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December 10, 2019

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ABSTRACT

In this paper, i have generalized the result of Murthy et al. [15] in metric spaces by replacing the containment condition and giving the shorter proof than of the authors in the main result.

Keywords and phrases: Common fixed point, Compatible mapping, Property (E.A.), Common property (E.A.), Occasionally weakly compatible maps, Coincidence points.

AMS (2010) subject classifications: 47H10, 54H25.

1. INTRODUCTION

Aamri and Moutawakil [1] introduced the concept of property (E.A.) which was perhaps inspired by the condition of compatibility introduced by Jungck [11] and further Imdad and Ali[10] extended this result . Recently Babu and Alemayehu [7, 8,9] proved common fixed point theorem for occasionally weakly compatible maps satisfying property (E.A.) using an inequality involving quadratic terms. Aliouche[4] proved a common fixed point theorem of Gregus type weakly compatible mappings satisfying generalized contractive conditions.

Abbas [2] established a common fixed point for Lipschitzian mapping satisfying rational contractive conditions.

2. PRELIMINARIES

Throughout this paper (X, d) is a metric space which is denoted by X.

Definition 2.1: [Jungck and Rhoades [13]]. Let A and S be selfmaps of a set X. If $Au = Su = \omega$ (say), $\omega \in X$, for some u in X, then u is called a coincidence point of A

and S and the set of coincidence points of A and S is denoted by C (A, S), and ω is called a point of coincidence of A and S.

Definition 2.2: Let A, B, S and T be self maps of a set X. If $u \in C(A,S)$ and $v \in C(B,T)$ for some $u, v \in X$ and $Au = Su = Bv = Tv = z$ (say), then z is called a common point of coincidence of the pairs (A. S) and (B. T).

Definition 2.3: The pair (A, S) is said to be

(I) Satisfy property ($E.A.$) [1] if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some t in X.

(II) Copatible [11] if $\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$, for some t in X whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$.

(III) Weakly compatible [12], if they commute at their coincidence point.

(IV) Occasionally weakly compatible (owc) [3,5,6] if $ASx = SAx$ for some $x \in C(A,S)$.

Remark 2.4

(I) Every compatible pair is weakly compatible but its converse need not be true [12].

(II) Weak compatibility and property (E. A.) are independent of each other [16].

(III) Every weakly compatible pair is occasionally weakly compatible but its converse need not be true [11].

(IV) Occasionally weakly compatible and property ($E.A.$) are independent of each other [8].

Definition 2.5: [14] Let (X, d) be a metric space and A, B, S and T be four selfmaps on X. The pairs (A,S) and (B,T) are said to satisfy common property ($E.A.$) if there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n$ for some t in X.

Remark 2.6: Let A,B,S and T be self maps of a set X. If the pairs (A, S) and (B, T) have common point of coincidence in X then $C(A, S) \neq \phi$ and $C(B, T) \neq \phi$. But converse is not true.

Example 2.7: Let $X = [0, \infty)$ with usual metric and A, B, S and T self maps on x and defined by $Ax = 1 - x^2; sx = 1 - x; Bx = \frac{1}{2} + x^2; Tx = \frac{1+x}{2}$ for all $x \in X$.

It is easy to observe that $C(A,S) = \{0,1\}$ and $C(B,T) = \left\{0, \frac{1}{2}\right\}$ but the pairs (A, S) and (B, T) not having common point of coincidence.

Remark 2.8: The converse of the remark 2.6 is true provided it satisfies inequality (3.1). This is given as in proposition 3.1 in section III.

Preposition 2.9: [2] Let A and S be two self maps of a set X and the pair (A, S) is satisfies occasionally weakly compatible (owc) condition. If the pairs (A, S) have unique point of coincidence $Ax = Sx = z$ then z is the unique common fixed point of A and S.

Proof: To be given $Ax = Sx = \{z\}$ (say) for any $x \in C(A,S)$ (2.1)
Since the pair (A, S) satisfies the property owc, therefore

$Az = ASx = SAx = Sz$ implies that $z \in C(A,S)$

From (2.1), we get $Az = Sz = z$. Hence proposition follows.

In 1996, Tas et al. [17] proved the following.

Theorem 2.10: Let A, B, S and T be selfmaps of a complete metric space (X,d) such that $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$ and satisfying the inequality.

$$\begin{aligned}[d(Ax,By)]^2 \leq C_1 \max&\left\{[d(Sx,Ax)]^2, [d(Ty,By)]^2, [d(Sx,Ty)]^2\right\} \\ &+C_2 \max\{d(Sx,Ax)d(Sx,By), d(Ty,Ax)d(Ty,By)\} \\ &+C_3 d(Sx,By)d(Ty,Ax)\end{aligned}$$

for all $x, y \in x$, where $C_1 + C_3, C_2, C_3 \geq 0, C_1 + 2C_2 < 1, C_1 + C_3 < 1$. Further, assume that the pairs (A,S) and (B,T) are compatible onX. If one of the mappings A,B,S and T is continuous then A,B,S and T have a unique common fixed point in X.

3. MAIN RESULTS

Proposition 3.1. Let *A, B, S* and *T* be self maps of a metric space (*X*, *d*) and satisfying the inequality.

$$[d(Ax,By)]^2 \le \alpha[d(Sx,Axd)(Ty,By)+d(Ty,Ax)d(Sx,By)]$$
$$+\beta[d(Sx,Ax)d(Sx,By)+d(Ty,By)d(Ty,Ax)]$$
$$+\gamma d(Sx,By)d(Ty,Ax) \qquad (3.1)$$

for all $x,y\in x$, where $\alpha,\beta,\gamma \ge 0$ and $\alpha+\gamma<1$. Then the pairs (A,S) and (B,T) have common point of coincidence in X if and only if $C(A,S)\ne\phi$ and $C(B,T)\ne\phi$.

Prof: If part: It is trivial

Only if part: Assume $C(A,S)\ne\phi$ and $C(B,T)\ne\phi$.
Then there is a $u\,\varepsilon\, C(A,S)$ and $v\,\varepsilon\, C(B,T)$ such that

$$Au=Su=p \qquad \text{(say)} \qquad (3.2)$$
$$Bv=Tv=q \qquad \text{(say)} \qquad (3.3)$$

on taking $x=u$ and $y=v$ in (3.1), we get

$$[d(Au,Bv)]^2 \le\propto [d(Su,Au)d(Tv,Bv)+d(Tv,Au)d(Su,Bv)]$$
$$+B[d(Su,Au)d(Su,Bv)+d(Tv,Bv)d(Tv,Au)]$$
$$+\upsilon d(Su,Bv)d(Tv,Au)$$

Using (3.2) and (3.3), we get

$[d(p,q)]^2 \le (\propto+\upsilon)[d(p,q)]^2$, a contradiction. Thus $p=q$

Therefore A,B,S and T have common point of coincidence in X.

In The proposition (2.1) of Babu et al. [9], we can obtain some more conclusions of in their paper. Therefore our result improves and strengthen proposition 3.1 and subsequent theorems in metric spaces.

Proposition 3.2: Let *A, B, S* and *T* be four self maps of a metric space (X,d) satisfying the inequality (3.1). Suppose that either

(i) $B(X)\subseteq S(X)$, the pair (B,T) satisfies property (E.A.) and $T(X)$ is a closed subspace of *X*; or

(ii) $A(X)\subseteq T(X)$, the pair (A,S) satisfies property $(E.A)$ and $S(X)$ is a closed subspace of X holds.

Then the pair (A,S) and (B,T) are satisfies the common property (*E*.A), also both the pairs (A,S) and (B,T) have common point of coincidence in *X*.

We have shorten the proof of theorem 2.2 of [9] by relaxing many lines:

Theorem 3.3: (Improved version of theorem 2.2of [9])
Let A,B,S and T are satisfying all the conditions given in proposition 3.2 with the following additional assumption.
The pairs (A,S) and (B,T) are owc on X.
Then A,B,S and T have a unique common fixed point in X.

Proof: By proposition 3.2 the pairs (A,S) and (B,T) have common point of coincidence. Therefore there is $u \in C(A,S)$ and $v \in C(B,T)$ such that

$$Au = Su = z \text{ (say)} = Bv = Tv \tag{3.4}$$

Now, we show that z is unique common point of coincidence of the pairs (A,S) and (B,T).
Let if possible z' is another point of coincidence of A,B,S and T. Then there is $u' \varepsilon C(A,S)$ and $v' \varepsilon C(B,T)$ such that

$$Au' = Su' = z' \text{ (say)} = Bv' = Tv' \tag{3.5}$$

Putting $x = u$ and $y = v'$ in inequality (3.1), we have

$$\begin{aligned}[d(Au,Bv')]^2 \le\ & \alpha[d(Su,Au)d(Tv',Bv') + d(Tv',Au)d(Su,Bv')] \\ & +\beta[d(Su,Au)d(Su,Bv') + d(Tv',Bv')d(Tv',Au)] \\ & +\gamma d(Su,Bv')d(Tv',Au)\end{aligned}$$

Now using (3.4) and (3.5), we get
$[d(z,z')]^2 \le (\alpha+\gamma)[d(z,z')]^2$, and arrive at a contradiction. Hence $z = z'$ and we have $C(A,S) = \{z\} = C(B,T)$. By proposition 2.9, z is the unique common fixed point of A,B,S and T in X.

Remark 3.4: Proposition 2.5 of [9] and theorem 2.6 of [9] are remain true, if we replace completeness of $S(X)$ and $T(X)$ by the completeness of $S(X) \cap T(X)$ in X. For this we have given an example 2.7 in the following manner without proof.
Now we rewriting the proposition 2.5 and theorem 2.6 of [9]

Proposition 3.5: Let *A, B, S* and *T* be four self maps of a metric. Space (X, d) satisfying the inequality (3.1) of proposition 3.1. Suppose that (A,S) and (B,T) satisfy a common property $(E.A)$ and $S(X) \cap T(X)$ are closed subset of X, then *A, B, S* and *T* have unique common point of coincidence. Theorem 3.6. In addition to the above proposition 3.5 on *A, B, S* and *T*, if both the pairs (A, S) and (B, T) are owc mapson X, then the point of coincidence is a unique common point of A,B,S and T.

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