Non-PDC Controller design for Uncertain Discrete-Time T-S Descriptor Models Subject to Input Saturation.

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Abstract— This paper is concerned with the stabilization of uncertain discrete-time descriptor models subject to input saturation and external disturbances. The design control strategy is based on Takagi-Sugeno (T-S) approach and a non parallel distributed compensation (non-PDC) control law. To synthesis the fuzzy controller, the stability conditions are derived using non-quadratic Lyapunov functions with respect to the given saturation constraint on the control input. The optimization problem is formulated in terms of linear matrix inequalities (LMIs). Numerical examples illustrate the efficiency of the proposed approaches.

I. INTRODUCTION

Recently, Takagi-Sugeno (T-S) models have been widely investigated to study nonlinear models [1]. Among nonlinear control theory, the T-S model-based approach has attracted great interest since it constitutes universal approximation of any smooth nonlinear function by a “blending” of some local linear system models. This method greatly facilitates observer/controller synthesis for complex nonlinear systems [2]. Based on this modeling technique, stability conditions have been obtained directly from Lyapunov methodology [3, 4]. For control design, the so-called parallel distributed compensation (PDC) has been the most commonly used scheme and remain to associate inferred state of output feedback to each local subsystem. Stabilization of T-S systems and the control design are investigated via the direct Lyapunov method. However, when fuzzy Lyapunov functions and parallel distributed compensation function (PDC) control are considered, the stabilization conditions are generally in terms of bilinear matrix inequalities especially for discrete-time T-S fuzzy [13]. To overcome such problem, a non-PDC control law can be applied based on non-quadratic Lyapunov function. Hence, more relaxed stabilization condition can be derived [3, 4-5]. The derived conditions are formulated into a set of linear matrix inequalities (LMI). These LMIs can be solved, when a solution exists, by classical convex optimization algorithms. In practice, many systems are physically described by nonlinear descriptor models. The T-S descriptor model representation has also the advantage to decrease the number of LMI constraints since it conserves nonlinearities in the left-hand side will keeping the original structure of the nonlinear model [7, 11-15].

Usually real physical applications suffer from actuator saturation and/or sensor saturation. Thus, a great attention has been given to the control design of T-S models with input saturation constraints [8-9]. Among the most popular works dealing with saturated input constraints, the convexity based approach to the saturation function (see [9-10] and the references therein). The interest of this approach is to consider a bounded ellipsoidal symmetric region of stability solved by a set of LMIs. Moreover, descriptor design approaches has been recently studied in [11] to deal with the problem of input saturated T-S systems using a polytopic representation of the saturation function. These design control approaches are based on state feedback or dynamic output feedback with anti-windup (AW) mechanisms.

In the present paper, stability analysis for uncertain discrete-time T-S descriptor models subject to input saturation and unknown disturbances is proposed. New LMI conditions are derived based on two Lyapunov functions with the Finsler’s lemma which allows decoupling the control law from the Lyapunov function. The $L_2$-gain performance is used to attenuate the extragenous disturbances and the derived conditions of asymptotic stability in the presence of input saturation are established and solved by means of LMI convex optimization.

This paper is organized as follows: Section II provides some useful notation and properties, it also introduces the uncertain discrete-time T-S descriptor model; Section III presents the LMI-based controller design for discrete-time T-S descriptor models subject to: uncertainties, input saturation and disturbances; Finally in section IV designed examples are given to demonstrate the effectiveness of the proposed approaches.

II. NOTATION AND PROBLEM STATEMENT

Given a set of nonlinear functions $h_i(\cdot) \geq 0, i \in \{1,\ldots,r\}$ having the convex sum property, $\sum_{i=1}^{r} h_i(\cdot) = 1$; a shorthand notation will be used in the sequel to represent convex sum of matrix expressions: $Y_h = \sum_{i=1}^{r} h_i(z(k))Y_i$ and $Y_v = \sum_{i=1}^{r} v_k(z(k))Y_k$ for single convex sum; $Y_{h+} = \sum_{i=1}^{r} h_i(z(k+1))Y_i = \sum_{i=1}^{r} h_i(z(k+1))Y_i$ for a delayed convex sum; $Y_{h-} = (\sum_{i=1}^{r} h_i(z(k))Y_i)^{-1}$ for the inverse of a convex sum; and, $Y_{hh} = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(z(k))h_j(z(k))Y_{ij}$ for a doubled rested convex sum. $H(A)$ denotes the Hermitian of the matrix $A$, i.e. $H(A) = A^* + A$. For a vector $x$, $x_k$ denotes its $k$th entry, and $x_k^*$ denotes $x(k+1)$. $n_k$ denotes the set $\{1,2,\ldots,r\}$. $I$ denotes the identity matrix. denotes the terms deduced by symmetry in symmetric block matrices.

The uncertain discrete-time T-S fuzzy model in the descriptor form subject to input saturation and external disturbances is given by following state equations:
\begin{equation}
E_p x_{k+} = (A_h + \Delta A_h)x_k + (B_h + \Delta B_h)\text{sat}(u(k)) + B_\omega \omega_k \quad (1)
\end{equation}

where \( x \in \mathbb{R}^n \) is the state vector, \( u \in \mathbb{R}^m \) is the control input vector, \( k \) is the current sample. Matrices \( A_i, B_i, i \in \mathbb{N}_r \) represent the \( i \)-th linear right-hand side model and \( E_v, k \in \{1, \ldots, r_e\} \), represents the \( k \)-th linear left-hand side model of T-S descriptor models. In the proposed work we suppose that \( E_v \) is regular matrix because it contains the inertia matrix which is motivated by mechanical systems [12]. The membership functions \( h_i(z(k)) \) satisfy the convex sum property and depend on the known premise variables. \( \Delta A_i(t) \in \mathbb{R}^{mxn} \) and \( \Delta B_i(t) \in \mathbb{R}^{mxm} \) contains the bounded uncertain terms which can be rewritten as:

\[
\Delta A_i(t) = H_{ai}D_{ai}(t)N_{ai} \quad \text{and} \quad \Delta B_i(t) = H_{bi}D_{bi}(t)N_{bi}
\]

with \( H_{ai}, H_{bi}, N_{ai}, \text{and} N_{bi} \) are known constant matrices and, \( D_{ai}(t) \) and \( D_{bi}(t) \) are unknown matrices functions bounded as: \( D_{ai}^T(t)D_{ai}(t) \leq I \) for all index \( i = a, b \).

The dead-zone nonlinearity \( \psi(\cdot): \mathbb{R}^m \to \mathbb{R}^m \) is defined by:

\[
\psi(u_k) = u_k - \text{sat}(u_k) \quad (2)
\]

However the saturation function \( \text{sat}: \mathbb{R}^m \to \mathbb{R}^m \) is written as:

\[
\text{sat}(u_k) = [\text{sat}_1(u_{k1}) \ldots \text{sat}_i(u_{ki}) \ldots \text{sat}_n(u_{kn})] \quad \text{and} \quad \text{sat}(u_{ki}) = \text{sign}(u_{ki}) \min(|u_{ki}|, u_{\text{max}})
\]

with \( u_{\text{max}} > 0 \), designate the saturation level.

**Remark 1:** In control applications, actuator saturation or control input saturation both in magnitude and rate usually degrades the performance of the closed-loop system, and leads to large overshoot, if the controller is designed without considering these kinds of nonlinearity. Recently, in the closed loop control system, the nonlinear system, behavior of the input saturation has been investigated as a convex combination of \( 2^m \) linear models in [10, 11].

In this work, in order to derive relaxed LMI conditions, the following lemmas will be introduced to reduce significantly the computational complexity.

**Relaxation Lemma** [14]: Let \( T_{ij}^R \) be matrices of appropriate dimensions, then for \( i, j \in \{1, \ldots, r\}, k \in \{1, \ldots, r_e\} \):

\[
\begin{align*}
\sum_{i=1}^r \sum_{j=1}^r \sum_{k=1}^{r_e} h_i(z(k)) h_j(z(k)) v_k(z(k)) & \leq 0 \\
T_{ij}^R & \leq 0
\end{align*}
\]

\[
\frac{2}{r-1} T_{ii}^R + T_{ij}^R + T_{ji}^R < 0, i \neq j \quad (4)
\]

**Finsler’s Lemma** [7]. Let \( x \in \mathbb{R}^n, q = q^T \in \mathbb{R}^{nxn} \) and \( R \in \mathbb{R}^{m \times n} \) such that \( \text{rank}(R) < n \), the following expressions are equivalent:

a) \( x^T G x < 0 \quad \forall x \in \mathbb{R}^n, x \neq 0, Rx = 0 \).

b) \( \exists \Theta \in \mathbb{R}^{m \times n}: q + \Theta R + R^T \Theta^T < 0 \).

**Property 1.** Let \( X = X^T > 0 \), and \( Y \) matrices of the appropriate size the following expressions holds:

\[
(Y - X)^T X^{-1}(Y - X) \geq 0 \Rightarrow Y^T X^{-1} Y \geq Y + Y^T - X
\]

**Lemma 1** [16]: Given matrices \( F_k \in \mathbb{R}^{m \times n} \), \( H_k \in \mathbb{R}^{n \times n} \) and \( W_k \in \mathbb{R}^{m \times n} \), for \( i, j, k \in \mathbb{N}_r, l \in \mathbb{N}_m \), let us define the following set:

\[
\mathcal{P}_u = \{ x \in \mathbb{R}^n : |(G_f x)^T \chi_j - W_k x_k | \leq u_{\text{max}} \}
\]

If \( x \in \mathcal{P}_u \), then \( \psi(u_k)^T S_{i,j}^{-1} \psi(u_k) - (W_k x_k) x_k \leq 0 \) holds of any positive diagonal matrices \( S_{i,j} \in \mathbb{R}^{m \times m} \), and for any scalar function \( \chi_{ij}, i, j \in \mathbb{N}_r \), satisfying the convex sum property.

**Assumption 1.** The validity domain \( \Omega_x \) of the system (1) is defined by:

\[
\Omega_x = \{ x \in \mathbb{R}^n : Q_m^T x \leq 1, \ m \in \mathbb{N}_q \}
\]

where the vectors \( Q_m \in \mathbb{R}^n \) are corresponding to the state constraints of system (1).

### III. MAI RESULTS

In this section, the objective is to design a non-PDC control law guarantying the desired control performance and the stability of the closed loop system, despite the presence of input control saturation and external disturbances. Accordingly, the proposed non-PDC controller law is given as follows:

\[
u_k = G_h v_i x_k \quad (7)
\]

In the following we consider that: \( A_h, \Delta A_h = \Delta \), \( B_h, \Delta B_h = \Delta \), \( E_v = \Delta \). The combination of the uncertain T-S descriptor model (1) with the control law (7) and the definition (2) yields:

\[
\begin{align*}
E x_{k+} &= A x_k + B \left( G_h v_i x_k - \psi(u_k) \right) + B_\omega \omega_k \\
y_k &= C x_k
\end{align*}
\]

Expression (8) can be rewritten as an equality constraint as:

\[
[\Delta + \Omega G_h v_i]^{-1} - E - B \omega [\psi(u_k)^T] = 0 \quad (9)
\]

To derive the stability conditions, two different Lyapunov functions will be considered [7]:

- \( V(x_k) = x_k^T P_h^{-1} x_k \) with \( P_h = P_h^T > 0, P_h^{-1} = X_h \)
- \( V(x_k) = x_k^T X_h^{-1} P_h X_h^{-1} x_k \) with \( P_h = P_h^T > 0 \)

\( A \) Case 1

The variation of the Lyapunov function in case 1 is calculated as:

\[
V(x_k) = x_k^T X_h x_k + x_k^T X_h x_k - 2 \cdot \psi(u_k)^T S_{i,j}^{-1} \psi(u_k) + \psi(u_k)^T S_{i,j}^{-1} W_v X_h^{-1} x_k + y_k \omega_k - y_k^T \omega_k < 0
\]

where the \( L_2 \) -norm of the output signal \( y_k \) is bounded as follows:
\[ \|y_k\| \leq \gamma \|\omega_k\|, \quad \forall k \geq 0. \] (12)

Consequently, the inequality (11) can be written as:

\[
\begin{bmatrix}
\chi_k x
\chi_k x+
\psi(u_k)
\omega_k
\end{bmatrix}
\begin{bmatrix}
- (X_h - C^T C) & 0 & 0 & 0
0 & X_h & 0 & 0
S_h^T W_h v_{hv} G_h v & -2 S_h^T v & 0
\end{bmatrix}
\begin{bmatrix}
\chi_k x
\chi_k x+
\psi(u_k)
\omega_k
\end{bmatrix} < 0 \quad (13)
\]

Via the Finsler's Lemma, equality (9) and inequality (13) results in:

\[
\begin{bmatrix}
M \\
N \\
0
\end{bmatrix}
\begin{bmatrix}
\mathbb{A} + \mathbb{B} G_h v_{hv}^{-1} & -E & -B_w
0 & 0 & 0
\end{bmatrix} + \begin{bmatrix}
\mathbb{A} + \mathbb{B} G_h v_{hv}^{-1} & -E & -B_w
0 & 0 & 0
\end{bmatrix} < 0
\]

(14)

where matrices \( M, N \in \mathbb{R}^{nxm} \) and \( N \in \mathbb{R}^{nxn} \) are free matrices fixed later on. Now let \( \lambda_{ij} = x_{hv} \), two results can be obtained depending on different congruence transformation of (14). The first one is stated in the following Lemma:

**Lemma 2.** The closed-loop uncertain T-S descriptor model (8) is asymptotically stable if there exist matrices \( P_j = P_j^T > 0, X_{jk}, G_{jk}, W_j \), a matrix \( Q_{m} \in \mathbb{R}^{m}, m \in \mathbb{N} \), and positive diagonal matrices \( S_{jk} \in \mathbb{R}^{mxm} \), for \( i, j, I \in \{1, ..., r\}, k \in \{1, ..., r\} \), the scalars \( \gamma, \alpha, \beta, \gamma_1 \) such that conditions 4 are satisfied with

\[
X_{hv}^T (X_h - C^T C) X_{hv} \begin{bmatrix}
0 & 0 & 0 & 0
0 & P_h X_{hv} P_h & 0 & 0
W_{hv} & -S_h^T \mathbb{B}^T & -2 S_h^T v & 0
0 & B_w^T & 0 & -\gamma I
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 0 & 0
0 & 0 & 0 & 0
0 & 0 & 0 & 0
0 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
\alpha & \beta & \gamma_1
0 & 0 & 0
0 & 0 & 0
0 & 0 & 0
\end{bmatrix} < 0
\]

(15a)

where

\[
\psi_{k11}^{ij} = \begin{bmatrix}
N_{ai} x_{jk} & 0 & 0 & 0
N_{bi} G_{jk} & 0 & 0 & 0
0 & 0 & 0 & 0
0 & 0 & 0 & 0
\end{bmatrix}
\]

(15c)

\[
\psi_{21}^{ij} = \begin{bmatrix}
\psi_{k11}^{ij} & \psi_{k21}^{ij}
\end{bmatrix} + \begin{bmatrix}
\psi_{k11}^{ij} & \psi_{k21}^{ij}
\end{bmatrix}
\]

(15d)

and

\[
\psi_{22}^{ij} = \begin{bmatrix}
0 & 0 & 0 & 0
0 & 0 & 0 & 0
0 & 0 & 0 & 0
0 & 0 & 0 & 0
\end{bmatrix}
\]

(16)

\[
- \chi_{jk}^T - \chi_{jk} + P_j \begin{bmatrix}
0 & 0 & 0 & 0
0 & 0 & 0 & 0
0 & 0 & 0 & 0
0 & 0 & 0 & 0
\end{bmatrix}
\]

(17)

\[
- \chi_{jk}^T - \chi_{jk} + P_j \begin{bmatrix}
0 & 0 & 0 & 0
0 & 0 & 0 & 0
0 & 0 & 0 & 0
0 & 0 & 0 & 0
\end{bmatrix}
\]

(17)

**Proof.** From inequality (16) it can be deduced that \( x \in \Omega_x \).

Furthermore, if condition (16) is satisfied, then it follows clearly that matrices \( X_{jk}, j, k \in \mathbb{N}_r, \) are regular since \( P_j > 0 \).

Besides, using Schur complement lemma [15] and matrix property 1, it can be deduced from (17) that

\[
G_{jk}^T P_j G_{jk} - \frac{(G_{jk}(t) - W_{jk}(t))^T (G_{jk}(t) - W_{jk}(t))}{(u_{jk}^\max)^2} \geq 0
\]

(18)

Pre and post-multiplying (2) with \( \chi_{jk}^T \) yields:

\[
p_j - \frac{(G_{jk}(t) - W_{jk}(t))^T (G_{jk}(t) - W_{jk})(t)}{(u_{jk}^\max)^2} \geq 0
\]

(19)

Then, it is easily observed that condition (19) implies the inclusion \( \Omega_x \subseteq \mathbb{P}_u \). Now, by using the congruence lemma property with the full rank matrix \( \text{diag}(X_{hv}^T, P_h, S_h^T, I) \), (14) yields:

\[
\begin{bmatrix}
X_{hv}^T (X_h - C^T C) X_{hv} \begin{bmatrix}
0 & 0 & 0 & 0
0 & P_h X_{hv} P_h & 0 & 0
W_{hv} & -S_h^T \mathbb{B}^T & -2 S_h^T v & 0
0 & B_w^T & 0 & -\gamma I
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 0 & 0
0 & 0 & 0 & 0
0 & 0 & 0 & 0
0 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
\alpha & \beta & \gamma_1
0 & 0 & 0
0 & 0 & 0
0 & 0 & 0
\end{bmatrix} < 0
\]

The main objective is to design a relaxed LMI optimization problem. For that a best choice is to consider \( M = 0 \) and \( N = X_{hv} \) then (20) yields:

\[
\psi_{hv} + \Delta \psi_{hv} < 0
\]

(21a)

with

\[
\psi_{hv} = \begin{bmatrix}
-\chi_{hv}^T (X_h - C^T C) X_{hv} & \psi_{11}^{ij} & \psi_{12}^{ij} & \psi_{13}^{ij}
A_h X_{hv} + B_h G_{hv} & -H (P_h E^T) + P_h & \psi_{21}^{ij} & \psi_{22}^{ij}
W_{hv} & -S_h^T B_w & -2 S_h^T v & 0
0 & B_w^T & 0 & -\gamma I
\end{bmatrix}
\]

(21b)

Recall that for any matrices \( G, \mathcal{K} \) and \( \mathcal{D}(t) \) of appropriate dimension satisfying \( \mathcal{D}(t) \mathcal{D}(t)^T \leq I \) and any positive scalar \( \alpha \), the following holds

\[
\mathcal{G}(t) \mathcal{K}^T + \mathcal{K} \mathcal{D}(t) \mathcal{G}^T \leq \alpha \mathcal{G}^T + \alpha^{-1} \mathcal{K} \mathcal{K}^T
\]

(21b)

Now, using uncertainties definitions, and by means of the previous inequality (21b), it can be stated that:

\[
\Delta \psi_{hv} < \begin{bmatrix}
0 & 0 & 0 & 0
0 & 0 & 0 & 0
0 & 0 & 0 & 0
0 & 0 & 0 & 0
\end{bmatrix}
\]

(21c)

with

\[
\Delta^2 \psi = \begin{bmatrix}
\begin{bmatrix}
\psi_{11}^{ij} & \psi_{12}^{ij} & \psi_{13}^{ij}
A_h X_{hv} + B_h G_{hv} & -H (P_h E^T) + P_h & \psi_{21}^{ij} & \psi_{22}^{ij}
W_{hv} & -S_h^T B_w & -2 S_h^T v & 0
0 & B_w^T & 0 & -\gamma I
\end{bmatrix}
\end{bmatrix}
\]

(21c)
Finally, applying property 1, the relaxation lemma and Schur complement lemma to (21a) by considering (21c), conditions (15) holds. This ends the proof.

Remark 2. A more general result can be reached by multiplying by \(\text{diag}(\hat{X}_h^T, G_{hh^+}, S_{hh^+})\) on the left hand side and by its transpose, on the right hand side of (14), gives:

\[
\begin{bmatrix}
\chi_h^T(X_h - C^TX_h) & 0 & 0 & 0 \\
0 & \chi_{hh^+}^T & 0 & 0 \\
W_{hh^+} & 0 & -S_{hh^+}B^T & 0 \\
0 & 0 & B_{\omega} & 0 \\
\end{bmatrix} + 
\end{bmatrix} 
\]

\[
\begin{bmatrix}
\chi_h^T & M \\
G_{hh^+} & N \\
0 & 0 \\
0 & 0 \\
\end{bmatrix} 
\]

\[
\begin{bmatrix}
A_{\chi_h} + B_{\chi_h} & -E_{\chi_h} & -B_{\omega}S_{\omega} & B_{\omega} + (* < 0) (22) \\
\end{bmatrix} 
\]

Note that a new matrix \(G_{hh^+}^T\) is introduced, thus adding extra degree of freedom to the inequality. Therefore, the following theorem can be stated:

Theorem 1. Given the uncertain T-S descriptor system (1) whose validity domain is characterized by a matrix \(Q_m \in \mathbb{R}^{n_r}, m \in \mathbb{N}_q\). If there exist positive definite matrices \(P_l \in \mathbb{R}^{n \times n}\), positive diagonal matrices \(S_{jr} \in \mathbb{R}^{m \times m}\), and matrices \(\chi_{jr} \in \mathbb{R}^{m \times n}, G_{jr} \in \mathbb{R}^{m \times n}, W_{jr} \in \mathbb{R}^{m \times n}, i,j,k \in \mathbb{N}_r\), and the scalars \(\gamma, \partial_1, \partial_2\), such that conditions 4 are satisfied with:

\[
\begin{bmatrix}
\phi_{i j l}^{11} & \psi_{i j l}^{21} & \psi_{i j l}^{22} \\
\end{bmatrix} 
\]

\[
\begin{bmatrix}
\phi_{i j l}^{11} = \\
-\mathcal{H} (\chi_{jr}) + P_l & (*) & (*) & (*) & (*) & (*) \\
0 & (*) & (*) & (*) & (*) & (*) \\
0 & (*) & (*) & (*) & (*) & (*) \\
0 & 0 & (*) & (*) & (*) & (*) \\
0 & 0 & (*) & (*) & (*) & (*) \\
0 & 0 & (*) & (*) & (*) & (*) \\
\end{bmatrix} 
\]

\[
\psi_{i j l}^{21} \text{ and } \psi_{i j l}^{22} \text{ are given by (15c) and (15d) respectively.} 
\]

and

\[
\begin{bmatrix}
P_l \\
Q_m P_l \\
1 \\
\end{bmatrix} \geq 0, m \in \mathbb{N}_q, j \in \mathbb{N}_r 
\]

\[
\begin{bmatrix}
-\mathcal{H} (\chi_{jr}) + P_l & (*) \\
0 & (*) \\
0 & (*) \\
0 & (*) \\
0 & (*) \\
\end{bmatrix} \leq 0, t \in \mathbb{N}_m, j,k \in \mathbb{N}_r 
\]

Proof. The result of Theorem 1 is derived from the proof of Lemma 2 by choosing \(M = 0\) and \(N = G_{hh^+}^T\). Thus, this proof is omitted here for brevity.

B. Case 2

Consider \(\chi(t) = \chi_h\) in (7). Then the variation of the Lyapunov function in case 2 is calculated as:

\[
\Delta V(x_k) = x_k^T X_h x_k + x_k^T P_h + \chi_h^T x_k + x_k^T P_h \chi_h^T x_k < 0 
\]

By considering lemma 1, the inequality (27) becomes:

\[
\Delta V(x_k) = -2\psi(u_k)^T S_{hh^+} \psi + \psi(u_k)^T S_{hh^+} W_{hh^+} \chi_h^T x_k + \gamma_k^T - \gamma_k \omega_k < 0 
\]

By developing inequality (28), with respect to the Finsler’s Lemma, and expressions (9) one can obtain:

\[
\begin{bmatrix}
-\chi_h^T(P_h - C^TX_h)^{-1} & 0 & 0 & 0 \\
0 & \chi_h^T P_h + \chi_h^T & 0 & 0 \\
S_{hh^+}^{-1} W_{hh^+} \chi_h^{-1} & -S_{hh^+}^{-1} B_{\omega} & 0 & -\gamma_k \\
0 & B_{\omega} & 0 & -\gamma_k \\
\end{bmatrix} 
\]

\[
\begin{bmatrix}
M \\
N \\
0 \\
0 \\
\end{bmatrix} 
\]

\[
\begin{bmatrix}
A_{\chi_h} + B_{\chi_h} & -E_{\chi_h} & -B_{\omega}S_{\omega} & B_{\omega} + (* < 0) (29) \\
\end{bmatrix} 
\]

Now, using the property of congruence with \(\text{diag}(X_h, \chi_h, S_{hh^+})\), (29) yields:

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix} 
\]

\[
\begin{bmatrix}
H_{hh^+}^T M \\
H_{hh^+}^T N \\
0 \\
0 \\
\end{bmatrix} 
\]

\[
\begin{bmatrix}
A_{\chi_h} + B_{\chi_h} & -E_{\chi_h} & -B_{\omega}S_{\omega} & B_{\omega} + (* < 0) \\
\end{bmatrix} 
\]

Accordingly, with respect to the same development of lemma 2 LMI conditions, the following theorem is obtained and provides conditions which allow the synthesis of the stabilization non-PDC controller satisfying the system performance described in the previous section.

Theorem 2. Given the uncertain T-S descriptor system (1) whose validity domain is characterized by a matrix \(Q_m \in \mathbb{R}^{n_r}, m \in \mathbb{N}_q\). If there exist positive definite matrices \(P_l \in \mathbb{R}^{n \times n}\), positive diagonal matrices \(S_{jr} \in \mathbb{R}^{m \times m}\), and matrices \(\chi_{jr} \in \mathbb{R}^{m \times n}, G_{jr} \in \mathbb{R}^{m \times n}, W_{jr} \in \mathbb{R}^{m \times n}, i,j,k \in \mathbb{N}_r\), and positive scalar \(\gamma, \partial_1, \partial_2\), such that conditions 4 are satisfied with:

\[
\begin{bmatrix}
\phi_{i j l}^{11} & (*) \\
\psi_{i j l}^{21} & \psi_{i j l}^{22} \\
\end{bmatrix} 
\]

with
\[
\begin{array}{ccc}
\psi_{ij}^{k1} & \psi_{ij}^{k2} & \psi_{ij}^{k3}
\end{array}
\]

\[
\begin{bmatrix}
-\mathbf{p}_j \\
\mathbf{C}_k \\
\mathbf{A}_k^{ij}
\end{bmatrix}
\] (15c) and (15d) respectively.

\[
\begin{bmatrix}
-\mathbf{p}_j \\
\mathbf{C}_k \\
\mathbf{A}_k^{ij}
\end{bmatrix}
\]

Theorem 2 are feasible, i.e., conditions in Theorem 2 are unfeasible. Solving the optimization LMI problem defined by Theorem 1 leads to:

\[
P_1 = \begin{bmatrix}
0.64 & -0.26 \\
-0.26 & 3.09
\end{bmatrix},
P_2 = \begin{bmatrix}
0.36 & 0.36 \\
0.36 & 4.76
\end{bmatrix}
\]

(33) Proof. The result of Theorem 2 is derived from the proof of Lemma 2 by choosing \( M = 0 \) and \( N = \chi_h^T = P_k^+ \). Thus, this proof is omitted here for brevity.

IV. ILLUSTRATIVE EXAMPLES

In this section, the proposed solutions are illustrated via the following two numerical examples.

Example 1. Consider the uncertain T-S descriptor model (1), with: \( r = r_e = 2 \) and \( a = -1.7 ; b = -0.4 ; c = 0.2172 \).

\[
A_1 = \begin{bmatrix}
0 & 0.5 \\
-1.5 & -3 + \left(\frac{b}{2}\right) \ast (1 - c)
\end{bmatrix},
B_1 = \begin{bmatrix}
-a/2 - 2 \\
1
\end{bmatrix},
\]

\[
A_2 = \begin{bmatrix}
0 & 0.5 \\
-1.5 & -3 + b \ast (1 - c)
\end{bmatrix},
B_2 = \begin{bmatrix}
a/2 - 2 \\
1
\end{bmatrix},
\]

\[
E_1 = \begin{bmatrix}
1 & 0 \\
-1 & 0.5
\end{bmatrix},
E_2 = \begin{bmatrix}
1 & -1 \\
1 & 0.5
\end{bmatrix},
B_w = \begin{bmatrix}
0.05 \\
0
\end{bmatrix}
\]

and \( C = [0 \; 1] \).

\[
H_{a1} = H_{a2} = \begin{bmatrix}
1 & 0 \\
1 & 1
\end{bmatrix};
N_{a1} = \begin{bmatrix}
0 & 0.5 \\
0 & \left(\frac{b}{2}\right) \ast (1 + c)
\end{bmatrix},
\]

\[
N_{a2} = \begin{bmatrix}
0 & 0.5 \\
0 & b \ast (1 + c)
\end{bmatrix};
\Delta a1 = \Delta a2 = 0.3 \ast \cos(2 \ast t).
\]

\[
H_{b1} = H_{b2} = \begin{bmatrix}
0 & 1 \\
1 & 1
\end{bmatrix};
\Delta b1 = \Delta b2 = 0.2 \ast \sin(3 \ast t)
\]

The membership functions are defined as follows:

\[
\begin{align*}
\nu_{1k} &= \cos(x_{2k})^2 + 2, \\
\nu_{2k} &= 1 - \nu_{1k}, \\
\end{align*}
\]

\[
\begin{align*}
\nu_{1k} &= \sin(x_{2k})^2 + 2, \\
\nu_{2k} &= 1 - \nu_{1k}, \\
\end{align*}
\]

The MFs satisfy the convex-sum property on the compact set \( \Delta = \{x_k : |x_{1k}| \leq 2, |x_{2k}| \leq 2\} \), and \( u^{max} = 0.05 \). The uncertain T-S descriptor system (1) is subject to amplitude-bounded disturbance \( \omega(t) \) defined by \( \omega(t) = 0.1 \ast \sin(10 \ast t) \). For this model, only the conditions of Theorem 1 are feasible, i.e., conditions in Theorem 2 are unfeasible.

Example 2. Consider an uncertain discrete-time T-S descriptor model as in (1) with \( r = r_e = 2 \), \( a = -1.7, b = -0.4, c = 0.2172 \) and

\[
A_1 = \begin{bmatrix}
-1.5 & -3 + b \ast (1 + c) \\
1 & 2
\end{bmatrix},
A_2 = \begin{bmatrix}
-1.5 & -3 + b \\
1 & 2
\end{bmatrix},
\]

\[
B_1 = \begin{bmatrix}
-1 \\
-2
\end{bmatrix},
B_2 = \begin{bmatrix}
1 \\
2
\end{bmatrix},
E_1 = \begin{bmatrix}
1 & 1 \\
1 & 1
\end{bmatrix},
E_2 = \begin{bmatrix}
1 & 1 \\
1 & 1
\end{bmatrix},
\]

\[
B_w = \begin{bmatrix}
0.1 \\
0.05
\end{bmatrix},
C = [0 \; 1];
H_{a1} = H_{a2} = \begin{bmatrix}
1 & 0.5 \\
1 & 1.75
\end{bmatrix};
\Delta a1 = \Delta a2 = 0.8 \ast \cos(3 \ast t)
\]

\[
H_{b1} = H_{b2} = \begin{bmatrix}
0 & 1 \\
1 & 1
\end{bmatrix};
\Delta b1 = \Delta b2 = 0.25 \ast \sin(2 \ast t)
\]

\[
F_1 = F_2 = [-1.70 \; -3.45],
F_3 = F_4 = [0.24 \; -0.79].
\]

\( \gamma = 1.42 \) and \( \delta_1 = \delta_2 = 1 \). Simulation results were obtained with initial conditions \( x(0) = [0.5 - 0.25]^T \) are shown in Figure 1 and 2. The obtained results illustrate the effectiveness of the proposed approach for the studied example.
The MFs are defined the same as in Example 1 and satisfy the convex-sum property on the compact set $\Delta = \{x_k:|x_{1k}| \leq 1,|x_{2k}| \leq 1\}$. and $\|\Delta\|_{\infty} = 0.05$. For this system LMI conditions in Theorem 1 are unfeasible, while applying LMI Theorem 2 gives the following gain matrices:

$$
P_1 = \begin{bmatrix} 6.73 & -0.44 \\ -0.44 & 0.29 \end{bmatrix}, P_2 = \begin{bmatrix} 0.89 & 1.60 \\ 1.60 & 0.62 \end{bmatrix},
H_1 = \begin{bmatrix} 0.98 & -0.53 \\ -0.47 & 0.77 \end{bmatrix}, H_2 = \begin{bmatrix} 0.91 & 0.79 \\ -0.32 & 0.77 \end{bmatrix},
F_1 = F_2 = [-0.74 \ -0.55], F_3 = F_4 = [-0.20 \ -0.44].$$

Simulation results with initial conditions $x(0) = [0.7 \ - 0.25]^T$ are depicted in Figure 3 and 4.

The result demonstrated that, despite the saturation limits the designed controller guarantees the closed loop system convergence to the zero with a good perturbation reject.

V. CONCLUSION

In this paper, a non quadratic Lyapunov functions is used to obtain sufficient conditions of asymptotic stability for nonlinear uncertain discrete-time systems represented by T-S descriptor models subject input saturation and external disturbances. The main advantage of the proposed approaches is to synthesize the control law by considering the saturation limits while achieving a guaranteed $L_2$-gain performance. A non PDC control law is used to achieve this objective. The controller gains are then obtained by solving an optimization problem under LMI constraints. Through two numerical examples the efficiency of the proposed techniques has been demonstrated.

REFERENCES


