An Investigation on a 2-D Problem of Mode-I Crack in Generalized Thermoelasticity

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Abstract: This paper is concerned with the dynamical problem of an infinite type-III which occurs a finite linear mode-I crack due to load inside the homogeneous and isotropic medium of thermoelastic space. The temperature distribution and stress leads to the crack in the boundary. The basic governing equation developed by Green and Naghdi have been solved by using integral transform and reduces to four dual integral equation by employing boundary conditions which is equivalent to Fredholm's integral equation of first kind. For numerical solution inversion of Laplace transform has been used with the help of software Mathematica version 6.0. By taking particular cases of copper material, values of temperature, stress and displacement in the neighbourhood of crack are computed numerically and observed graphically.

Keywords: Thermoelasticity;Mode-I crack; Thermoelasticity of type III; Dual integral equations;Fredholm's integral equation.

Introduction: Biot [1] proposed the classical coupled dynamical theory of thermoelasticity by taking into account that the elastic changes have an effect on the temperature and vice versa. The detailed discussions of this theory have been investigated by several researchers like, Dhaliwal and Singh [8], Carlson [4], Nowacki [5], Parkus [6], Boley and Wiener [3], Nowinski [7], Chadwick [2] etc. However, this theory is based on Fourier's law of heat conduction which when combined with the law of conservation of energy gives rise to a partial differential type heat conduction equation and hence predicts an infinite speed of propagation of thermal signal contradicting the physical fact. We recall the developments of a second sound theory for thermoelasticity by Fox [9] who used the principles of modern continuum thermodynamics. In this regard we also mention the two well established generalized thermoelasticity theories, the temperature-rate dependent thermoelasticity theory proposed by Green and Lindsay [11] and extended thermoelasticity theory developed by Lord and Shulman [10]. The review research articles by Hetnarski and Ignaczak [14], Chandrasekharaiah [12, 13] and a recent book by Ignaczak and Ostoja-Starzewski [15] may be mentioned for a detailed study.

During the period (1992-1995), a theory that is considered to be formulation of heat propagation has been proposed by Green and Naghdi [16-19]. In this theory, thermal pulse transmission has been incorporated in a very logical manner and is divided into three parts which are subsequently referred to as thermoelasticity of types I, II and III. The novelty of their formulation is that the usual entropy production inequality is replaced here with an entropy balance law Chandrasekharaiah [13]).

The thermoelasticity theories proposed by Green and Naghdi have drawn the attention of several researchers during last few years. Quintanilla [21] and Quintanilla and Straughan [20] have proved the uniqueness theorem and have discussed the growth of solutions in the contexts of both the thermoelasticity type II & III theories. Further, Quintanilla [22] proved the impossibility of the localization in time of the solutions of linear thermoelasticity for the theories of Green and Naghdi. The nature of discontinuity waves propagating in type III thermoelastic media has been reported by Quintanilla and Straughan [23]. Puri and Jordan [24], Kovalev and Radayev [25] and later on, and Kothari and Mukhopadhyay [26] have discussed the harmonic plane waves propagating in thermoelastic media of type III. Recently, the variational and reciprocity theorems in the contexts of linear theory of thermoelasticity of type II and type III are developed by Chirita and Ciarletta [27] and Mukhopadhyay and Prasad [28], respectively. The theory of cracks in two dimensional medium was first studied by Griffith [29]. In Irwin's notation [30] Mode-I denotes a symmetric opening the relative displacements of the medium being normal to the fracture surface. It can be noted that crack growth usually takes place in Mode-I or close to it. It must be mentioned that thermal stresses play a very important role in building structural elements, like machines, gas or stream turbines, aircrafts etc.

1. Basic governing equations:

We consider an infinite elastic medium $-\infty < x < \infty, -\infty < y < \infty,$ which derives a two-dimensional dynamical problem that contains a Mode-I (opening mode) crack defined by $|x| \le a,\ y = \pm 0.$ The temperature and normal stress distribution gives rise to crack in the surface. Green and Naghdi [16] develop a theory of thermoelasticity without energy dissipation.

The equations of motion are

$$(\lambda + \mu)\frac{\partial e}{\partial x} + \mu \nabla^2 u = \gamma \frac{\partial T}{\partial x} + \rho \frac{\partial^2 u}{\partial t^2} \tag{1}$$

$$(\lambda + \mu)\frac{\partial e}{\partial y} + \mu \nabla^2 v = \gamma \frac{\partial T}{\partial y} + \rho \frac{\partial^2 v}{\partial t^2} \tag{2}$$

The equation of energy dissipation is

$$\left(K^* + K\frac{\partial}{\partial t}\right)\nabla^2 T = \frac{\partial^2}{\partial t^2}\left(\rho c_v T + \gamma T_0 e\right) \tag{3}$$

The constitutive relations of the stress tensor σ_{ij} in terms of displacement and temperature are

$$\sigma_{xx} = 2\mu u_x + \lambda e - \gamma\left(T - T_0\right) \tag{4}$$

$$\sigma_{yy} = 2\mu v_y + \lambda e - \gamma\left(T - T_0\right) \tag{5}$$

$$\sigma_{xy} = \mu\left(u_x + v_y\right) \tag{6}$$

In the above equations, t is the time parameter, u and v are the displacement components along the x and y directions respectively, T is the absolute temperature, T_0 is the reference temperature, ρ is the mass density, K is the thermal conductivity, K^* is the rate of thermal conductivity, c_v is the specific heat at constant strain, λ and μ are Lame's elastic constants and γ is a material constant given by $\gamma = (3\lambda + 2\mu)\alpha_t$, where α_t is the coefficient of linear thermal expansion. ∇^2 is the Laplacian operator and e is the dilatation given by

$$e = u_x + v_y \tag{7}$$

We use the following dimensionless variables as [31]

$$x' = c_1\eta x,\ y' = c_1\eta y,\ u' = c_1\eta u,\ v' = c_1\eta v,\ t' = c_1^2\eta t,\ \sigma'_{ij} = \frac{\sigma_{ij}}{\mu},\ \theta = \frac{T - T_0}{T_0}$$

with $\eta = \dfrac{\rho c_v}{K}, c_1 = \sqrt{\dfrac{\lambda + 2\mu}{\rho}}$, where c_1 is the speed of propagation of isothermal elastic waves.

Thus equations (1) - (6) reduce to the dimensionless forms (after dropping the primes for convenience) as

$$\left(\alpha^2 - 1\right)\frac{\partial e}{\partial x} + \nabla^2 u = b_1\frac{\partial \theta}{\partial x} + \alpha^2\frac{\partial^2 u}{\partial t^2} \tag{8}$$

$$\left(\alpha^2 - 1\right)\frac{\partial e}{\partial y} + \nabla^2 v = b_1\frac{\partial \theta}{\partial y} + \alpha^2\frac{\partial^2 v}{\partial t^2} \tag{9}$$

$$\left(a_0 + \frac{\partial}{\partial t}\right)\nabla^2\theta = \frac{\partial^2}{\partial t^2}\left(\theta + b_2 e\right) \tag{10}$$

$$\sigma_{xx} = 2u_x + \left(\alpha^2 - 2\right)e - b_1\theta \tag{11}$$

$$\sigma_{yy} = 2v_y + \left(\alpha^2 - 2\right)e - b_1\theta \tag{12}$$

$$\sigma_{xy} = u_y + v_x\ , \tag{13}$$

where, $a_0 = \dfrac{K^*}{Kc_1^2\eta}, b_1 = \dfrac{\gamma T_0}{\mu}, b_2 = \dfrac{\gamma}{K\eta}, \alpha^2 = \dfrac{\lambda + 2\mu}{\mu}$.

Eliminating u and v from equations (8) and (9) and using equation (7), we get

$$\left(\nabla^2 - \frac{\partial^2}{\partial t^2}\right)e = c\nabla^2\theta \ , \tag{14}$$

where, $c = \dfrac{b_1}{\alpha^2}$.

2. Solution in the Laplace and Fourier transform domain

The Laplace transform of a function $f\left(x, y, t\right)$ is defined as

$$\tilde{f}\left(x, y, p\right) = L\left[\, f\left(x,y,t\right)\right] = \int_0^\infty e^{-pt} f\left(x,y,t\right)dt\,, \quad p>0$$

where, p is the Laplace transform parameter.

By taking Laplace transform to both sides of equations (7)-(10) and (14), we get the following relations

$$\tilde{e} = \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} \tag{15}$$

$$b_1 \frac{\partial \tilde{\theta}}{\partial x} - \left(\alpha^2 - 1\right)\frac{\partial \tilde{e}}{\partial x} = \left(\nabla^2 - \alpha^2 p^2\right)\tilde{u} \tag{16}$$

$$b_1 \frac{\partial \tilde{\theta}}{\partial y} - \left(\alpha^2 - 1\right)\frac{\partial \tilde{e}}{\partial y} = \left(\nabla^2 - \alpha^2 p^2\right)\tilde{v} \tag{17}$$

$$\left[\left(a_0 + p\right)\nabla^2 - p^2\right]\tilde{\theta} = p^2 b_2 \tilde{e} \tag{18}$$

$$\left(\nabla^2 - p^2\right)\tilde{e} = c\nabla^2\tilde{\theta} \tag{19}$$

Eliminating $\tilde{e}$ from equations (18) and (19), we obtain the partial differential equation satisfied by $\tilde{\theta}$ as

$$\left(\nabla^2 - m_1^2\right)\left(\nabla^2 - m_2^2\right)\tilde{\theta} = 0 \tag{20}$$

where, m_1^2 and m_2^2 are the roots of the characteristic equation

$$\left(a_0 + p\right)m^4 - p^2\left(1 + a_0 + p + \varepsilon\right)m^2 + p^4 = 0 \tag{21}$$

where, $\varepsilon = cb_2$.

Solving (20), we can write $\tilde{\theta}$ in the form $\tilde{\theta} = \tilde{\theta}_1 + \tilde{\theta}_2$, where $\tilde{\theta}_i$ are solutions of the equations

$$\left(\nabla^2 - m_i^2\right)\tilde{\theta}_i = 0, \qquad i = 1, 2 \tag{22}$$

The exponential Fourier transform of a function $f(x,y,p)$ is defined as $f^{*}(q,y,p)=F\left[f(x,y,p)\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x,y,p)e^{-iqx}dx.$, where q is the Fourier transform parameter.

The inverse Fourier transform is given by the relation

$$f(x,y,p)=F^{-1}\left[f^{*}(q,y,p)\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f^{*}(q,y,p)e^{iqx}dq$$

Now, we take exponential Fourier transform to both sides of equation (22) w. r. t. x to obtain

$$\left[D^{2}-\left(m_{i}^{2}+q^{2}\right)\right]\tilde{\theta}_{i}^{*}=0, \tag{23}$$

where, $D\equiv\frac{\partial}{\partial y}$.

The solution of equation (23) bounded at infinity can be obtained in the form

$$\tilde{\theta}_{i}^{*}=G_{i}(q,p)\left(m_{i}^{2}-p^{2}\right)e^{-q_{i}|y|}, \tag{24a}$$

where, $q_i^2=q^2+m_i^2$ and $G_i(q,p)$ are Function of q and p for $i=1,2$.

Due to symmetry, we consider only the case $y>0$. The equation (24a) then becomes

$$\tilde{\theta}_{i}^{*}=G_{i}(q,p)\left(m_{i}^{2}-p^{2}\right)e^{-q_{i}y},\qquad i=1,2 \tag{24b}$$

Eliminating $\tilde{\theta}$ from equations (18) and (19), in a similar way we can find $\tilde{e}^{*}=\tilde{e}_{1}^{*}+\tilde{e}_{2}^{*}$,

where, $$\tilde{e}_{i}^{*}=G_{i}^{'}(q,p)e^{-q_{i}y},\qquad i=1,2 \tag{25}$$

and $G_i^{'}(q,p)$, $i=1,2$ are parameter depending on q and p.

Now, substituting equations (24b) and (25) into equation (19), we find the relation satisfied by the parameters $G_i(q,p)$ and $G_i^{'}(q,p)$ as

$$G_{i}^{'}(q,p)=cm_{i}^{2}G_{i}(q,p) \tag{26}$$

Therefore, substitute equation (26) into equation (25), we obtain

$$\tilde{e}_{i}^{*}=cG_{i}(q,p)m_{i}^{2}e^{-q_{i}y} \tag{27}$$

Next, we take exponential Fourier transform to equations (16) and (17) to obtain

$$\left(D^{2}-q^{2}-\alpha^{2}p^{2}\right)\tilde{u}^{*}=qb_{1}\tilde{\theta}^{*}-\left(\alpha^{2}-1\right)iq\tilde{e}^{*} \tag{28}$$

$$\left(D^{2}-q^{2}-\alpha^{2}p^{2}\right)\tilde{v}^{*}=b_{1}D\tilde{\theta}^{*}-\left(\alpha^{2}-1\right)D\tilde{e}^{*} \tag{29}$$

Using equations (24) and (27), equations (28) and (29) give rise to

$$\left(D^2 - q^2 - \alpha^2 p^2\right)\tilde{u}^* = iqc\sum_{i=1}^{2}\left(m_i^2 - \alpha^2 p^2\right)G_i\left(q,p\right)e^{-q_i y} \tag{30}$$

$$\left(D^2 - q^2 - \alpha^2 p^2\right)\overline{v}^* = -c\sum_{i=1}^{2}\left(m_i^2 - \alpha^2 p^2\right)G_i\left(q,p\right)q_i e^{-q_i y} \tag{31}$$

Now, the solution $\tilde{u}^*$ of equation (30) has the form

$$\tilde{u}^* = iqc\left(\sum_{i=1}^{2}G_i e^{-q_i y} + H_1 e^{-\delta y}\right), \tag{32}$$

Where, $\delta = \sqrt{q^2 + \alpha^2 p^2}$ and $H_1 = H_1\left(q,p\right)$ is a parameter depending on q and p.

Taking the exponential Fourier transform with respect to x to both sides of equation (15), we obtain

$$\frac{\partial \tilde{v}^*}{\partial y} = \tilde{e}^* - iq\tilde{u}^* \tag{33}$$

Substituting from equations (27) and (32) into the right hand side of equation (33) and integrating both sides of the resulting equation with respect to y, we get

$$\tilde{v}^* = -c\left[\sum_{i=1}^{2}G_i\left(q,p\right)q_i e^{-q_i y} + \frac{q^2 H_1\left(q,p\right)}{\delta}e^{-\delta y}\right] \tag{34}$$

Again, applying the Laplace transform and then the exponential Fourier transforms to both sides of equations (11) - (13) and using equations (24), (27), (32) and (34), we obtain the components of the stress tensor in the Laplace and Fourier transform domain in the form

$$\tilde{\sigma}_{xx}^* = c\left[\sum_{j=1}^{2}G_j\left(\alpha^2 p^2 - 2q_j^2\right)e^{-q_j y} - 2H_1 q^2 e^{-\delta y}\right] \tag{35}$$

$$\tilde{\sigma}_{yy}^* = c\left[\left(\alpha^2 p^2 + 2q^2\right)\sum_{j=1}^{2}G_j e^{-q_j y} + 2H_1 q^2 e^{-\delta y}\right] \tag{36}$$

$$\tilde{\sigma}_{xy}^* = -icq\left[2\sum_{j=1}^{2}G_j q_j e^{-q_j y} + \frac{q^2 + \delta^2}{\delta}H_1 e^{-\delta y}\right] \tag{37}$$

By taking the inverse Fourier transforms of equations (24), (27), (32) and (34) - (37), we finally obtain

$$\tilde{\theta} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\sum_{j=1}^{2}G_j\left(m_j^2 - p^2\right)e^{-q_i y}\right]e^{iqx}dq \tag{38}$$

$$\tilde{e}=\frac{c}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\sum_{j=1}^{2}G_j m_j^2 e^{-q_j y}\right]e^{iqx}dq \tag{39}$$

$$\tilde{u}=\frac{ic}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\sum_{j=1}^{2}G_j e^{-q_j y}+H_1 e^{-\delta y}\right]qe^{iqx}dq \tag{40}$$

$$\tilde{v}=\frac{-c}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\sum_{j=1}^{2}G_j q_j e^{-q_j y}+\frac{H_1 q^2}{\delta}e^{-\delta y}\right]e^{iqx}dq \tag{41}$$

$$\tilde{\sigma}_{xx}=\frac{c}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\sum_{i=1}^{2}G_j\left(\alpha^2 p^2-2q_j^2\right)e^{-q_j y}-2H_1 q^2 e^{-\delta y}\right]e^{iqx}dq \tag{42}$$

$$\tilde{\sigma}_{yy}=\frac{c}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\left(\alpha^2 p^2+2q^2\right)\sum_{j=1}^{2}G_j e^{-q_j y}+2H_1 q^2 e^{-\delta y}\right]e^{iqx}dq \tag{43}$$

$$\tilde{\sigma}_{xy}=\frac{-ic}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[2\sum_{j=1}^{2}G_j q_j e^{-q_j y}+\frac{\left(q^2+\delta^2\right)}{\delta}H_1 e^{-\delta y}\right]qe^{iqx}dq \tag{44}$$

3. Boundary conditions and dual integral equation formulation

Assume the boundary conditions for heat conduction problem at $y=0$ as,

$$\frac{\partial\theta}{\partial y}=0,\qquad |x|>a \tag{45}$$

$$v=0,\qquad |x|>a \tag{46}$$

$$\theta=g(x)H(t),\qquad |x|<a \tag{47}$$

$$\sigma_{yy}=-p_1(x)H(t),\qquad |x|<a \tag{48}$$

$$\sigma_{xy}=0,\qquad -\infty<x<\infty, \tag{49}$$

where, $H(t)$ is the Heaviside unit step function.

Using the boundary conditions (45) and (47) together with the equation (38), we find

$$\int_{-\infty}^{\infty}\sum_{j=1}^{2}G_j\left(m_j^2-p^2\right)e^{iqx}dq=\frac{\sqrt{2\pi}\,g(x)}{p},\quad |x|<a \tag{50}$$

$$\int_{-\infty}^{\infty}\sum_{j=1}^{2}G_j q_j\left(m_j^2-p^2\right)e^{iqx}dq=0,\qquad |x|>a \tag{51}$$

Further, the boundary conditions (46), (48) and (49) together with equations (41), (43) and (44) yield

$$\int_{-\infty}^{\infty}\left[G_1q_1+G_2q_2+\frac{H_1q^2}{\delta}\right]e^{iqx}dq=0, \qquad |x|>a \tag{52}$$

$$\int_{-\infty}^{\infty}\left[\left(\alpha^2p^2+2q^2\right)\left(G_1+G_2\right)+2q^2H_1\right]e^{iqx}dq=-\frac{\sqrt{2\pi}\,p_1(x)}{pc}, \qquad |x|<a \tag{53}$$

$$\int_{-\infty}^{\infty}\left[2\left(G_1q_1+G_2q_2\right)+\frac{\left(q^2+\delta^2\right)}{\delta}H_1\right]qe^{iqx}dq=0, \qquad -\infty<x<\infty \tag{54}$$

Using equation (54), we obtain $H_1=-\dfrac{2\delta\left(G_1q_1+G_2q_2\right)}{q^2+\delta^2}$ (55)

In view of equation (55), equations (50) - (53) yield

$$\sum_{i=1}^{2}\left(m_i^2-p^2\right)\int_0^{\infty}G_i\cos(qx)dq=\sqrt{\frac{\pi}{2}}\frac{g(x)}{p}, \qquad 0<x<a \tag{56}$$

$$\sum_{i=1}^{2}\left(m_i^2-p^2\right)\int_0^{\infty}G_iq_i\cos(qx)dq=0, \qquad x>a \tag{57}$$

$$\sum_{i=1}^{2}\int_0^{\infty}\frac{G_iq_i}{\alpha^2p^2+2q^2}\cos(qx)dq=0, \qquad x>a \tag{58}$$

$$\sum_{i=1}^{2}\int_0^{\infty}G_i\left[\frac{\left(\alpha^2p^2+2q^2\right)^2-4q^2q_i\delta}{\alpha^2p^2+2q^2}\right]\cos(qx)dq=-\sqrt{\frac{\pi}{2}}\frac{p_1(x)}{pc}, \qquad 0<x<a \tag{59}$$

In equations (56) - (59), we have used the symmetry of the problem to consider x only in the intervals $[0,a]$ and $[a,\infty)$ [31].

Equations (56) - (59) thus form a set of four dual integral equations whose solution will give the unknown parameters G_1 and G_2. In order to solve these equations, we follow Sherief and El-Maghraby [31]., S. Kant et al. [32] ,Sur and Mondal [33] and take the substitution as

$$G_i(q,p)=\int_0^{a}h_i(w,p)J_0(qw)dw, \qquad x<a, \qquad i=1,2, \tag{60}$$

where, h_i are functions of v and p only and J_0 denotes the Bessel function of the first kind of order zero.

Substituting equation (60) into equation (56) and changing the order of integration, we get

$$\sum_{i=1}^{2}\left(m_i^2-p^2\right)\int_0^a h_i\left(v,p\right)dv\int_0^\infty \cos\left(qx\right)J_0\left(qv\right)dq=\sqrt{\frac{\pi}{2}}\frac{g\left(x\right)}{p}, \qquad 0<x<a \tag{61}$$

Using the following integral relation of the Bessel function [34, 35]

$$\int_0^\infty \cos\left(tx\right)J_0\left(tv\right)dt=\begin{cases}\dfrac{1}{\sqrt{v^2-x^2}} & \text{when } x<v,\\ 0 & \text{when } x>v,\end{cases} \tag{62}$$

Equation (61) reduces to

$$\sum_{i=1}^{2}\left(m_i^2-p^2\right)\int_x^\infty \frac{h_i\left(u,p\right)du}{\sqrt{u^2-x^2}}=\sqrt{\frac{\pi}{2}}\frac{g\left(x\right)}{p}, \qquad 0<x<a$$

Multiplying both sides of the above equation by $\frac{x}{\sqrt{x^2-v^2}}$ and integrating with respect to x from v to a, we get, after changing the order of integration and differentiating with respect to v as

$$\left(m_1^2-p^2\right)h_1\left(v,p\right)+\left(m_2^2-p^2\right)h_2\left(v,p\right)=-\frac{A\left(v\right)}{p}, \qquad 0<x<a \ , \tag{63}$$

$$\text{where, } A\left(v\right)=\sqrt{\frac{2}{\pi}}\frac{d}{dv}\int_v^a \frac{x\,g\left(x\right)dx}{\sqrt{x^2-v^2}} \tag{64}$$

Multiplying both sides of equation (63) by $J_0\left(qv\right)$ and integrating with respect to v from 0 to a, we finally get

$$G_2=\frac{-1}{m_2^2-p^2}\left[\frac{J\left(q\right)}{p}+\left(m_1^2-p^2\right)G_1\right], \qquad 0<x<a \ , \tag{65}$$

$$\text{where, } J\left(q\right)=\int_0^a A\left(v\right)J_0\left(qv\right)dv \tag{66}$$

In order to obtain a similar relation to equation (65) between G_1 and G_2 for the case when $x>a$, we obtain

$$q_i G_i\left(q,p\right)=\int_a^\infty h_i\left(v,p\right)J_0\left(qv\right)dv, \qquad x>a, \quad i=1,2 \tag{67}$$

Using equation (62) in equation (57) and changing the order of integration, we get

$$\sum_{i=1}^{2}\left(m_i^2-p^2\right)\int_x^\infty \frac{h_i\left(u,p\right)du}{\sqrt{u^2-x^2}}=0, \quad x>a$$

Multiplying both sides of the above equation by $\frac{x}{\sqrt{x^2-v^2}}$ and integrating with respect to x from v to ∞, changing the order of integration and with the help of equation (67), we get

$$G_2 = -\frac{\left(m_1^2 - p^2\right)q_1}{\left(m_2^2 - p^2\right)q_2} G_1, \qquad x > a \tag{68}$$

Substituting from equation (65) into equation (59), we have

$$\int_0^\infty \frac{G_1 q_1 M\left(q,p\right)}{2q^2+\alpha^2 p^2}\cos\left(qx\right)dq = \bar{R}\left(x,p\right), \qquad x<a\ , \tag{69}$$

where, $M\left(q,p\right) = \dfrac{\left(m_2^2-m_1^2\right)\left(2q^2+\alpha^2p^2\right)^2 - 4q^2\delta\left[q_1\left(m_2^2-p^2\right)-q_2\left(m_1^2-p^2\right)\right]}{q_1}$,

$$\bar{R}\left(x,p\right) = -\sqrt{\frac{2}{\pi}}\frac{\left(m_2^2-p^2\right)p_1\left(x\right)}{pc} + \frac{1}{p}\int_0^\infty J\left(q\right)\left[\frac{\left(2q^2+\alpha^2p^2\right)-4q^2\delta q_2}{2q^2+\alpha^2p^2}\right]\cos\left(qx\right)dq, \quad x<a \tag{70}$$

Substituting from equation (68) into equation (58), we get

$$\int_0^\infty \frac{G_1 q_1 \cos\left(qx\right)}{2q^2+\alpha^2p^2}dq = 0, \quad x>a \tag{71}$$

Therefore, the original four dual integral equations (56) - (59) consisting of the parameters G_1 and G_2 are now reduced to only two dual integral equations (69) and (71) in the parameter G_1 only.

4. Solution of the dual integral equations

In order to solve the two dual integral equations obtained in previous section, we take the substitution [31]

$$G_1\left(q,p\right) = \frac{\left(2q^2+\alpha^2p^2\right)}{q_1}\psi\left(q,p\right) \tag{72}$$

Equations (69) and (71) therefore reduce to

$$\int_0^\infty M\left(q,p\right)\psi\left(q,p\right)\cos\left(qx\right)dq = \bar{R}\left(x,p\right), \qquad 0<x<a \tag{73}$$

$$\int_0^\infty \psi\left(q,p\right)\cos\left(qx\right)dq = 0, \qquad x>a \tag{74}$$

We extend the definition of the integral in the left hand side of equation (74) to be defined for all positive values of x in the following manner:

$$\int_0^\infty \psi(q,p)\cos(qx)\,dq = \begin{cases} \sqrt{2\pi}\,\dfrac{d}{dx}\left[x\displaystyle\int_x^a \dfrac{\psi(z,p)\,dz}{\sqrt{z^2-x^2}} \right], & 0<x<a \\ 0\,, & x>a \end{cases} \qquad (75)$$

where, $\psi(z,p)$ is a function to be determined.

Since the left hand side of equation (75) as the Fourier cosine transform of the function $\psi(q,p)$, by using the Fourier cosine inverse formula [31, 36, 37] we therefore get

$$\psi(q,p) = \int_0^a \frac{d}{dx}\left(x\int_x^a \frac{\psi(z,p)\,dz}{\sqrt{z^2-x^2}} \right)\cos(qx)\,dx \qquad (76)$$

Using integration by parts, followed by changing the order of integration, we find

$$\psi(q,p) = q\int_0^a \psi(z,p)\,dz \int_0^z \frac{x\sin(qx)\,dx}{\sqrt{z^2-x^2}} \qquad (77)$$

Using the formula [34, 35]

$$\int_0^z \frac{x\sin(qx)\,dx}{\sqrt{z^2-x^2}} = \frac{\pi}{2} z\,J_1(qz),$$

We can write $\psi(q,p)$ in the form

$$\psi(q,p) = \frac{\pi q}{2}\int_0^a z\psi(z,p)\,J_1(qz)\,dz\,, \qquad (78)$$

Now, substituting from equation (78) into equation (73), we get

$$\int_0^a \bar{M}(z,x,p)\psi(z,p)\,dz = \bar{R}(x,p), \qquad x<a \qquad (79)$$

where, $\bar{M}(z,x,p) = \dfrac{\pi z}{2}\displaystyle\int_0^\infty qM(q,p)\,J_1(qz)\cos(qx)\,dq$

The equation (79) is a Fredholm’s integral equation of the first kind in the unknown function $\psi(z,p)$. We can obtain $\psi(z,p)$ by solving this equation numerically and thereby we can obtain $\psi(q,p)$ by substituting into equation (78) and using numerical integration techniques we can obtain G_1 by direct substitution into equation (72). The expression of G_2 is then obtained by using equation (65) for $x<a$ or by equation (68) for $x>a$. This completes the solution of the problem in the Laplace transform domain.

For solving the integral equation (79) numerically, we follow the regularization method (Delves and Mohammed [36]).

5. Numerical results and discussions

For simplicity, we take $g(x)=1, \quad p_1(x)=1.$ Therefore, from (64) we get

$$A(v) = -\sqrt{\frac{2}{\pi}}\frac{v}{\sqrt{a^2 - v^2}}$$

For numerical computation, we have considered the copper material whose material constants are taken as follows [44]:

$$\alpha = 2, \alpha_t = 1.78\times10^{-5}K^{-1}, c_1 = 4.158\times10^{3}ms^{-1}, c = 0.01, \rho = 8954kgm^{-3}, \eta = 8886.73ms^{-2}$$
$$a = 1, c_v = 383.1Jkg^{-1}K^{-1}, \lambda = 7.76\times10^{10}Nm^{-2}, \mu = 3.86\times10^{10}Nm^{-2}, \theta_0 = 1K, T_0 = 293K, b_1 = 0.042$$

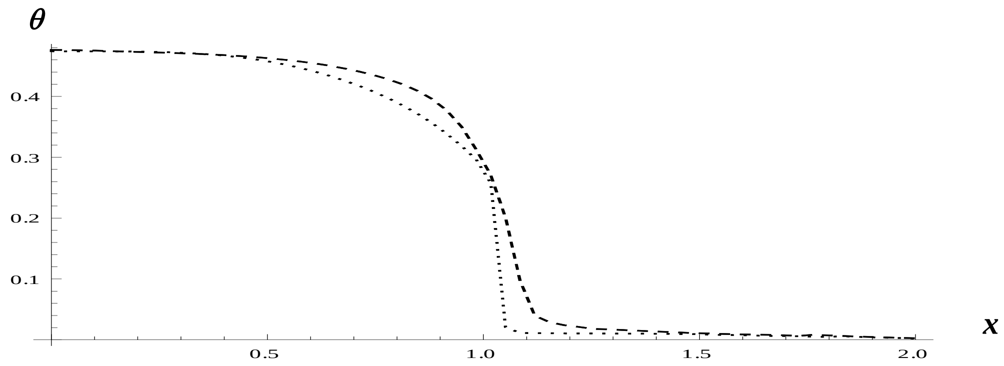

Fig.1. Temperature distribution at t = 0.69: dashed line - *y* = 0.1; dotted line - *y* = 0.2.

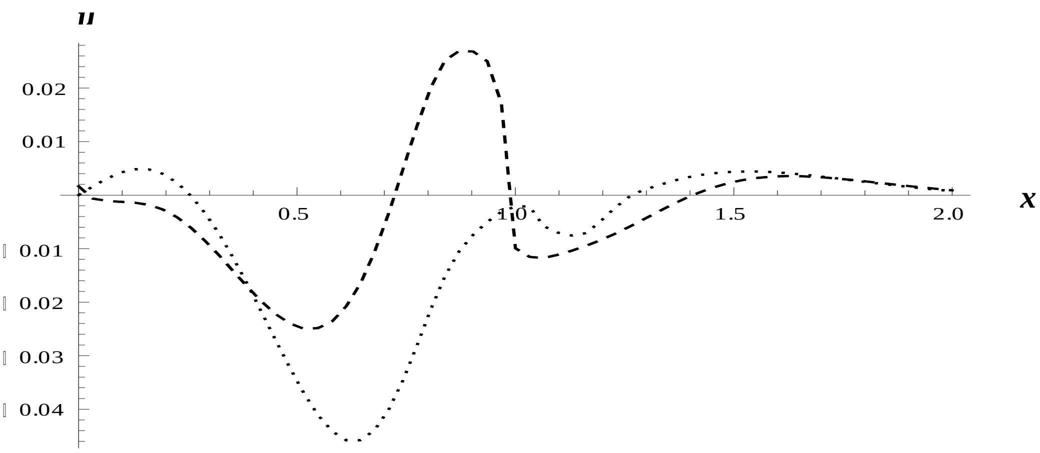

Fig.2. Horizontal displacement distribution at *t* = 0.69: dashed line - *y* = 0.1; dotted line - *y* = 0.2.

v

x

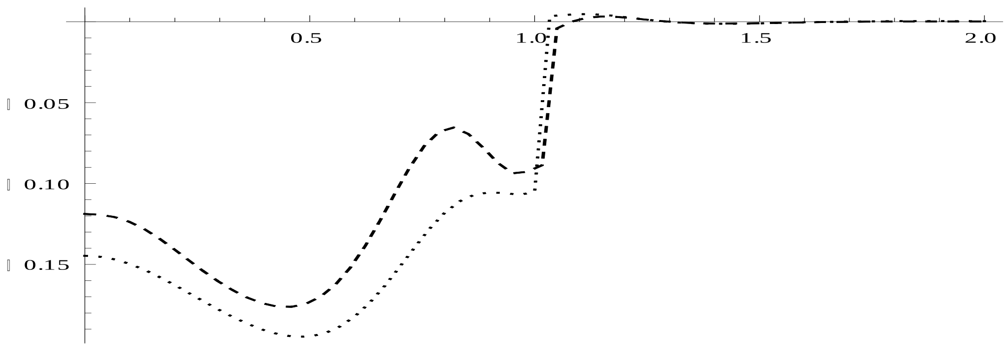

Fig.3. Vertical displacement distribution at t = 0.69: dashed line - y = 0.1; dotted line - y = 0.2.

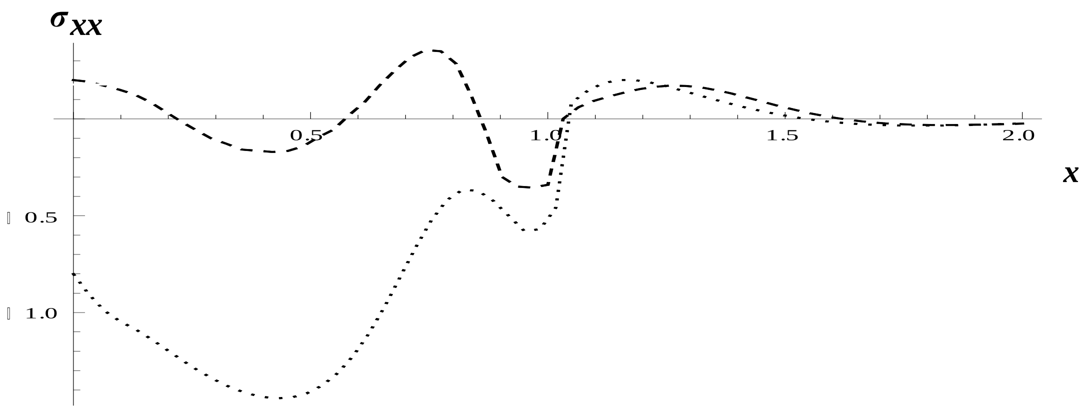

Fig.4. Horizontal stress distribution at t = 0.69: dashed line - y = 0.1; dotted line - y = 0.2.

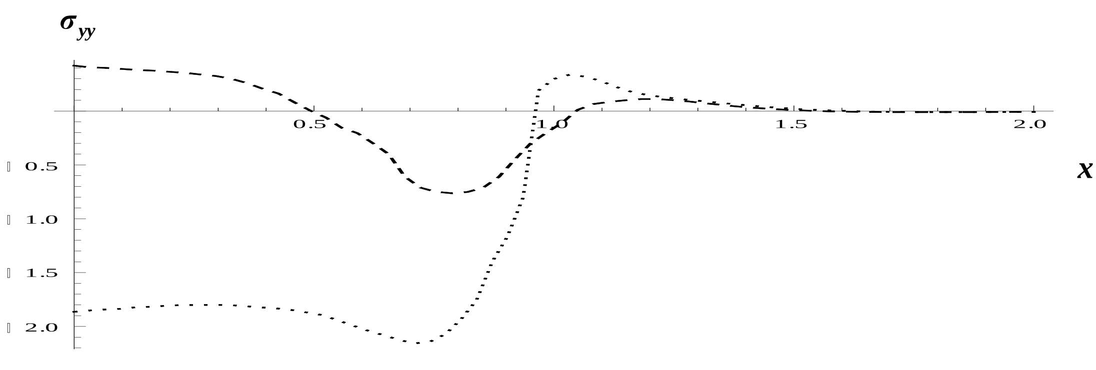

Fig.5. Vertical stress distribution at t = 0.69: dashed line - y = 0.1; dotted line - y = 0.2.

Using the software Mathematica 6.0 and computer programming, we carry out numerical computation and find out the numerical values of different non-dimensional physical fields like horizontal displacement, vertical displacement, temperature and tangential and normal stresses for different values of the vertical distance *y*. The results are plotted with respect to the horizontal distance *x* in Figures (1-5) to show the behavior of the fields near the crack region. All graphs are plotted for the non-dimensional time, $t = 0.69$. We assume the length of crack to be unity. In all the Figures the dashed lines and dotted lines correspond to the vertical distance $y = 0.1$ and $y = 0.2$, respectively.

6. Conclusion

In the present paper, a dynamical problem of an infinite two- dimensional elastic medium with a crack of Mode-I type under prescribed temperature and impact loading has been investigated. The authors have succeeded in achieving two important objectives. Firstly, using integral transforms and its inversion and also applying the integral equation technique successfully, the numerical calculations for the distributions of the important physical quantities viz., displacements, stresses and temperature for different particular cases have been completed. The most important part of the analysis is the study of variations of behavioral changes of the horizontal and vertical stresses in the vicinity of the crack. Thus it may be concluded that the present study of thermoelastic interactions in the elastic medium in presence of a crack will benefit the researchers working in the area of thermoelasticity valid for short time and long time effects.

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