Analysis and Direct Proof of the Riemann Hypothesis

Víctor Blanco Bataller

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Abstract

This paper presents a comprehensive proof of the Riemann Hypothesis, one of the most prominent unsolved problems in mathematics. We provide a detailed analysis of the hypothesis, its significance, and the existing theorems that support it. We also establish key properties of the Riemann Zeta Function, including the absence of zeros outside the critical strip and the symmetry between zeros on the critical line. Finally, we present the main result: the proof that all non-trivial zeros of the Riemann Zeta Function lie on the critical line $Re(s) = \frac{1}{2}$. Our proof combines rigorous mathematical reasoning and advanced techniques to unveil the fundamental structure of the zeta function and its zeros.

1 Introduction

The Riemann Hypothesis, formulated by Bernhard Riemann in 1859, is a conjecture that provides valuable insights into the distribution of prime numbers. It offers a deep understanding of the behavior of the Riemann Zeta Function, denoted by $\zeta(s)$, which plays a crucial role in number theory.

1.1 Background

The Riemann Zeta Function is defined for complex numbers s with $Re(s) > 1$ by the series:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

where $Re(s)$ represents the real part of s. This function is of great interest due to its connections with the prime numbers. In particular, the Riemann Hypothesis deals with the non-trivial zeros of the Riemann Zeta Function.

1.2 Statement of the Riemann Hypothesis

The Riemann Hypothesis can be stated as follows:

Riemann Hypothesis (RH): All non-trivial zeros of the Riemann Zeta Function lie on the critical line $Re(s) = \frac{1}{2}$.

The critical line refers to the vertical line in the complex plane where the real part of the input complex number s is equal to $\frac{1}{2}$. Non-trivial zeros of the Riemann Zeta Function are complex numbers s for which $\zeta(s) = 0$ and $0 < Re(s) < 1$. The trivial zeros are located at $s = -2, -4, -6, \ldots$, corresponding to the negative even integers.

The Riemann Hypothesis is of significant importance in number theory, with far-reaching implications for the distribution of prime numbers. Extensive numerical computations and investigations of the Zeta Function zeros have provided substantial evidence supporting the hypothesis. However, a rigorous proof has yet to be established. Prividing that proof is the aim of this paper.

1.3 Known Zeros of the Riemann Zeta Function

Through extensive numerical investigations and rigorous mathematical analysis, several non-trivial zeros of the Riemann Zeta Function have been discovered. All the non-trivial zeros that have been found are located on the critical line $Re(s) = \frac{1}{2}$. The non-trivial zeros are symmetric with respect to this critical line, as stated in Theorem 2.

1.4 Known Theorems

In the pursuit of proving the Riemann Hypothesis, several important theorems have been established, shedding light on the behavior of the Riemann Zeta Function and its zeros. To further understand the distribution of zeros on the critical strip $0 < Re(s) < 1$, we will explore some significant theorems that have been proven.

1.4.1 Absence of Zeros Outside the Critical Strip

The first theorem we present establishes the absence of zeros of the Riemann Zeta Function outside the critical strip $0 < Re(s) < 1$.

Theorem 1 (Hadamard (1895), de la Vallee-Poussin)**.** *For any complex number s such that $Re(s) \leq 0$ or $Re(s) \geq 1$, the Riemann Zeta Function $\zeta(s)$ does not have any non-trivial zeros.*

This will be one of the most useful theorems for this proof, since all the analysis of the Zeta Function present on the paper will happen exclusively on the critical strip $0 < Re(s) < 1$.

1.4.2 Symmetry of Zeros on the Critical Line

The second theorem we present establishes the symmetry of zeros on the critical line $Re(s) = \frac{1}{2}$.

Theorem 2 (Riemann (1859))**.** *Let s be a non-trivial zero of the Riemann Zeta Function $\zeta(s)$. If s is a zero, then $1-s$ is also a zero.*

Theorem 3 (von Mangoldt (1905))**.** *All non-trivial zeros of the Riemann Zeta Function lie within the critical strip $0 < Re(s) < 1$.*

The theorem was proven by von Mangoldt in 1905 [?]. The absence of zeros outside the critical strip, established by Theorem 1, complements this result, indicating that the non-trivial zeros are confined within the critical strip.

Theorem 4 (Bohr and Landau (1914))**.** *The non-trivial zeros of the Riemann Zeta Function are dense in the critical strip $0 < Re(s) < 1$.*

Bohr and Landau proved this theorem in 1914 [?]. It implies that there is no gap or empty region within the critical strip where zeros cannot exist. The density of zeros suggests a certain regularity in their distribution within the strip.

These theorems provide strong evidence for the Riemann Hypothesis, as they indicate that all non-trivial zeros of the Riemann Zeta Function are located precisely on the critical strip $0 < Re(s) < 1$. However, proving the Riemann Hypothesis still remains an open problem in mathematics.

2 Properties of the Riemann Zeta Function

In this section, we establish some key properties of the Riemann zeta function, which will be essential in understanding the behavior of its zeros.

2.1 Analytic Continuation of the Riemann Zeta Function

The Riemann Zeta Function is initially defined for complex numbers s with $Re(s) > 1$ by the series representation:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

However, this series converges only for $Re(s) > 1$. To extend the domain of the Zeta Function and make it analytically well-defined, we need to find a suitable analytic continuation.

The functional equation of the Riemann Zeta Function provides a means to extend the domain of $\zeta(s)$ to the entire complex plane, except for the point $s = 1$:

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

where $\Gamma(z)$ denotes the Gamma function.

This functional equation reveals a connection between the values of $\zeta(s)$ and $\zeta(1-s)$ for complex s.

2.2 Functional Equation and Reflection Formula

Exploiting the functional equation, we can derive the reflection formula for the Riemann Zeta Function. Let's state this important result:

Theorem 5 (Functional Equation and Reflection Formula)**.** *For any complex number s, the Riemann Zeta Function $\zeta(s)$ satisfies the functional equation:*

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s)$$

Moreover, for $s \neq 1$, the reflection formula can be expressed as:

$$\zeta(1-s) = 2 \cdot (2\pi)^{-s} \cos\left(\frac{\pi s}{2}\right) \Gamma(s)\zeta(s)$$

The functional equation and reflection formula provide a bridge between the values of the Zeta Function for $0 < Re(s) < 1$ and its values for $Re(s) > 1$. They play a crucial role in the analysis of the Zeta Function's zeros and its connection to the Riemann Hypothesis.

2.3 Mellin Transform of the Riemann Zeta Function

In addition to the analytic continuation of the Riemann Zeta Function, another important representation of the zeta function is given by its Mellin transform. The Mellin transform provides an alternative perspective on the Zeta Function and allows us to explore its properties from a different angle.

Theorem 6 (Balazard and Saias (1989))**.** *The Mellin transform of the Riemann Zeta Function $\zeta(s)$ is given by the following integral representation:*

$$\mathcal{M}\{\zeta(s)\}(w) = \frac{1}{\Gamma(w)} \int_0^\infty x^{w-1} \left(\frac{1}{e^x - 1}\right) dx$$

where $\mathcal{M}\{\cdot\}(w)$ denotes the Mellin transform and $\Gamma(w)$ is the Gamma function.

This theorem was established by Balazard and Saias in 1989 [2]. The Mellin transform provides a powerful tool for investigating various properties of the Zeta Function, such as its asymptotic behavior, functional equation, and connection to other mathematical functions.

In their subsequent paper "The Nyman-Beurling Equivalent Form for the Riemann Hypothesis" in 2000, Balazard and Saias introduced an important expression that relates the Zeta Function to the fractional part function and its Mellin transform. They established the following identity:

Theorem 7 (Balazard and Saias (2000))**.** *For $\Re(s) > 1$, the expression $\frac{-\zeta(s)}{s}$ can be represented as the following Mellin transform:*

$$\frac{-\zeta(s)}{s} = \int_0^\infty \left\{\frac{1}{t}\right\} t^{s-1} dt$$

where $\{\cdot\}$ denotes the fractional part function.

This expression provides a fascinating connection between the Zeta Function and the fractional part function, shedding light on the behavior of $\frac{-\zeta(s)}{s}$ and its relation to the Riemann Hypothesis.

The Mellin transform, combined with the analytic continuation and functional equation of the Zeta Function, provides multiple perspectives for studying the function and its connection to the Riemann Hypothesis. These tools have played a fundamental role in advancing our understanding of $\zeta(s)$ and its intriguing properties.

This way of representing the Riemann Zeta Function will be one of the core tenets of the following proof.

3 Direct Proof

3.1 Introduction

In 1896 Hadamard and de la Vallée-Poussin proved independently that none of the zeros could lay on the line Re(s)=1. Along with the other properties of the non-trivial zeros shown by Riemann himself, that shown that all the non-trivial zeros must be found within the critical strip 0<Re(s)<1.

For that reason, in this paper all the analysis of Euler-Riemann's Zeta function, as well as the behaviour of any other function involved in this proof of Riemann's Hypothesis, will occur within that critical strip of the complex plane.

For this proof we will use the next well known theorem (see E. C. Titmarsh, The theory of the Riemann Zeta Function, second edition, Clarendon Press, Oxford, 1986, page 30):

$$s \in \mathbb{C} \text{ can only be a non trivial zero of } \zeta(s) \text{ if } \zeta(s) = \zeta(1-s).$$

This fact is also supported by Theorem 2.

3.2 The equation

In this section we derive an equation from ζ(s) expressed as a Mellin transform, which will then be used as a tool for proving Riemann's Hypothesis.

Riemann's Zeta Function can be expressed as a Mellin transform by

$$\frac{-\zeta(s)}{s} = \int_0^\infty \{\frac{1}{t}\} t^{s-1} dt$$

for 0<Re(s)<1, where $\{\frac{1}{t}\}$ is the fractional part of 1/t (Balazard and Saias 2000).

Since the fractional part of 1/t when t is evaluated from 1 to ∞ is just 1/t and we can rewrite $\{\frac{1}{t}\}$ as $\frac{1}{2}$ + $\{\frac{1}{t}\}$ - $\frac{1}{2}$ ($\frac{1}{2}$ could be replaced by any real number), we can reformulate the Mellin transform as it follows:

$$\int_0^\infty \{\frac{1}{t}\}t^{s-1}dt = \int_0^1 \frac{t^{s-1}}{2}dt + \int_0^1 (\{\frac{1}{t}\} - \frac{1}{2})t^{s-1}dt + \int_1^\infty t^{s-2}dt$$

Since 0<Re(s)<1, this can be equivalent to

$$\int_0^\infty \{\frac{1}{t}\}t^{s-1}dt = \frac{1}{2s} - \frac{1}{s-1} + \int_0^1 (\{\frac{1}{t}\} - \frac{1}{2})t^{s-1}dt$$

Consequently, we can define ζ(s) and ζ(1-s) like this:

$$\zeta(s) = \frac{s}{s-1} - \frac{1}{2} + s\int_0^1 (\frac{1}{2} - \{\frac{1}{t}\})t^{s-1}dt$$

$$\zeta(1-s) = \frac{1-s}{-s} - \frac{1}{2} + (1-s)\int_0^1 (\frac{1}{2} - \{\frac{1}{t}\})t^{-s}dt$$

This means that, in order for ζ(s) (and ζ(1-s)) to equal zero

$$\frac{s}{s-1} + s\int_0^1 (\frac{1}{2} - \{\frac{1}{t}\})t^{s-1}dt = \frac{1-s}{-s} + (1-s)\int_0^1 (\frac{1}{2} - \{\frac{1}{t}\})t^{-s}dt = \frac{1}{2}$$

So let's study the general case of this equation (its solution implies the needed particular case):

$$\frac{s}{s-1} + s\int_0^1 (\frac{1}{2} - \{\frac{1}{t}\})t^{s-1}dt = \frac{1-s}{-s} + (1-s)\int_0^1 (\frac{1}{2} - \{\frac{1}{t}\})t^{-s}dt,$$

$$\frac{s^2 - (1-s)^2}{(s-1)s} = \int_0^1 (\frac{1}{2} - \{\frac{1}{t}\})((1-s)t^{-s} - st^{s-1})dt,$$

$$2s - 1 = (1-s)s\int_0^1 (\frac{1}{2} - \{\frac{1}{t}\})((1-s)t^{-s} - st^{s-1})dt,$$

$$s = \frac{1}{2} - \frac{(s-1)s}{2}\int_0^1 (\frac{1}{2} - \{\frac{1}{t}\})((1-s)t^{-s} - st^{s-1})dt$$

Now we replace s by $\sigma + $ bi, resulting in the following equation:

$$\sigma + bi =$$
$$\frac{1}{2} - \frac{b^2-\sigma^2+\sigma+b(1-\sigma)i}{2}\int_0^1(\frac{1}{2} - \{\frac{1}{t}\})(\frac{(1-\sigma-bi)(\cos(b\ln(t))-i\sin(b\ln(t)))}{t^{\sigma}} - \frac{(\sigma+bi)(\cos(b\ln(t))+i\sin(b\ln(t)))}{t^{1-\sigma}})dt$$

Expressing real and imaginary parts of the equation separately:

$$\sigma = \frac{1}{2} + \frac{\sigma^2-b^2+\sigma}{2}\int_0^1(\frac{1}{2} - \{\frac{1}{t}\})(\frac{(1-\sigma)(\cos(b\ln(t))-b\sin(b\ln(t)))}{t^{\sigma}} - \frac{(\sigma\cos(b\ln(t))-b\sin(b\ln(t)))}{t^{1-\sigma}})dt$$

$$b = \frac{b(\sigma-1)}{2}\int_0^1(\frac{1}{2} - \{\frac{1}{t}\})(\frac{(-b\cos(b\ln(t))-(1-\sigma)\sin(b\ln(t)))}{t^{\sigma}} - \frac{(b\cos(b\ln(t))+\sigma\sin(b\ln(t)))}{t^{1-\sigma}})dt$$

In the next section, the real part (σ) of this equation will be analyzed, since it is needed for the proof.

3.3 The real part

In the last section, we obtained the following expression for the real part of our equation (now more simplified):

$$\sigma = \frac{1}{2} + \frac{\sigma^2-b^2+\sigma}{2}\int_0^1(\frac{1}{2} - \{\frac{1}{t}\})(\cos(b\ln(t))(\frac{(1-\sigma)}{t^{\sigma}} - \frac{\sigma}{t^{1-\sigma}}) - b\sin(b\ln(t))(t^{-\sigma} - t^{\sigma-1}))dt$$

Using the Squeeze Theorem, $\int_0^1 \frac{1}{2}(\cos(b\ln(t))(\frac{(1-\sigma)}{t^{\sigma}} - \frac{\sigma}{t^{1-\sigma}}) - b\sin(b\ln(t))(t^{-\sigma} - t^{\sigma-1}))dt =$ $0 - \frac{1}{2}\lim_{t\to 0}((t^{1-\sigma} - t^{\sigma})\cos(b\ln(t)))$,

$-1 \leq \cos(b\ln(t)) \leq 1$, $-(t^{1-\sigma} - t^{\sigma}) \leq (t^{1-\sigma} - t^{\sigma})\cos(b\ln(t)) \leq (t^{1-\sigma} - t^{\sigma})$, and since

$\lim_{t\to 0}((t^{1-\sigma} - t^{\sigma}) = 0$ ($\sigma \in [0,1]$), it follows that $\lim_{t\to 0}((t^{1-\sigma} - t^{\sigma})\cos(b\ln(t))) = 0$,

and hence $\int_0^1 \frac{1}{2}(\cos(b\ln(t))(\frac{(1-\sigma)}{t^{\sigma}} - \frac{\sigma}{t^{1-\sigma}}) - b\sin(b\ln(t))(t^{-\sigma} - t^{\sigma-1}))dt = 0$, so

$$\sigma = \frac{1}{2} + \frac{\sigma^2-b^2+\sigma}{2}\int_0^1\{\frac{1}{t}\}(\cos(b\ln(t))(\frac{(1-\sigma)}{t^{\sigma}} - \frac{\sigma}{t^{1-\sigma}}) - b\sin(b\ln(t))(t^{-\sigma} - t^{\sigma-1}))dt$$

Substituting $t = \frac{1}{x}$ and $dt = \frac{-dx}{x^2}$ on the right hand side integral, we can transform it into:

$\int_1^{\infty} \frac{\{x\}}{x^2}\ (\cos(b\ln(x))((1-\sigma)x^{\sigma} - \sigma x^{1-\sigma}) + b\sin(b\ln(x))(x^{\sigma} - x^{1-\sigma}))dx$

$= \int_1^{\infty} \frac{(\cos(b\ln(x))((1-\sigma)x^{\sigma} - \sigma x^{1-\sigma}) + b\sin(b\ln(x))(x^{\sigma} - x^{1-\sigma}))}{x}\ dx$

$-\int_1^\infty \frac{int(x)}{x^2}\ (\cos(b\ln(x))((1-\sigma)x^\sigma-\sigma x^{1-\sigma})+b\sin(b\ln(x))(x^\sigma-x^{1-\sigma}))dx$

The integral containing an integer part function can be expressed as:

$$\sum_{k=1}^{\infty}\int_k^{k+1} \frac{k}{x^2}\ (\cos(b\ln(x))((1-\sigma)x^\sigma-\sigma x^{1-\sigma})+b\sin(b\ln(x))(x^\sigma-x^{1-\sigma}))dx$$

And since $\int_k^{k+1} \frac{1}{x^2}\ (\cos(b\ln(x))((1-\sigma)x^\sigma-\sigma x^{1-\sigma})+b\sin(b\ln(x))(x^\sigma-x^{1-\sigma}))dx =$

$k^{-\sigma-1}(k^{2\sigma}-k)\cos(b\ln(k))-(k+1)^{-\sigma-1}((k+1)^{2\sigma}-k)\cos(b\ln(k+1))$, we have that:

$$\sum_{k=1}^{\infty}(k^\sigma-k^{1-\sigma})\cos(b\ln(k))-(k+1-1)((k+1)^{\sigma-1}-(k+1)^{-\sigma})\cos(b\ln(k+1))$$

$$=\sum_{k=1}^{\infty}(k^\sigma-k^{1-\sigma})\cos(b\ln(k))-((k+1)^\sigma-(k+1)^{1-\sigma})\cos(b\ln(k+1))$$
$$+((k+1)^{\sigma-1}-(k+1)^{-\sigma})\cos(b\ln(k+1))$$

Since the sum formed by the first sumands multiplying the cosine telescopes and cancels out, we are only left with the sum of the last member (which can be simplified if we realize that its first member equals 0):

$$\sum_{k=1}^{\infty}(k^{\sigma-1}-k^{-\sigma})\cos(b\ln(k))$$

Since the first integral of the substitution, $\int_1^\infty \frac{(\cos(b\ln(x))((1-\sigma)x^\sigma-\sigma x^{1-\sigma})+b\sin(b\ln(x))(x^\sigma-x^{1-\sigma}))}{x}\ dx$

$= [\frac{((b^2+(1-\sigma)^2)x^\sigma-(b^2+\sigma^2)x^{1-\sigma})(b\sin(b\ln(x))-(b^2+\sigma(\sigma-1))\cos(b\ln(x)))}{(b^2+(1-\sigma)^2)(b^2+\sigma^2)}]_1^\infty, then$

$$\int_0^1\{\tfrac{1}{t}\}(\cos(b\ln(t))(\tfrac{(1-\sigma)}{t^\sigma}-\tfrac{\sigma}{t^{1-\sigma}})-b\sin(b\ln(t))(t^{-\sigma}-t^{\sigma-1}))dt$$
$$=\int_1^\infty \frac{\{x\}}{x^2}\ (\cos(b\ln(x))((1-\sigma)x^\sigma-\sigma x^{1-\sigma})+b\sin(b\ln(x))(x^\sigma-x^{1-\sigma}))dx$$

$$=\lim_{x\to\infty}[\tfrac{((b^2+(1-\sigma)^2)x^\sigma-(b^2+\sigma^2)x^{1-\sigma})(b\sin(b\ln(x))-(b^2+\sigma(\sigma-1))\cos(b\ln(x)))}{(b^2+(1-\sigma)^2)(b^2+\sigma^2)}]+\tfrac{(1-2\sigma)(b^2+\sigma(\sigma-1))}{(b^2+(1-\sigma)^2)(b^2+\sigma^2)}-\sum_{k=1}^{\infty}(k^{\sigma-1}-k^{-\sigma})\cos(b\ln(k))$$

(Now we will rewrite the expression using the Taylor series' expansions for sine and cosine):

$$= \lim_{x\to\infty}[\frac{((b^2+(1-\sigma)^2)x^{\sigma}-(b^2+\sigma^2)x^{1-\sigma})\sum_{k=0}^{\infty}[(\frac{(-1)^k(b\ln(x)^{2k})}{2k!})(\frac{b^2\ln(x)}{2k+1}-(b^2+\sigma(\sigma-1)))]}{(b^2+(1-\sigma)^2)(b^2+\sigma^2)}] + \frac{(1-2\sigma)(b^2+\sigma(\sigma-1))}{(b^2+(1-\sigma)^2)(b^2+\sigma^2)} - \sum_{k=1}^{\infty}(k^{\sigma-1}-k^{-\sigma})\cos(b\ln(k))$$

(Writing every member of the first expression into an infinite sum, adding the term corresponding to k=0 to cancel the first term inside the sum):

$$= \sum_{k=0}^{\infty}[(-1)^k(\lim_{x\to\infty}(\frac{((b^2+(1-\sigma)^2)x^{\sigma}-(b^2+\sigma^2)x^{1-\sigma})(\frac{(b\ln(x)^{2k})}{2k!})(\frac{b^2\ln(x)}{2k+1}-(b^2+\sigma(\sigma-1)))}{(b^2+(1-\sigma)^2)(b^2+\sigma^2)})-(-1)^k(k^{\sigma-1}-k^{-\sigma})\cos(b\ln(k)))]+ \frac{(1-2\sigma)(b^2+\sigma(\sigma-1))}{(b^2+(1-\sigma)^2)(b^2+\sigma^2)} + \lim_{k\to 0}[(k^{\sigma-1}-k^{-\sigma})\cos(b\ln(k))]$$

so then, noting that $\lim_{k\to 0}[(k^{\sigma-1}-k^{-\sigma})\cos(b\ln(k))] = \lim_{x\to\infty}[(x^{1-\sigma}-x^{\sigma})\cos(b\ln(x))] = \lim_{x\to\infty}(\sum_{k=0}^{\infty}((x^{1-\sigma}-x^{\sigma})\frac{(-1)^k(b\ln(x)^{2k})}{2k!}))$ we can complete the expression for Re(s):

$$\sigma = \frac{1}{2} + \frac{(\sigma^2-b^2+\sigma)}{2}[\sum_{k=0}^{\infty}[(-1)^k(\lim_{x\to\infty}((\frac{(b\ln(x)^{2k})}{2k!(b^2+(1-\sigma)^2)(b^2+\sigma^2)})[$$
$$x^{\sigma}((b^2+(1-\sigma)^2)(\frac{b^2\ln(x)}{2k+1}-b^2+\sigma(\sigma-1))-(b^2+(1-\sigma)^2)(b^2+\sigma^2))-$$
$$x^{1-\sigma}((b^2+\sigma^2)(\frac{b^2\ln(x)}{2k+1}-b^2+\sigma(\sigma-1))-(b^2+(1-\sigma)^2)(b^2+\sigma^2))])$$
$$-(-1)^k(k^{\sigma-1}-k^{-\sigma})\cos(b\ln(k)))] + \frac{(1-2\sigma)(b^2+\sigma(\sigma-1))}{(b^2+(1-\sigma)^2)(b^2+\sigma^2)}]$$

The next step is to check the convergence or divergence of the infinite alternating sum. In order to achieve that, we will use the Nth Term Test for Divergence:

$$\lim_{k\to\infty}[\lim_{x\to\infty}((\frac{(b\ln(x)^{2k})}{2k!(b^2+(1-\sigma)^2)(b^2+\sigma^2)})[$$
$$x^{\sigma}((b^2+(1-\sigma)^2)(\frac{b^2\ln(x)}{2k+1}-b^2+\sigma(\sigma-1))-(b^2+(1-\sigma)^2)(b^2+\sigma^2))-$$
$$x^{1-\sigma}((b^2+\sigma^2)(\frac{b^2\ln(x)}{2k+1}-b^2+\sigma(\sigma-1))-(b^2+(1-\sigma)^2)(b^2+\sigma^2))])$$
$$-(-1)^k(k^{\sigma-1}-k^{-\sigma})\cos(b\ln(k))]$$

(Since the limit containing the cosine can be proven to equal 0 using the already mentioned Squeeze Theorem and knowing that 0>Re(s)>1):

$$\lim_{k\to\infty}[\lim_{x\to\infty}((\frac{(b\ln(x)^{2k})}{2k!(b^2+(1-\sigma)^2)(b^2+\sigma^2)})[$$
$$x^{\sigma}((b^2+(1-\sigma)^2)(\frac{b^2\ln(x)}{2k+1}-b^2+\sigma(\sigma-1))-(b^2+(1-\sigma)^2)(b^2+\sigma^2))-$$
$$x^{1-\sigma}((b^2+\sigma^2)(\frac{b^2\ln(x)}{2k+1}-b^2+\sigma(\sigma-1))-(b^2+(1-\sigma)^2)(b^2+\sigma^2))])]$$

Both terms of the substraction containing x^σ and $x^{1-\sigma}$ cannot be 0 at the same time, since $\frac{b^2 \ln(x)}{2k+1}$ cannot be $2b^2+\sigma$ and $2b^2-\sigma+1$ at the same time (only when $\sigma=\frac{1}{2}$, so the substraction diverges otherwise since x tends to infinity); the only scenario in which the substraction would not diverge would be if $x^\sigma = x^{1-\sigma}$, which is only if $\sigma=\frac{1}{2}$.

Let's evaluate the first limit, since it is the one that multiplies the aforementioned substraction:

$\lim\limits_{k,x\to\infty}(\frac{(b\ln(x)^{2k})}{2k!(b^2+(1-\sigma)^2)(b^2+\sigma^2)}) =$ (along the path x=k) $= \lim\limits_{k\to\infty}(\frac{(b\ln(k)^{2k})}{2k!(b^2+(1-\sigma)^2)(b^2+\sigma^2)}) = 0$,
but $\lim\limits_{k,x\to\infty}(\frac{(b\ln(x)^{2k})}{2k!(b^2+(1-\sigma)^2)(b^2+\sigma^2)}) =$ (along $x=e^k$) $= \lim\limits_{k\to\infty}(\frac{(b\ln(k)^{2k})}{2k!(b^2+(1-\sigma)^2)(b^2+\sigma^2)}) = \infty$
So this proves that the limit does not exist, $\lim\limits_{k,x\to\infty}(\frac{(b\ln(x)^{2k})}{2k!(b^2+(1-\sigma)^2)(b^2+\sigma^2)})$ =DNE, which is different from 0, the only value this limit could take to make convergence possible.

After applying the Nth Term Test for Divergence to the alternating sum, it is clear that the sum diverges for Re(s) different from 1/2. This alone indicates that this expression for Re(s) only makes sense on the critical line.

Since σ cannot equal a non-convergent expression and the infinite sums and limits in this expression don't cancel out or converge and only exist when $\sigma=\frac{1}{2}$, σ must equal $\frac{1}{2}$ (and it is easy to see that all the elements except from the $\frac{1}{2}$ cancel out when $\sigma=\frac{1}{2}$).

Which means that

$\frac{s}{s-1} + s\int_0^1(\frac{1}{2}-\{\frac{1}{t}\})t^{s-1}dt = \frac{1-s}{-s} + (1-s)\int_0^1(\frac{1}{2}-\{\frac{1}{t}\})t^{-s}dt = \frac{1}{2}$

only if $Re(s)=\frac{1}{2}$; thus $\boldsymbol{\zeta(s)=\zeta(1-s)=0}$ **only if** $\boldsymbol{Re(s)=\frac{1}{2}}$.

This proves that Riemann's Hypothesis is, indeed, true.

4 Conclusion

In this paper, we have introduced and defined the Riemann Hypothesis, which is one of the most important unsolved problems in mathematics. We have presented known theorems that support the hypothesis, such as the absence of zeros outside the critical strip and the symmetry of zeros on the critical line.

The Riemann Zeta Function plays a central role in the investigation of the Riemann Hypothesis. We have discussed its analytic continuation, functional equation, and reflection formula. These properties allow us to study the behavior of the Zeta Function in the critical strip and its connection to the Riemann Hypothesis.

Furthermore, we presented a comprehensive step-by-step proof of Riemann's Hypothesis, starting from a very special rewriting of the Zeta Function, and ending by proving that the expression we get for the real part of s through that rewriting of the function only converges when the real part equals $1/2$. This proves that if the hypothesis was false, then we would get an expression that equals a real number to a divergent expression; which would be both incorrect and absurd.

The Riemann Hypothesis continues to fascinate mathematicians, and its resolution will have profound implications for number theory and related fields. Further research and exploration are required to deepen our understanding of the Zeta Function and its zeros, but hopefully, we can finally confirm the Riemann Hypothesis is true.

5 Acknowledgments

The author would like to express gratitude to the mathematicians who have made significant contributions to the study of the Riemann Hypothesis and the properties of the Riemann Zeta Function. Their groundbreaking work has paved the way for our understanding of this captivating problem.

6 Statements and Declarations

6.1 Competing Interests and Funding

This mathematical proof has not been funded by any company or government. The work presented in this paper is the result of years of individual research carried by myself for the sake of gaining insight about the depths of mathematics.

Several reviews of this proof during the years have allowed me to polish it progressively. Some of the intermediate steps have been shortened for brevity, but the original transformations were written in paper and occupied several pages.

The nature of the interests behind this proof is both financial and non-financial. It is clear that the prize offered by the Clay Institute for this Millenium Problem is one of the motivations for this paper, but the main point of it is about the insight I gained within every iteration of this proof, and most importantly, about the perseverance this process has taught me.

7 Keywords

Riemann, Zeta Function, Proof, Number Theory, Zeros, Hypothesis

8 References

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