{res} \int_{\mathbb{T}^1\times\mathbb{S}^1} \lambda^{-p}\frac{\partial^2 a(x;\lambda)}{\partial x^2}\frac{\partial b(x;\lambda)}{\partial x}dxdy \tag{1.4}$$

for any vector fields $\tilde{a} = a(x,y;\lambda)\frac{\partial}{\partial x}, \tilde{b} = b(x,y;\lambda)\frac{\partial}{\partial x} \in \tilde{\mathcal{G}}$ on $\mathbb{T}^1$, parameterized by means of the spatial parameter $y \in \mathbb{S}^1$ and a fixed integer $p \in \mathbb{Z}$.

Yet, the scheme, based on the central extension technique, does not allow [7] to construct effectively commuting to each other spatially multidimensional

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linear differential expressions and, thereby, generate completely integrable nonlinear equations in partial derivatives. Taking into account this fact, we will describe below a direct scheme of describing infinite hierarchies of commuting to each other spatially multidimensional linear vector field equations, generating completely integrable nonlinear Hamiltonian systems on functional manifolds, many of which are important for applications in modern mathematical physics.

2. The Lie-algebraic structures and integrable Hamiltonian systems

The integrable dynamical systems related to the central extension, mentioned above, were described in detail in [9]. Concerning a further generalization of the multi-dimensional case related to the loop group $\tilde{\mathcal{G}}$ for $n \in \mathbb{Z}_+$ one can proceed in the following [7] natural way: as the Lie algebra $\tilde{\mathcal{G}} = \widetilde{diff}(\mathbb{T}^n)$ consists of the elements, depending additionally on the "*spectral*" variable $\lambda \in \mathbb{C}^1$, one can extend the basic Lie structure on $\tilde{\mathcal{G}} = \widetilde{diff}(\mathbb{T}^n)$ to the generalized Lie algebra $\bar{\mathcal{G}} := diff_{hol}(\mathbb{T}^n_{\mathbb{C}})$ of vector fields on the complexified torus $\mathbb{T}^n_{\mathbb{C}}$. This Lie algebra has elements representable as $\bar{a}(x;\lambda) :=< a(x;\lambda), \frac{\partial}{\partial \mathrm{x}} >= \sum_{j=1}^{n} a_j(x;\lambda)\frac{\partial}{\partial x_j} + a_0(x;\lambda)\frac{\partial}{\partial \lambda} \in \bar{\mathcal{G}}$ for some holomorphic in $\lambda \in \mathbb{D}^1_{\pm}$ vectors $a(x;\lambda) \in \mathbb{E}\times\mathbb{E}^n$ for all $x \in \mathbb{T}^n$, where $\frac{\partial}{\partial \mathrm{x}} := (\frac{\partial}{\partial \lambda}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n})^{\intercal}$ is the generalized Euclidean vector gradient with respect to the vector variable $\mathrm{x} := (\lambda, x) \in \mathbb{T}^n_{\mathbb{C}}$.

It is now important to mention that the Lie algebra $\bar{\mathcal{G}}$ also naturally splits into the direct sum of two subalgebras:

$$\bar{\mathcal{G}} = \bar{\mathcal{G}}_+ \oplus \bar{\mathcal{G}}_-, \tag{2.1}$$

allowing to introduce on it the classical $\mathcal{R}$-structure:

$$[\bar{a}, \bar{b}]_{\mathcal{R}} := [\mathcal{R}\bar{a}, \bar{b}] + [\bar{a}, \mathcal{R}\bar{b}] \tag{2.2}$$

for any $\bar{a}, \bar{b} \in \bar{\mathcal{G}}$, where

$$\mathcal{R} := (P_+ - P_-)/2, \tag{2.3}$$

and

$$P_{\pm}\bar{\mathcal{G}} := \bar{\mathcal{G}}_{\pm} \subset \bar{\mathcal{G}}. \tag{2.4}$$

The space $\bar{\mathcal{G}}^* \simeq \Lambda^1(\mathbb{T}^n_{\mathbb{C}})$, adjoint to the Lie algebra $\bar{\mathcal{G}}$ of vector fields on $\mathbb{T}^n_{\mathbb{C}}$, can be functionally identified with $\bar{\mathcal{G}}$ subject to the metric

$$(\bar{l}, \bar{a}) = \frac{1}{2\pi i}\oint_{\mathbb{S}^1} d\lambda (l, a)_H, \tag{2.5}$$

for arbitrary $\bar{l} :=< l(x;\lambda), d\mathrm{x} >:= \sum_{j=\overline{0,n}} l_j(x;\lambda)d\mathrm{x}_j \in \bar{\mathcal{G}}^*$, $\bar{a} :=< a(x;\lambda), \partial/\partial\mathrm{x}>=< \sum_{j=\overline{0,n}} a_j(x;\lambda), \frac{\partial}{\partial \mathrm{x}_j} \in \bar{\mathcal{G}}$, where $(l,a)_H = \int_{\mathbb{T}^n} dx < l(x;\lambda), a(x;\lambda) >$.

Now for arbitrary $f, g \in \mathrm{D}(\bar{\mathcal{G}}^*)$, one can determine two Lie–Poisson type brackets

$$\{f, g\} := (\bar{l}, [\nabla f(\bar{l}), \nabla g(\bar{l})]) \tag{2.6}$$

and

$$\{f, g\}_{\mathcal{R}} := (\bar{l}, [\nabla f(\bar{l}), \nabla g(\bar{l})]_{\mathcal{R}}), \tag{2.7}$$

where at any seed element $\bar{l} \in \bar{\mathcal{G}}^*$ the gradient element $\nabla f(\bar{l})$ and $\nabla g(\bar{l}) \in \bar{\mathcal{G}}$ are calculated with respect to the metric (2.5).

Now let us assume that a smooth function $\gamma \in \mathrm{I}(\bar{\mathcal{G}}^*)$ is a Casimir invariant, that is

$$ad^*_{\nabla\gamma(\bar{l})}\bar{l} = 0 \tag{2.8}$$

for a chosen seed element $\bar{l} \in \bar{\mathcal{G}}^*$. As the adjoint mapping $ad^*_{\nabla f(\bar{l})}\bar{l}$ for any $f \in \mathrm{D}(\bar{\mathcal{G}}^*)$ can be rewritten in the reduced form as

$$ad^*_{\nabla f(\bar{l})}(\bar{l}) = \left\langle \frac{\partial}{\partial \mathrm{x}}, \nabla f(l) \right\rangle \bar{l} + \sum_{j=1}^{n} \left\langle \left\langle l, \frac{\partial}{\partial \mathrm{x}} \nabla f(l) \right\rangle, d\mathrm{x} \right\rangle, \tag{2.9}$$

where $\nabla f(\bar{l}) :=< \nabla f(l), \frac{\partial}{\partial \mathrm{x}} >$. For the Casimir function $\gamma \in \mathrm{D}(\bar{\mathcal{G}}^*)$ the condition (2.8) is then equivalent to the equation

$$l \left\langle \frac{\partial}{\partial \mathrm{x}}, \nabla\gamma(l) \right\rangle + \left\langle \nabla\gamma(l), \frac{\partial}{\partial \mathrm{x}} \right\rangle l + \left\langle l, (\frac{\partial}{\partial \mathrm{x}} \nabla\gamma(l)) \right\rangle = 0, \tag{2.10}$$

which should be solved analytically. In the case when an element $\bar{l} \in \bar{\mathcal{G}}^*$ is singular as $|\lambda| \to \infty$, one can consider the general asymptotic expansion

$$\nabla\gamma := \nabla\gamma^{(p)} \sim \lambda^p \sum_{j \in \mathbb{Z}_+} \nabla\gamma_j^{(p)} \lambda^{-j} \tag{2.11}$$

for some suitably chosen $p \in \mathbb{Z}_+$, and upon substituting (2.11) into the equation (2.10), one can proceed to solving it recurrently.

Now let $h^{(y)}, h^{(t)} \in \mathrm{I}(\bar{\mathcal{G}}^*)$ be such Casimir functions for which the Hamiltonian vector field generators

$$\nabla h_+^{(y)}(l) := (\nabla\gamma^{(p_y)}(l))_+, \qquad \nabla h_+^{(t)}(l) := (\nabla h^{(p_t)}(l))_+ \tag{2.12}$$

are, respectively, defined for special integers $p_y, p_t \in \mathbb{Z}_+$. These invariants generate, owing to the Lie–Poisson bracket (2.7), the following commuting flows:

$$\partial l/\partial t = - \left\langle \frac{\partial}{\partial \mathrm{x}}, \nabla h_+^{(t)}(l) \right\rangle l - \left\langle l, (\frac{\partial}{\partial \mathrm{x}} \nabla h_+^{(t)}(l)) \right\rangle \tag{2.13}$$

and

$$\partial l/\partial y = - \left\langle \frac{\partial}{\partial \mathrm{x}}, \nabla h_+^{(y)}(l) \right\rangle l - \left\langle l, (\frac{\partial}{\partial \mathrm{x}} \nabla h_+^{(y)}(l)) \right\rangle >, \tag{2.14}$$

where $y, t \in \mathbb{R}$ are the corresponding evolution parameters. Since the invariants $h^{(y)}, h^{(t)} \in \mathrm{I}(\bar{\mathcal{G}}^*)$ commute with respect to the Lie–Poisson bracket (2.7), the flows (2.13) and (2.14) also commute, implying that the corresponding Hamiltonian vector field generators

$$\bar{A}_{\nabla h_+^{(t)}} := \left\langle \frac{\partial}{\partial \mathrm{x}}, \nabla h_+^{(t)}(l) \right\rangle, \qquad \bar{A}_{\nabla h_+^{(y)}} := \left\langle \frac{\partial}{\partial \mathrm{x}}, \nabla h_+^{(y)}(l) \right\rangle \tag{2.15}$$

satisfy the Lax compatibility condition

$$\frac{\partial}{\partial y} \bar{A}_{\nabla h_+^{(t)}} - \frac{\partial}{\partial t} \bar{A}_{\nabla h_+^{(y)}} = [\bar{A}_{\nabla h_+^{(t)}}, \bar{A}_{\nabla h_+^{(y)}}] \tag{2.16}$$

for all $y, t \in \mathbb{R}$. On the other hand, the condition (2.16) is equivalent to the compatibility condition of two linear equations

$$(\frac{\partial}{\partial t} + \bar{A}_{\nabla h_+^{(t)}})\psi = 0, \qquad (\frac{\partial}{\partial y} + \bar{A}_{\nabla h_+^{(y)}})\psi = 0 \tag{2.17}$$

for a function $\psi \in C^2(\mathbb{T}^n_{\mathbb{C}}; \mathbb{C})$ for all $y, t \in \mathbb{R}$ and any $\lambda \in \mathbb{C}$.

The above can be formulated as the following key result:

Proposition 2.1. *Let a seed vector field be $\bar{l} \in \bar{\mathcal{G}}^*$ and $h^{(y)}, h^{(t)} \in \mathrm{I}(\bar{\mathcal{G}}^*)$ be Casimir functions subject to the metric $(\cdot,\cdot)$ on the loop Lie algebra $\bar{\mathcal{G}}$ and the natural coadjoint action on the loop co-algebra $\bar{\mathcal{G}}^*$. Then the following dynamical systems*

$$\partial\bar{l}/\partial y = -ad^*_{\nabla h_+^{(y)}(\bar{l})}\bar{l}, \quad \partial\bar{l}/\partial t = -ad^*_{\nabla h_+^{(t)}(\bar{l})}\bar{l} \tag{2.18}$$

are commuting Hamiltonian flows for all $y,t\in\mathbb{R}$. Moreover, the compatibility condition of these flows is equivalent to the vector fields representation

$$(\partial/\partial t+\bar{A}_{\nabla h_{+}^{(t)}})\psi=0,\qquad(\partial/\partial y+\bar{A}_{\nabla h_{+}^{(y)}})\psi=0,\tag{2.19}$$

where $\psi\in C^{2}(\mathbb{R}^{2}\times\mathbb{T}_{\mathbb{C}}^{n};\mathbb{C})$ and the vector fields $\bar{A}_{\nabla h_{+}^{(y)}},\bar{A}_{\nabla h_{+}^{(t)}}\in\bar{\mathcal{G}}$ are given by the expressions (2.15) and (2.12).

Remark 2.2. As mentioned above, the expansion (2.11) is effective if a chosen seed element $\bar{l}\in\bar{\mathcal{G}}^{*}$ is singular as $|\lambda|\to\infty$. In the case when it is singular as $|\lambda|\to 0$, the expression (2.11) should be replaced by the expansion

$$\nabla\gamma^{(p)}(l)\sim\lambda^{-p}\sum_{j\in\mathbb{Z}_{+}}\nabla\gamma_{j}^{(p)}(l)\lambda^{j}\tag{2.20}$$

for suitably chosen integers $p\in\mathbb{Z}_{+}$, and the reduced Casimir function gradients then are given by the Hamiltonian vector field generators

$$\nabla h_{-}^{(y)}(l):=\lambda(\lambda^{-p_{y}-1}\nabla\gamma^{(p_{y})}(l))_{-},\qquad\nabla h_{-}^{(t)}(l):=\lambda(\lambda^{-p_{t}-1}\nabla\gamma^{(p_{t})}(l))_{-}\tag{2.21}$$

for suitably chosen positive integers $p_{y},p_{t}\in Z_{+}$ and the corresponding Hamiltonian flows are, respectively, written as $\partial\bar{l}/\partial t=ad^{*}_{\nabla h_{-}^{(t)}(\bar{l})}\bar{l}$, $\partial\bar{l}/\partial y=ad^{*}_{\nabla h_{-}^{(y)}(\bar{l})}\bar{l}$.

It is also worth of mentioning that, following Ovsienko's scheme [10, 11], one can consider a wider class of integrable heavenly equations, realized as compatible Hamiltonian flows on the semidirect product of the holomorphic loop Lie algebra $\bar{\mathcal{G}}$ of vector fields on the torus $\mathbb{T}_{\mathbb{C}}^{n}$ and its regular co-adjoint space $\bar{\mathcal{G}}^{*}$, supplemented with naturally related cocycles.

3. The Lax-Sato type integrable systems and related conformal structure generating equations

3.1. Example: Einstein–Weyl metric equation. Define $\bar{\mathcal{G}}^{*}=diff_{hol}(\mathbb{T}_{\mathbb{C}}^{1})$ and take the seed element

$$\tilde{l}=(u_{x}\lambda-2u_{x}v_{x}-u_{y})\,dx+\left(\lambda^{2}-v_{x}\lambda+v_{y}+v_{x}^{2}\right)d\lambda,$$

which generates with respect to the metric (2.5) the gradient of the Casimir invariants $h^{(p_{t})},h^{(p_{y})}\in\mathrm{I}(\bar{\mathcal{G}}^{*})$ in the form

$$\nabla h^{(p_{t})}(l)\sim\lambda^{2}(0,1)^{\intercal}+(-u_{x},v_{x})^{\intercal}\lambda+(u_{y},u-v_{y})^{\intercal}+O(\lambda^{-1}),\tag{3.1}$$

$$\nabla h^{(p_{y})}(l)\sim\lambda(0,1)^{\intercal}+(-u_{x},v_{x})^{\intercal}+(u_{y},-v_{y})^{\intercal}\lambda^{-1}+O(\lambda^{-2})$$

as $|\lambda|\to\infty$ at $p_{t}=2$, $p_{y}=1$. For the gradients of the Casimir functions $h^{(t)},h^{(y)}\in\mathrm{I}(\bar{\mathcal{G}}^{*})$, determined by (2.12) one can easily obtain the corresponding Hamiltonian vector field generators

$$\tilde{A}_{\nabla h_{+}^{(t)}}=\left\langle\nabla h_{+}^{(t)}(l),\frac{\partial}{\partial\mathrm{x}}\right\rangle=(\lambda^{2}+\lambda v_{x}+u-v_{y})\frac{\partial}{\partial x}+(-\lambda u_{x}+u_{y})\frac{\partial}{\partial\lambda},$$

$$\tilde{A}_{\nabla h_{+}^{(y)}}=\left\langle\nabla h_{+}^{(y)}(l),\frac{\partial}{\partial\mathrm{x}}\right\rangle=(\lambda+v_{x})\frac{\partial}{\partial x}-u_{x}\frac{\partial}{\partial\lambda},\tag{3.2}$$

satisfying the compatibility condition (2.16), which is equivalent to the set of equations

$$\left\{\begin{array}{l}u_{xt}+u_{yy}+(uu_{x})_{x}+v_{x}u_{xy}-v_{y}u_{xx}=0,\\[2ex] v_{xt}+v_{yy}+uv_{xx}+v_{x}v_{xy}-v_{y}v_{xx}=0,\end{array}\right.\tag{3.3}$$

describing general integrable Einstein–Weyl metric equations [4].

As is well known [8], the invariant reduction of (3.3) at $v=0$ gives rise to the famous dispersionless Kadomtsev–Petviashvili equation

$$(u_t+uu_x)_x+u_{yy}=0, \tag{3.4}$$

for which the reduced vector field representation (2.17) follows from (3.2) and is given by the vector fields

$$\bar{A}_{\nabla h_+^{(t)}}=(\lambda^2+u)\frac{\partial}{\partial x}+(-\lambda u_x+u_y)\frac{\partial}{\partial \lambda}, \tag{3.5}$$

$$\bar{A}_{\nabla h_+^{(y)}}=\lambda\frac{\partial}{\partial x}-u_x\frac{\partial}{\partial \lambda},$$

satisfying the compatibility condition (2.16), equivalent to the equation (3.4). In particular, one derives from (2.17) and (3.5) the vector field compatibility relationships

$$\begin{gathered}\frac{\partial\psi}{\partial t}+(\lambda^2+u)\frac{\partial\psi}{\partial x}+(-\lambda u_x+u_y)\frac{\partial\psi}{\partial\lambda}=0\\ \frac{\partial\psi}{\partial y}+\lambda\frac{\partial\psi}{\partial x}-u_x\frac{\partial\psi}{\partial\lambda}=0,\end{gathered} \tag{3.6}$$

satisfied for $\psi\in C^2(\mathbb{R}^2\times\mathbb{T}^1_{\mathbb{C}};\mathbb{C})$ and any $y,t\in\mathbb{R},(x,\lambda)\in\mathbb{T}^1_{\mathbb{C}}$.

3.2. **The modified Einstein–Weyl metric equation.** This equation system is

$$u_{xt}=u_{yy}+u_xu_y+u_x^2w_x+uu_{xy}+u_{xy}w_x+u_{xx}a, \tag{3.7}$$

$$w_{xt}=uw_{xy}+u_yw_x+w_xw_{xy}+aw_{xx}-a_y,$$

where $a_x:=u_xw_x-w_{xy}$, and was recently derived in [14]. In this case we take also $\bar{\mathcal{G}}^*=diff_{hol}(\mathbb{T}^1_{\mathbb{C}})$, yet for a seed element $\tilde{l}\in\bar{\mathcal{G}}$ we choose the form

$$\tilde{l}=[\lambda^2u_x+(2u_xw_x+u_y+3uu_x)\,\lambda+2u_x\partial_x^{-1}u_xw_x\ +2u_x\partial_x^{-1}u_y\ + \tag{3.8}$$

$$+3u_xw_x{}^2+2u_yw_x+6uu_xw_x+2uu_y+3u^2u_x-2au_x]dx+$$

$$+[\lambda^2+(w_x+3u)\,\lambda+2\partial_x^{-1}u_xw_x\ +2\partial_x^{-1}u_y\ +w_x{}^2+3uw_x+3u^2-a]d\lambda,$$

which with respect to the metric (2.5) generates two Casimir invariants $\gamma^{(j)}\in \mathrm{I}(\bar{\mathcal{G}}^*), j=\overline{1,2}$, whose gradients are

$$\nabla\gamma^{(2)}(l)\sim\lambda^2[(u_x,-1)^\intercal+(uu_x+u_y,-u+w_x)^\intercal\lambda^{-1}\ + \tag{3.9}$$

$$+\,(0,uw_x-a)^\intercal\lambda^{-2}]+O(\lambda^{-1})\ ,$$

$$\nabla\gamma^{(1)}(l)\sim\lambda[(u_x,-1)^\intercal+(0,w_x)^\intercal\ \lambda^{-1}]+O(\lambda^{-1}),$$

as $|\lambda|\to\infty$ at $p_y=1,p_t=2$. The corresponding gradients of the Casimir functions $h^{(t)},h^{(y)}\in\mathrm{I}(\mathcal{G}^*)$, determined by (2.12), generate the Hamiltonian vector field expressions

(3.10)

$$\nabla h_+^{(y)}:=\nabla\gamma^{(1)}(l)|_+=(u_x\lambda,-\lambda+w_x)^\intercal,$$

$$\nabla h_{2,+}^{(t)}=\nabla\gamma^{(2)}(l)|_+=(u_x\lambda^2+(uu_x+u_y)\lambda,-\lambda^2+(w_x-u)\lambda+uw_x-a)^\intercal.$$

Now one easily obtains from (3.10) the compatible Lax system of linear equations

(3.11)
$$\frac{\partial\psi}{\partial y}+(-\lambda+w_x)\frac{\partial\psi}{\partial x}+u_x\lambda\frac{\partial\psi}{\partial\lambda}=0,$$

$$\frac{\partial\psi}{\partial t}+\ (-\lambda^2+(\ w_x-u)\lambda+uw_x-a)\frac{\partial\psi}{\partial x}+(u_x\lambda^2+(uu_x+u_y)\lambda)\frac{\partial\psi}{\partial\lambda}=0,$$

satisfied for $\psi\in C^2(\mathbb{R}^2\times\mathbb{T}^1_{\mathbb{C}};\mathbb{C})$ and any $y,t\in\mathbb{R}$, $(x,\lambda)\in\mathbb{T}^1_{\mathbb{C}}$.

3.3. **Example: The Dunajski heavenly equations.** This equation, suggested in [3], generalizes the corresponding anti-self-dual vacuum Einstein equation, which is related to the Plebański metric and the celebrated Plebański [12, 5] second heavenly equation. To study the integrability of the Dunajski equations

$$u_{x_1t}+u_{yx_2}+u_{x_1x_1}u_{x_2x_2}-u^2_{x_1x_2}-v=0, \tag{3.12}$$

$$v_{x_1t}+v_{x_2y}+u_{x_1x_1}v_{x_2x_2}-2u_{x_1x_2}v_{x_1x_2}=0,$$

where $(u,v)\in C^\infty(\mathbb{R}^2\times\mathbb{T}^2;\mathbb{R}^2)$, $(y,t;x_1,x_2)\in\mathbb{R}^2\times\mathbb{T}^2$, we define $\bar{\mathcal{G}}^*:=diff^*_{hol}(\ \mathbb{T}^2_{\mathbb{C}})$ and take the following as a seed element $\tilde{l}\in\bar{\mathcal{G}}^*$
(3.13)
$$\tilde{l}=(\lambda+v_{x_1}-u_{x_1x_1}+u_{x_1x_2})dx_1+(\lambda+v_{x_2}+u_{x_2x_2}-u_{x_1x_2})dx_2+(\lambda-x_1-x_2)d\lambda.$$

With respect to the metric (2.5), the gradients of two functionally independent Casimir invariants $h^{(p_y)},h^{(p_y)}\ \in \mathrm{I}(\bar{\mathcal{G}}^*)$ can be obtained as $|\lambda|\to\infty$ in the asymptotic form as

$$\nabla h^{(p_y)}\ (l)\sim\lambda(0,1,0)^\intercal+(-v_{x_1},-u_{x_1x_2},u_{x_1x_1})^\intercal\ +O(\lambda^{-1}), \tag{3.14}$$

$$\nabla h^{(p_t)}\ (l)\sim\lambda(0,0,-1)^\intercal+(v_{x_2},u_{x_2x_2},-u_{x_1x_2})^\intercal\ +O(\lambda^{-1})$$

at $p_t=1=p_y$. Upon calculating the Hamiltonian vector field generators

$$\nabla h^{(y)}_+:=\nabla h^{(p_y)}\ (l)|_+=(-v_{x_1},\lambda-u_{x_1x_2},u_{x_1x_1})^\intercal, \tag{3.15}$$

$$\nabla h^{(t)}_+:=\nabla h^{(p_t)}\ (l)|_+=(v_{x_2},u_{x_2x_2},-\lambda-u_{x_1x_2})^\intercal,$$

following from the Casimir functions gradients (3.14), one easily obtains the following vector fields

$$\bar{A}_{\nabla h^{(t)}_+}=<\nabla h^{(t)}_+,\frac{\partial}{\partial \mathrm{x}}>=u_{x_2x_2}\frac{\partial}{\partial x_1}-(\lambda+u_{x_1x_2})\frac{\partial}{\partial x_2}+v_{x_2}\frac{\partial}{\partial\lambda}, \tag{3.16}$$

$$\bar{A}_{\nabla h^{(y)}_+}=<\nabla h^{(y)}_+,\frac{\partial}{\partial \mathrm{x}}>=(\lambda-u_{x_1x_2})\frac{\partial}{\partial x_1}+u_{x_1x_1}\frac{\partial}{\partial x_2}-v_{x_1}\frac{\partial}{\partial\lambda},$$

satisfying the Lax compatibility condition (2.16), which is equivalent to the vector field compatibility relationships

$$\frac{\partial\psi}{\partial t}+u_{x_2x_2}\frac{\partial\psi}{\partial x_1}-(\lambda+u_{x_1x_2})\frac{\partial\psi}{\partial x_2}+v_{x_2}\frac{\partial\psi}{\partial\lambda}=0,$$

(3.17)

$$\frac{\partial\psi}{\partial y}+(\lambda-u_{x_1x_2})\frac{\partial\psi}{\partial x_1}+u_{x_1x_1}\frac{\partial\psi}{\partial x_2}-v_{x_1}\frac{\partial\psi}{\partial\lambda}=0,$$

satisfied for $\psi\in C^\infty(\mathbb{R}^2\times\mathbb{T}^2_{\mathbb{C}};\mathbb{C})$, any $(y,t)\in\mathbb{R}^2$ and all $(x_1,x_2;\lambda)\in\mathbb{T}^2_{\mathbb{C}}$. As was mentioned in [1], the Dunajski equations (3.12) generalize both the dispersionless Kadomtsev–Petviashvili and Plebański second heavenly equations, and is also a Lax integrable Hamiltonian system.

3.4. **First conformal structure generating equation:** $u_{yy}+u_{xt}u_y-u_tu_{xy}=0.$. The seed element $\tilde{l}\in\tilde{\mathcal{G}}^*$ in the form

$$\tilde{l}=[u_t^{-2}(1-\lambda)\lambda^{-1}+u_y^{-2}\lambda(\lambda-1)^{-1}]dx, \tag{3.18}$$

where $u\in C^2(\mathbb{T}^1\times\mathbb{R}^2;\mathbb{R})$, $x\in\mathbb{T}^1$, $\lambda\in\mathbb{C}\backslash\{0,1\}$ and "d" denotes the full differential, generates two independent Casimir functionals $\gamma^{(1)}$ and $\gamma^{(2)}\in I(\tilde{\mathcal{G}}^*)$, whose gradients have the following asymptotic expansions:

$$\nabla\gamma^{(1)}(l)\simeq u_y+O(\mu^2),$$

as $|\mu|\to 0$, $\mu:=\lambda-1$, and

$$\nabla\gamma^{(2)}(l)\simeq u_t+O(\lambda^2),$$

as $|\lambda|\to 0$. The commutability condition

$$[X^{(y)},X^{(t)}]=0 \tag{3.19}$$

of the vector fields

$$X^{(y)}:=\partial/\partial y+\nabla h^{(y)}(l),\qquad X^{(t)}=\partial/\partial t+\nabla h^{(t)}(l), \tag{3.20}$$

where

$$\begin{aligned}\nabla h^{(y)}(\tilde{l})&:=-(\mu^{-1}\nabla\gamma^{(1)}(\tilde{l}))|_-=-\frac{u_y}{\lambda-1}\frac{\partial}{\partial x},\\ \nabla h^{(t)}(\tilde{l})&:=-(\lambda^{-1}\nabla\gamma^{(2)}(\tilde{l}))|_-=-\frac{u_t}{\lambda}\frac{\partial}{\partial x},\end{aligned} \tag{3.21}$$

leads to the heavenly type equation

$$u_{yt}+u_{xt}u_y-u_{xy}u_t=0.$$

Its Lax-Sato representation is the compatibility condition for the first order partial differential equations

$$\begin{aligned}&\frac{\partial\psi}{\partial y}-\frac{u_y}{\lambda-1}\frac{\partial\psi}{\partial x}=0,\\ &\frac{\partial\psi}{\partial t}-\frac{u_t}{\lambda}\frac{\partial\psi}{\partial x}=0,\end{aligned} \tag{3.22}$$

where $\psi\in C^2(\mathbb{T}^1\times\mathbb{R}^2;\mathbb{R})$.

3.5. **Second conformal structure generating equation:** $u_{xt}+u_xu_{yy}-u_yu_{xy}=0.$. For a seed element $\tilde{l}\in\tilde{\mathcal{G}}^*$ in the form

$$\tilde{l}=[u_x^2+2u_x^2(u_y+\alpha)\lambda^{-1}+u_x^2(3u_y^2+4\alpha u_y+\beta)\lambda^{-2}]dx, \tag{3.23}$$

where $u\in C^2(\mathbb{T}^1\times\mathbb{R}^2;\mathbb{R})$, $x\in\mathbb{T}^1$, $\lambda\in\mathbb{C}\setminus\{0\}$,and $\alpha,\beta\in\mathbb{R}$, there a one independent Casimir functional $\gamma^{(1)}\in I(\tilde{\mathcal{G}}^*)$ with the following asymptotic as $|\lambda|\to 0$ expansion of its functional gradient:

$$\nabla\gamma^{(1)}(l)\simeq c_0u_x^{-1}+(-c_0u_y+c_1)u_x^{-1}\lambda+(-c_1u_y+c_2)u_x^{-1}\lambda^2+O(\lambda^3),$$

where $c_r\in\mathbb{R}$, $r=\overline{1,2}$. If one assumes that $c_0=1$, $c_1=0$ and $c_2=0$, then we obtain two functionally independent gradient elements

$$\begin{aligned}\nabla h^{(y)}(\tilde{l})&:=-(\lambda^{-1}\nabla\gamma^{(1)}(\tilde{l}))|_-=-\frac{1}{\lambda u_x}\frac{\partial}{\partial x},\\ \nabla h^{(t)}(\tilde{l})&:=(\lambda^{-2}\nabla\gamma^{(1)}(\tilde{l}))|_-=\left(\frac{1}{\lambda^2u_x}-\frac{u_y}{\lambda u_x}\right)\frac{\partial}{\partial x}.\end{aligned} \tag{3.24}$$

The corresponding commutability condition (3.35) of the vector fields (**??**) givse rise to the following heavenly type equation:

$$u_{xt}+u_xu_{yy}-u_yu_{xy}=0, \tag{3.25}$$

whose linearized Lax-Sato representation is given given by the first order system

$$\begin{aligned} &\frac{\partial\psi}{\partial y} - \frac{1}{\lambda u_x}\frac{\partial\psi}{\partial x} = 0, \\ &\frac{\partial\psi}{\partial t} + \left(\frac{1}{\lambda^2 u_x} - \frac{u_y}{\lambda u_x}\right)\frac{\partial\psi}{\partial x} = 0 \end{aligned} \tag{3.26}$$

of linear vector field equations on a function $\psi \in C^2(\mathbb{T}^1 \times \mathbb{R}^2; \mathbb{R})$.

3.6. **Inverse first Shabat reduction heavenly equation.** A seed element $\tilde{l} \in \tilde{\mathcal{G}}^*$ in the form

$$\tilde{l} = (a_0 u_y^{-2} u_x^2 (\lambda+1)^{-1} + a_1 u_x^2 + a_1 u_x^2 \lambda) dx, \tag{3.27}$$

where $u \in C^2(\mathbb{T}^1 \times \mathbb{R}^2; \mathbb{R})$, $x \in \mathbb{T}^1$, $\lambda \in \mathbb{C} \setminus \{-1\}$, and $a_0, a_1 \in \mathbb{R}$, generates two independent Casimir functionals $\gamma^{(1)}$ and $\gamma^{(2)} \in I(\tilde{\mathcal{G}}^*)$, whose gradients have the following asymptotic expansions:

$$\nabla\gamma^{(1)}(l) \simeq u_y u_x^{-1} - u_y u_x^{-1}\mu + O(\mu^2), \tag{3.28}$$

as $|\mu| \to 0$, $\mu := \lambda + 1$, and

$$\nabla\gamma^{(2)}(l) \simeq u_x^{-1} + O(\lambda^{-2}), \tag{3.29}$$

as $|\lambda| \to \infty$. If wu put, by definition,

$$\begin{aligned} \nabla h^{(y)}(\tilde{l}) &:= (\mu^{-1}\nabla\gamma^{(1)}(\tilde{l}))|_- = -\frac{\lambda}{\lambda+1}\frac{u_y}{u_x}\frac{\partial}{\partial x}, \\ \nabla h^{(t)}(\tilde{l}) &:= (\lambda\nabla\gamma^{(2)}(\tilde{l}))|_+ = \frac{\lambda}{u_x}\frac{\partial}{\partial x}, \end{aligned} \tag{3.30}$$

the commutability condition (3.35) of the vector fields (**??**) leads to the heavenly equation

$$u_{xy} + u_y u_{tx} - u_{ty} u_x = 0, \tag{3.31}$$

which can be obtained as a result of the simultaneous changing of independent variables $\mathbb{R} \ni x \rightleftarrows t \in \mathbb{R}$, $\mathbb{R} \ni y \rightleftarrows x \in \mathbb{R}$ and $\mathbb{R} \ni t \rightleftarrows y \in \mathbb{R}$ in the first Shabat reduction heavenly equation. The corersponding Lax-Sato representation is given by the compatibility condition for the first order vector field equations equations

$$\begin{aligned} &\frac{\partial\psi}{\partial y} - \frac{\lambda}{\lambda+1}\frac{u_y}{u_x}\frac{\partial\psi}{\partial x} = 0, \\ &\frac{\partial\psi}{\partial t} + \frac{\lambda}{u_x}\frac{\partial\psi}{\partial x} = 0, \end{aligned} \tag{3.32}$$

where $\psi \in C^2(\mathbb{T}^1 \times \mathbb{R}^2; \mathbb{R})$.

3.7. **First Plebański heavenly equation.** The seed element $\tilde{l} \in \tilde{\mathcal{G}}^*$ in the form

$$\tilde{l} = \lambda^{-1}(u_{yx_1} dx_1 + u_{yx_2} dx_2) = \lambda^{-1} du_y, \tag{3.33}$$

where $u \in C^2(\mathbb{T}^2 \times \mathbb{R}^2; \mathbb{R})$, $(x_1, x_2) \in \mathbb{T}^2$, $\lambda \in \mathbb{C}\setminus\{0\}$ and "d" designates a full differential, generates two independent Casimir functionals $\gamma^{(1)}$ and $\gamma^{(2)} \in$ I$(\tilde{\mathcal{G}}^*)$, whose gradients have the following asymptotic expansions:

$$\begin{aligned} \nabla\gamma^{(1)}(l) &\sim (-u_{yx_2}, u_{yx_1},)^\top + O(\lambda), \\ \nabla\gamma^{(2)}(l) &\sim (-u_{tx_2}, u_{tx_1})^\top + O(\lambda), \end{aligned} \tag{3.34}$$

as $|\lambda| \to 0$. The commutability condition

$$[\partial/\partial y + \nabla h_-^{(y)}(l), \partial/\partial t + \nabla h_-^{(t)}(l)] = 0 \tag{3.35}$$

of the vector fields $\partial/\partial y+\nabla h_-^{(y)}(l)$ and $X^{(t)}=\partial/\partial t+\nabla h_-^{(t)}(l)$, where

$$\nabla h_-^{(y)}(\tilde l):=(\lambda^{-1}\nabla\gamma^{(1)}(\tilde l))|_-=-\frac{u_{yx_2}}{\lambda}\frac{\partial}{\partial x_1}+\frac{u_{yx_1}}{\lambda}\frac{\partial}{\partial x_2},\tag{3.36}$$
$$\nabla h_-^{(t)}(\tilde l):=(\lambda^{-1}\nabla\gamma^{(2)}(\tilde l))|_-=-\frac{u_{tx_2}}{\lambda}\frac{\partial}{\partial x_1}+\frac{u_{tx_1}}{\lambda}\frac{\partial}{\partial x_2},$$

leads to the first Plebański heavenly equation [2]

$$u_{yx_1}u_{tx_2}-u_{yx_2}u_{tx_1}=1.\tag{3.37}$$

Its Lax-Sato representation (3.35) entails the compatibility condition for the first order partial differential equations

$$\frac{\partial\psi}{\partial y}-\frac{u_{yx_2}}{\lambda}\frac{\partial\psi}{\partial x_1}+\frac{u_{yx_1}}{\lambda}\frac{\partial\psi}{\partial x_2}=0,$$
$$\frac{\partial\psi}{\partial t}-\frac{u_{tx_2}}{\lambda}\frac{\partial\psi}{\partial x_1}+\frac{u_{tx_1}}{\lambda}\frac{\partial\psi}{\partial x_2}=0,$$

where $\psi\in C^\infty(\mathbb{T}^2\times\mathbb{R}^2;\mathbb{C})$.

Remark 1. Taking into account that the condition for Casimir invariants is equivalent to the system of nonhomogeneous linear first order partial differential equations for the vector-function $l=(l_1,l_2)^\intercal$, the corresponding seed-element can be chosen in another forms. The asymptotic expansions (3.34) are also true for such seed-elements as

$$\tilde l=\lambda^{-1}du_t,$$

and

$$\tilde l=\lambda^{-1}(du_y+du_t).$$

The above described scheme can be easily generalized for all $m=2n$, where $m\in\mathbb{N}$ and $n>2$. In this case one has $2n$ independent Casimir functionals $\gamma^{(j)}\in\mathrm{I}(\tilde{\mathcal{G}}^*)$, where $j=\overline{1,2n}$, with the following asymptotic expansions for their gradients:

$$\nabla\gamma^{(1)}(l)\sim(-u_{yx_2},u_{yx_1},\underbrace{0,\dots,0}_{2n-2})^\top+O(\lambda),$$
$$\nabla\gamma^{(2)}(l)\sim(-u_{tx_2},u_{tx_1},\underbrace{0,\dots,0}_{2n-2})^\top+O(\lambda),$$
$$\nabla\gamma^{(3)}(l)\sim(0,0,-u_{yx_4},u_{yx_3},\underbrace{0,\dots,0}_{2n-4})^\top+O(\lambda),$$
$$\nabla\gamma^{(4)}(l)\sim(0,0,-u_{tx_4},u_{tx_3},\underbrace{0,\dots,0}_{2n-4})^\top+O(\lambda),$$
$$\dots,$$
$$\nabla\gamma^{(2k-1)}(l)\sim(\underbrace{0,\dots,0}_{2n-2},-u_{yx_{2k}},u_{yx_{2n-1}})^\top+O(\lambda),$$
$$\nabla\gamma^{(2k)}(l)\sim(\underbrace{0,\dots,0}_{2n-2},-u_{tx_{2k}},u_{tx_{2n-1}})^\top+O(\lambda).$$

If we put

$$\nabla h_-^{(y)}(\tilde l):=(\lambda^{-1}(\nabla\gamma^{(1)}(\tilde l)+\dots+\nabla\gamma^{(2k-1)}(\tilde l)))|_-=$$
$$=-\frac{u_{yx_2}}{\lambda}\frac{\partial}{\partial x_1}+\frac{u_{yx_1}}{\lambda}\frac{\partial}{\partial x_2}+\dots-\frac{u_{yx_{2k}}}{\lambda}\frac{\partial}{\partial x_{2k-1}}+\frac{u_{yx_{2k-1}}}{\lambda}\frac{\partial}{\partial x_{2k}},$$
$$\nabla h_-^{(t)}(\tilde l):=(\lambda^{-1}(\nabla\gamma^{(2)}(\tilde l)+\dots+\nabla\gamma^{(2k)}(\tilde l)))|_-=$$
$$=-\frac{u_{tx_2}}{\lambda}\frac{\partial}{\partial x_1}+\frac{u_{tx_1}}{\lambda}\frac{\partial}{\partial x_2}+\dots-\frac{u_{tx_{2k}}}{\lambda}\frac{\partial}{\partial x_{2k-1}}+\frac{u_{tx_{2k-1}}}{\lambda}\frac{\partial}{\partial x_{2k}},$$

the commutability condition (3.35) of the vector fields (3.35) leads to the corresponding multi-dimensional analogs of the first Plebański heavenly equation:

$$\sum_{j=1}^{n}(u_{yx_{2j-1}}u_{tx_{2j}} - u_{yx_{2j}}u_{tx_{2j-1}}) = 1.$$

3.8. **Modified Plebański equation.** For the seed element $\tilde{l} \in \tilde{\mathcal{G}}^*$ in the form

$$\begin{aligned} \tilde{l} &= (\lambda^{-1}u_{x_1y} + u_{x_1x_1} - u_{x_1x_2} + \lambda)dx_1 + \\ &+ (\lambda^{-1}u_{x_2y} + u_{x_1x_2} - u_{x_2x_2} + \lambda)dx_2 = \\ &= d(\lambda^{-1}u_y + u_{x_1} - u_{x_2} + \lambda x_1 + \lambda x_2). \end{aligned} \tag{3.38}$$

where $u \in C^2(\mathbb{T}^2 \times \mathbb{R}^2;\mathbb{R})$, $(x_1,x_2) \in \mathbb{T}^2, \lambda \in \mathbb{C} \setminus \{0\}$, there exist two independent Casimir functionals $\gamma^{(1)}$ and $\gamma^{(2)} \in \mathrm{I}(\tilde{\mathcal{G}}^*)$ with the following gradient asymptotic expansions:

$$\nabla\gamma^{(1)}(l) \sim (u_{yx_2}, -u_{yx_1})^\top + O(\lambda),$$

as $|\lambda| \to 0$, and

$$\nabla\gamma^{(2)}(l) \sim (0,-1)^\top + (-u_{x_2x_2}, u_{x_1x_2})^\top\lambda^{-1} + O(\lambda^{-2}),$$

as $|\lambda| \to \infty$. In the case, when

$$\begin{aligned} \nabla h_-^{(y)}(\tilde{l}) &:= (\lambda^{-1}\nabla\gamma^{(1)}(\tilde{l}))|_- = \frac{u_{yx_2}}{\lambda}\frac{\partial}{\partial x_1} - \frac{u_{yx_1}}{\lambda}\frac{\partial}{\partial x_2}, \\ \nabla h_-^{(t)}(\tilde{l}) &:= (\lambda\nabla\gamma^{(2)}(\tilde{l}))|_+ = -u_{x_2x_2}\frac{\partial}{\partial x_1} + (u_{x_1x_2} - \lambda)\frac{\partial}{\partial x_2}, \end{aligned}$$

the commutability condition of the vector fields $\partial/\partial y + \nabla h_-^{(y)}(\tilde{l})$ and $\partial/\partial t + \nabla h_-^{(t)}(\tilde{l})$ leads to the modified Plebański heavenly equation [2]:

$$u_{yt} - u_{yx_1}u_{x_2x_2} + u_{yx_2}u_{x_1x_2} = 0,$$

with the Lax-Sato representation given by the first order partial differential equations

$$\begin{aligned} &\frac{\partial\psi}{\partial y} - \frac{u_{yx_2}}{\lambda}\frac{\partial\psi}{\partial x_1} + \frac{u_{yx_1}}{\lambda}\frac{\partial\psi}{\partial x_2} = 0, \\ &\frac{\partial\psi}{\partial t} - u_{x_2x_2}\frac{\partial\psi}{\partial x_1} + (u_{x_1x_2} - \lambda)\frac{\partial\psi}{\partial x_2} = 0 \end{aligned}$$

for functions $\psi \in C^2(\mathbb{T}^2 \times \mathbb{R}^2;\mathbb{C})$.

3.9. **Husain heavenly equation.** The seed element $\tilde{l} \in \tilde{\mathcal{G}}^*$ in the form

$$\tilde{l} = \frac{d(u_y + iu_t)}{\lambda - i} + \frac{d(u_y - iu_t)}{\lambda + i} = \frac{2(\lambda du_y - du_t)}{\lambda^2 + 1}, \qquad i^2 = -1, \tag{3.39}$$

where $u \in C^2(\mathbb{T}^2 \times \mathbb{R}^2;\mathbb{R})$, $(x_1,x_2) \in \mathbb{T}^2$, $\lambda \in \mathbb{C}\backslash\{-i;i\}$, generates two independent Casimir functionals $\gamma^{(1)}$ and $\gamma^{(2)} \in \mathrm{I}(\tilde{\mathcal{G}}^*)$, with the following gradient asymptotic expansions:

$$\nabla\gamma^{(1)}(l) \sim \frac{1}{2}(-u_{yx_2} - iu_{tx_2}, u_{yx_1} + iu_{tx_1})^\top + O(\mu), \qquad \mu := \lambda - i,$$

as $|\mu| \to 0$, and

$$\nabla\gamma^{(2)}(l) \sim \frac{1}{2}(-u_{yx_2} + iu_{tx_2}, u_{yx_1} - iu_{tx_1})^\top + O(\xi), \qquad \xi := \lambda + i,$$

as $|\xi| \to 0$. In the case, when

$$
\begin{aligned}
\nabla h_-^{(y)}(\tilde{l}) &:= (\mu^{-1}\nabla\gamma^{(1)}(\tilde{l}) + \xi^{-1}\nabla\gamma^{(2)}(\tilde{l}))|_- = \\
&= \frac{1}{2\mu}\left((-u_{yx_2} - iu_{tx_2})\frac{\partial}{\partial x_1} + (u_{yx_1} + iu_{tx_1})\frac{\partial}{\partial x_2}\right) + \\
&+ \frac{1}{2\xi}\left((-u_{yx_2} + iu_{tx_2})\frac{\partial}{\partial x_1} + (u_{yx_1} - iu_{tx_1})\frac{\partial}{\partial x_2}\right) = \\
&= \frac{u_{tx_2} - \lambda u_{yx_2}}{\lambda^2+1}\frac{\partial}{\partial x_1} + \frac{\lambda u_{yx_1} - u_{tx_1}}{\lambda^2+1}\frac{\partial}{\partial x_2},
\end{aligned}
$$

$$
\begin{aligned}
\nabla h_-^{(t)}(\tilde{l}) &:= (-\mu^{-1}i\nabla\gamma^{(1)}(\tilde{l}) + \xi^{-1}i\nabla\gamma^{(2)}(\tilde{l}))|_- = \\
&= \frac{1}{2\mu}\left((-u_{tx_2} + iu_{yx_2})\frac{\partial}{\partial x_1} + (u_{tx_1} - iu_{yx_1})\frac{\partial}{\partial x_2}\right) + \\
&+ \frac{1}{2\xi}\left(-(u_{tx_2} + iu_{yx_2})\frac{\partial}{\partial x_1} + (u_{tx_1} + iu_{yx_1})\frac{\partial}{\partial x_2}\right) = \\
&= -\frac{u_{yx_2} + \lambda u_{tx_2}}{\lambda^2+1}\frac{\partial}{\partial x_1} + \frac{u_{yx_1} + \lambda u_{tx_1}}{\lambda^2+1}\frac{\partial}{\partial x_2},
\end{aligned}
$$

the commutability condition (3.35) of the vector fields $\partial/\partial y + \nabla h_-^{(y)}(\tilde{l})$ and $\partial/\partial t + \nabla h_-^{(t)}(\tilde{l})$ leads to the Husain heavenly equation [2]:

$$
u_{yy} + u_{tt} + u_{yx_1}u_{tx_2} - u_{yx_2}u_{tx_1} = 0,
$$

with the Lax-Sato representation given by the first order partial differential equations

$$
\begin{aligned}
\frac{\partial\psi}{\partial y} + \frac{u_{tx_2} - \lambda u_{yx_2}}{\lambda^2+1}\frac{\partial\psi}{\partial x_1} + \frac{\lambda u_{yx_1} - u_{tx_1}}{\lambda^2+1}\frac{\partial\psi}{\partial x_2} &= 0, \\
\frac{\partial\psi}{\partial t} - \frac{u_{yx_2} + \lambda u_{tx_2}}{\lambda^2+1}\frac{\partial\psi}{\partial x_1} + \frac{u_{yx_1} + \lambda u_{tx_1}}{\lambda^2+1}\frac{\partial\psi}{\partial x_2} &= 0,
\end{aligned}
$$

where $\psi \in C^2(\mathbb{T}^2 \times \mathbb{R}^2; \mathbb{C})$.

4. Conclusion

We succeeded in applying the Lie-algebraic approach to studying vector fields on the complexified n-dimensional torus and the related Lie-algebraic structures, which made it possible to construct a wide class of multidimensional dispersionless integrable systems, describing conformal structure generating equations of modern mathematical physics.

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