On Semi − IS Regular Sets, AB_IS Sets and Decompositions of Continuity, R_IS C_IS − Continuity

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ON SEMI − IS REGULAR SETS, ABIS − SETS AND DECOMPOSITIONS OF
CONTINUITY, RISCI − CONTINUITY

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Abstract: In this paper, we introduce the notion of a semi − IS regular set and
investigate some of its properties. We show that it is weaker than the notion of a
regular-IS-closed set. Additionally, we also introduce the notion of an ABIS − set
by using the semi − IS regular set and study some of their properties.

Key words and Phrases: semi -IS- open sets, pre-IS- open sets, α − IS - open sets, regular-IS-closed sets, semi − IS regular sets and ABIS − sets.

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1 Introduction

Ideal in topological spaces have been considered since 1966 by Kuratowski [7] and
Vaidyanathaswamy [12]. After several decades, in 1990, Jankovic and Hamlet [4] investigated
the topological ideals which is the generalization of general topology. Whereas in 2010, Khan
and Noiri [5] introduced and studied the concept of semi local functions. In 2014, Shanthi and
Rameshkumar introduced semi -IS- open sets, pre-IS- open sets, α − IS - open sets, BIS - set [9]
and regular-IS-closed set [11]. In this paper we introduce the notion of a semi − IS regular set
and investigate some of its properties. We show that it is weaker than the notion of a regular-IS-
closed set. Besides, we introduce the notion of an ABIS − set by using the semi − IS regular set
and study some of their properties. This eventually shows that it is stronger than BIS − set.

Let (X, τ) be a topological space and I is an ideal of subset of X. An ideal I on a
topological space (X, τ) is a collection of nonempty subsets of X which satisfies (i) A ∈ I and
B ⊆ A implies B ∈ I and (ii) A ∈ I and B ∈ I implies A ∪ B ∈ I. Given a topological space (X, τ)
with an ideal I on X and if ϕ(X) is the set of all subsets of X, a set operator ( ) : ϕ(X) → ϕ(X)
called the local function of A with respect to τ and I, is defined as follows: for $A \subset X$, $A^*(I, \tau) = \{ x \in X | U \cap A \notin I \text{forevery} U \in \tau(x) \}$ where $\tau(x) = \{ U \in \tau / x \in U \}$ (Kuratowski 1966). A Kuratowski closure operator $cl^"(\cdot)$ for a topology $\tau^*(I, \tau)$, called the *-topology, finer than $\tau$ is defined by $cl^"(A) = A \cup A^*(I, \tau)$ (Vaidyanathaswamy, 1945). When there is no chance for confusion, we will simply write $A^*$ for $A^*(I, \tau)$ and $\tau^*$ or $\tau^*(I)$ for $\tau^*(I, \tau)$. If I is an ideal on X, then $(X, \tau, I)$ is called an ideal space.

2 Preliminaries

Definition 2.1. Let $(X, \tau)$ be a topological space. A subset A of X is said to be semi-open [8] if there exists an open set U in X such that $U \subset A \subset cl(U)$. The complement of a semi-open set is said to be semi-closed. The collection of semi-open (resp. semi-closed) sets in X is denoted by SO(X) (resp. SC(X)).

The semi closure of A in $(X, \tau)$ is denoted by $scl(A)$, and the semi-interior of A in $(X, \tau)$ is called an ideal space.

Definition 2.2. For $A \subset X$, $A_*(I, \tau) = \{ x \in X | U \cap A \notin I \text{forevery} U \in SO(X, x) \}$ is called the semi-local function [5] of A with respect to I and $\tau$ where $SO(X, x) = \{ U \in SO(X) : x \in U \}$. We write $A_*$ instead of $A_*(I, \tau)$. It is given in [1] that $\tau^*(I, \tau)$ is a topology on X, generated by the subbasis $\{ U - E : U \in SO(X), E \in I \}$ or equivalently $\tau^*(I) = \{ U \subset X : cl^{"}(X - U) = X - U \}$. The closure operator $cl^{"}$ for a topology $\tau^{"}(I)$ is defined as follows: for $A \subset X$, $cl^{\"}(A) = A \cup A_*$ and $int^{\"}(A)$ denote the interior of the set A in $(X, \tau^{\"}, I)$. It is known that $\tau \subset \tau^*(I) \subset \tau^{\"}(I)$.

A subset A of $(X, \tau, I)$ is called semi-*--perfect [6] (resp. semi--dense in itself [6], semi--closed [6]) if $A = A_*$ (resp. $A \subset A_*$, $A_* \subset A$).

Lemma 2.3 [5]. Let $(X, \tau, I)$ be an ideal space and $A, B \subset X$.

Then for the semi-local function the following properties hold:

(i) If $A \subset B$, then $A_* \subset B_*$. 

(ii) If $U \in \tau$, then $U \cap A_* \subset (U \cap A)_*$. 

(iii) $(A_*)_* = A_*$. 

(iv) $(A \cup B)_* = A_* \cup B_*$. 

Definition 2.4

A subset A of an ideal space $(X, \tau, I)$ is said to be

(i) *-I--open [3] if $A \subset int(cl^*(int(A)))$. 


(ii) pre−I − open [2] if \( A \subset \text{int} \left( \text{cl}^* (A) \right) \).

(iii) Semi−I − open [3] if \( A \subset \text{cl}^* \left( \text{int} (A) \right) \).

Definition 2.5
A subset \( A \) of an ideal space \((X, \tau, I)\) is said to be

(i) \( \alpha − I_S \) − open [9] if \( A \subset \text{int} \left( \text{cl}^{**} (\text{int} (A)) \right) \).

(ii) pre−I_S − open [9] if \( A \subset \text{int} \left( \text{cl}^{**} (A) \right) \).

(iii) semi−I_S − open [9] if \( A \subset \text{cl}^{**} \left( \text{int} (A) \right) \).

(iv) \( \alpha^* − I_S \) − set [9] if \( \text{int} \left( \text{cl}^{**} (\text{int} (A)) \right) = \text{int} (A) \)

(v) \( C_{I_S} \) − set [9] if \( A = U \cap V \), where \( U \in \tau \) and \( V \) is an \( \alpha^* − I_S \) − set.

(vi) regular − I_S − closed [11] if \( A = \text{int} (A) \).

(vii) \( \tau − I_S \) − set [9] if \( \text{int} (A) = \text{int} \left( \text{cl}^{**} (A) \right) \)

(viii) \( B_{I_S} \) − set [9] if \( A = U \cap V \), where \( U \in \tau \) and \( V \) is an \( \tau − I_S \) − set.

The family of all \( \alpha − I_S \) − open (resp. semi − I_S − open, pre−I_S − open) sets in an ideal space \((X, \tau, I)\) denoted by \( dISO(X) \) (resp. \( SISO(X) \), \( PISO(X) \)).

Lemma 2.6 [10] Let \((X, \tau, I)\) be an ideal space and \( A \subset X \).

If \( U \) is open in \((X, \tau, I)\), then \( U \cap \text{cl}^{**} (A) \subset \text{cl}^{**} (U \cap A) \)

Lemma 2.7 [10] Let \((X, \tau, I)\) be an ideal space. A subset \( A \) of \( X \) is \( \alpha − I_S \) − open if and only if it is semi − I_S − open and pre−I_S − open.

3 Regular-I_S-closed sets

Proposition 3.1 For a subset \( A \) of an ideal space \((X, \tau, I)\), the following properties hold:

(i) Every regular − I_S − closed set is \( \alpha − I_S \) − open and semi − I_S − open.

(ii) Every regular − I_S − closed set is semi−*− perfect.

Proof: (i) Let \( A \) be a regular − I_S − closed set. Then we have
\[
\text{cl}^{**} \left( \text{int} (A) \right) = \text{int} (A) \cup \left( \text{int} (A) \right) = \text{int} (A) \cup A = A.
\]
Thus, \( \text{int} \left( \text{cl}^{**} \left( \text{int} (A) \right) \right) = \text{int} (A) \) and \( A \subset \text{cl}^{**} \left( \text{int} (A) \right) \). Therefore, \( A \) is \( \alpha − I_S \) − open and semi − I_S − open.
(ii) Let $A$ be a regular $-I_s$-closed set. Then we have $A = (\text{int} (A))$. Since $\text{int} (A) \subset A_s (\text{int} (A)), \subset A_s$ by Lemma 2.3. Then we have $A = (\text{int} (A)) = A_s$. On the other hand, by Lemma 2.3 it follows from $A = (\text{int} (A))$, that $A_s = (((\text{int} (A))),) \subset (\text{int} (A))$. Therefore, we obtain $A = A_s$. This shows that $A$ is semi-**- perfect.

Corollary 3.2 Every regular $-I_s$-closed set is **-semi dense in itself and semi-**-closed.

Proof: The proof is obvious from Proposition 3.1.

Proposition 3.3 Let $A$ be a subset of an ideal space $(X, \tau, I)$. Then $A$ is a regular $-I_s$-closed set if and only if $A$ is a semi $-I_s$-open set and a semi-**-perfect set.

Proof: Let $A$ be a semi $-I_s$-open set and a semi-**-perfect set. Since $A$ is a semi $-I_s$-open set, we have $A \subset cl^{**} (\text{int} (A))$.

Respectively, by using Lemma 1.3 (i), (iv), (iii)

$A_s \subset (cl^{**} (\text{int} (A))), = (\text{int} (A) \cup \text{int} (A)), = (\text{int} (A)) \cup (\text{int} (A)), \subset (\text{int} (A)),$

hence we have $A_s \subset (\text{int} (A))$. On the other hand, since $\text{int} (A) \subset A$, by lemma (2.3), we have $(\text{int} (A)) \subset A_s$. Therefore, we obtain $A = (\text{int} (A))$. Also by hypothesis, since $A$ is a semi-**-perfect set, we have $A = A_s$. So $A = A_s = (\text{int} (A))$. That is $A = (\text{int} (A))$, and hence $A$ is a regular $-I_s$-closed set.

Converse follows from Proposition 3.1.

4. Semi $-I_s$-regular sets and its properties

Definition 4.1. A subset $A$ of an ideal space $(X, \tau, I)$ is said to be semi $-I_s$-regular if $A$ is both a $t-I_s$-set and a semi $-I_s$-open set.

We will denote the family of all semi $-I_s$-regular sets of $(X, \tau, I)$ by $S_{is}R(X, \tau)$ if there is no chance for confusion with the ideal.

Remark 4.2 Note first that $t-I_s$-sets and semi $-I_s$-regular sets are independent concepts as shown in the following examples.

Example 4.3 Let $X = \{a, b, c, d\}, \tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, d\}, \{a, b, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{b, c\}, \{b, d\}, \{c, d\}\}$ and $I = \{\{b\}, \{c\}, \{b, c\}\}$

(i) Let $A = \{a, b\}$. Then $A$ is a semi $-I_s$-open set but it is not $t-I_s$-set. Because $\text{int} (A) = \{a, b\}, (\text{int} (A)) = \{a, d\}, cl^{**} (\text{int} (A)) = \{a, b, d\} \supset \{a, b\} = A$. Hence $A$ is semi $-I_s$-open.

Therefore, $A = (\text{int} (A))$. Hence $A$ is not a $t-I_s$-set.
(ii) Let \( A = \{b, d\} \). Then \( A \) is a \( t - I_S \) set but it is not a semi \( -I_S \) open set. Because 
\( \text{int}(A) = \{b\}, A_s = d, \text{int} \left( cl^{**}(A) \right) = \text{int}(A \cup A_s) = \text{int} \{b, d\} = \{b\} = \text{int}(A) \). Hence \( A \) is a \( t - I_S \) set.

On the other hand, since \( \text{int}(A) = \{b\}, (\text{int}(A))_s = \phi, \text{cl}^{**}(\text{int}(A)) = \{b\} \).

This shows that \( A \) is not a semi \( -I_S \) open set.

Proposition 4.4 Every semi \( -** \) closed set is a \( t - I_S \) set.

Proof: Let \( A \) be a semi \( -** \) closed set. Then \( A_s \subseteq A \). Therefore \( cl^{**}(A) = A \) implies that
\( \text{int}(cl^{**}(A)) = \text{int}(A) \). This shows that \( A \) is a \( t - I_S \) set.

Proposition 4.5 For a subset \( A \) of an ideal space \((X, \tau, I)\), the following properties hold:

(i) Every regular \( -I_S \) closed set is a semi \( -I_S \) regular set.

(ii) Every semi \( -I_S \) regular set is a semi \( -I_S \) open set.

(iii) Every semi \( -I_S \) regular set is a \( t - I_S \) set.

Proof: (i) Let \( A \) be a regular \( -I_S \) closed set. Then by Proposition 3.1, Corollary 3.2 and by Proposition 3.4, \( A \) is a semi \( -I_S \) regular set.

(ii) and (iii) are obvious.

Remark 4.6 The converse of Proposition 4.5 need not be true as the following examples show.

Example 4.7 Let \((X, \tau, I)\) be the same ideal space as in Example 4.3.

(i) Let \( A = \{a, b, d\} \). Then \( A \) is semi \( -I_S \) regular, which is not a regular \( -I_S \) closed set.

Because \( A \) is open and hence \( A \) is a semi \( -I_S \) open set. But 
\( A_s = \{a, d\}, cl^{**}(A) = \{a, b, d\}, \text{int} \left( cl^{**}(A) \right) = \{a, b, d\} = \text{int}(A) \).

This shows that \( A \) is \( t - I_S \) set.

Therefore, \( A \) is a semi \( -I_S \) regular set. As \( A \) is open, we also have 
\( (\text{int}(A))_s = A_s = \{a, d\} \neq \{a, b, d\} = A \).

Hence, \( A \) is not a regular \( -I_S \) closed set.

(ii) Let \( A = \{a, b\} \). Then \( A \) is a semi \( -I_S \) open set which is not a semi \( -I_S \) regular set.

According to Example 2.1, \( A \) is a semi \( -I_S \) open set but it is not a \( t - I_S \) set. Therefore, \( A \) is not a semi \( -I_S \) regular set.

(iii) Let \( A = \{b, d\} \). Then \( A \) is a \( t - I_S \) set which is not a semi \( -I_S \) regular set. According to Example 4.3, \( A \) is a \( t - I_S \) set but it is not a semi \( -I_S \) open set. Therefore, \( A \) is not a semi \( -I_S \) regular set.

Remark 4.8 Since every semi \( -** \) perfect set is semi \( -** \) closed and every semi \( -I_S \) regular set is a semi \( -I_S \) open set, a semi \( -** \) closed (hence semi \( -** \) perfect) set and a semi \( -I_S \) open (hence semi \( -I_S \) regular) set are independent concepts as shown in the following examples.

Example 4.9 Let \((X, \tau, I)\) be the same ideal space as in Example 4.3.
(i) Let \( A = \{b, d\} \). Then \( A_s = \{d\} \subseteq \{b, d\} = A \). hence \( A \) is a semi − **− closed set. But
\[
\text{int} (A) = \{b\}, \text{int} (A) = \phi, \text{cl}^{**} (\text{int} (A)) = \{b\}.
\]
This shows that \( A \) is not a semi − \( I_s \) − open set.

(ii) Let \( A = \{a\} \). Then \( \text{int} (A) = \{a\}, \text{int} (A) = \{a, d\}, \text{cl}^{**} (\text{int} (A)) = \{a, d\} \supseteq \{a\} = A \). Hence \( A \) is a semi − \( I_s \) − open set. But \( A_s = \{a, d\} \not\subseteq \{a\} = A \).This shows that \( A \) is not a semi − **− closed set.

5. \( AB_{I_s} \) − set

Definition 5.1 A subset \( A \) of an ideal space \((X, \tau, I)\) is said to be a \( AB_{I_s} \) − set if \( A = U \cap V \),
where \( U \in \tau \) and \( V \) is a semi − \( I_s \) − regular set.

We will denote the family of all \( AB_{I_s} \) − set of \((X, \tau, I)\) by \( AB_{I_s} (X, \tau) \), if there is no chance for confusion with the ideal.

Proposition 5.2 For a subset \( A \) of an ideal space \((X, \tau, I)\), the following properties hold:

(i) Every open set is a \( AB_{I_s} \) − set.

(ii) Every semi − \( I_s \) − regular set is a \( AB_{I_s} \) − set.

(iii) Every \( AB_{I_s} \) − set

Proof: (i),(ii) Since \( X \in \tau \cap S_{I_s} R(X, \tau) \), the statements are clear.

(iii) Since every semi − \( I_s \) − regular set is a \( t − I_s \) − set, it is obvious.

Proposition 5.3 Let \((X, \tau, I)\) be an ideal space. Then every \( AB_{I_s} \) − set is semi − \( I_s \) − open.

Proof: Let \( A \) be a \( AB_{I_s} \) − set. Then \( A = U \cap V \), where \( U \in \tau \) and \( V \) is a semi − \( I_s \) − regular set.

By Definition 4.1, \( V \) is also semi − \( I_s \) − open. Since \( V \) is semi − \( I_s \) − open,
\[
A = U \cap V \subseteq U \cap \text{cl}^{**} (\text{int} (V)) \subseteq \text{cl}^{**} (U \cap \text{int} (V)) = \text{cl}^{**} (\text{int} (U \cap V)) = \text{cl}^{**} (\text{int} (A))
\]
and hence \( A \subseteq \text{cl}^{**} (\text{int} (A)) \). This shows that \( A \) is semi − \( I_s \) − open.

Proposition 5.4 For a subset \( A \) of an ideal space \((X, \tau, I)\), the following properties are equivalent:

(i) \( A \) is an open set

(ii) \( A \) is an \( \alpha − I_s \) − open set and an \( AB_{I_s} \) − set.

(iii) \( A \) is a pre − \( I_s \) − open set and an \( AB_{I_s} \) − set.

Proof: (i) \( \Rightarrow \) (ii) and (ii) \( \Rightarrow \) (iii) are obvious.

(iii) \( \Rightarrow \) (i) Let \( A \) be a pre − \( I_s \) − open set and an \( AB_{I_s} \) − set.

Then, since \( A \) is a pre − \( I_s \) − open set, we have \( A \subseteq \text{int} (\text{cl}^{**} (A)) \). Furthermore, because \( A \) is an \( AB_{I_s} \) − set, we have \( A = U \cap V \), where \( U \) is open and \( V \) is a semi − \( I_s \) − regular set. Since \( V \) is a semi − \( I_s \) − regular set, \( V \) is also a \( t − I_s \) − set. Thus \( \text{int} (V) = \text{int} (\text{cl}^{**} (V)) \)

Now \( A \subseteq \text{int} (\text{cl}^{**} (A)) \)

Since $A \subseteq U$, we have
\[ A = U \cap A \subseteq U \cap \left[ \text{int} \left( \text{cl}^{**}(U) \right) \cap \text{int}(V) \right] = U \cap \left[ \text{int} \left( \text{cl}^{**}(U) \right) \right] \cap \text{int}(V) = U \cap \text{int}(V) \]
and $A \subseteq U \cap \text{int}(V)$. Since $U$ is an open set, we have $A \subseteq U \cap \text{int}(V) \subseteq \text{int}(A)$. Thus $A \in \tau$

6 Decomposition of continuity

Definition 6.1 A function $f : (X, \tau, I) \to (Y, \sigma)$ is said to be semi-$$**$$ perfectly continuous (resp. semi-$I_{S}$-regular continuous, $I_{S}$-continuous, $AB_{I_{S}}$-continuous, $R_{I_{S}}$-continuous), semi-$I_{S}$-continuous [11], semi-$I_{S}$-continuous [9], $B_{I_{S}}$-continuous [9], $\alpha - I_{S}$-continuous [9], pre-$I_{S}$-continuous [9]) if for every $V \in \sigma$, $f^{-1}(V)$ is semi-$$**$$ perfect (resp. semi-$I_{S}$-regular set, $I_{S}$-set, regular-$I_{S}$-closed set, semi-$I_{S}$-open set, $B_{I_{S}}$-set, $\alpha - I_{S}$-open set, pre-$I_{S}$-open set) set of $(X, \tau, I)$.

Theorem 6.2 For a function $f : (X, \tau, I) \to (Y, \sigma)$, the following are equivalent.
(i) $f$ is $R_{I_{S}}$-continuous.
(ii) $f$ is semi-$I_{S}$-continuous and semi-$$**$$ perfectly continuous.

Proof: The proof is obvious from Proposition 3.3

Theorem 6.3. For a function $f : (X, \tau, I) \to (Y, \sigma)$, the following properties hold.
(i) If $f$ is $R_{I_{S}}$-continuous, then $f$ is semi-$I_{S}$-regular continuous,
(ii) If $f$ is semi-$I_{S}$-regular continuous, then $f$ is semi-$I_{S}$-continuous
(iii) If $f$ is semi-$I_{S}$-continuous, then $f$ is $I_{S}$-continuous.

Proof: The proof is obvious from Proposition 4.5

Theorem 6.4. For a function $f : (X, \tau, I) \to (Y, \sigma)$, the following properties hold.
(i) If $f$ is continuous, then $f$ is $AB_{I_{S}}$-continuous.
(ii) If $f$ is semi-$I_{S}$-regular continuous, then $f$ is $AB_{I_{S}}$-continuous.
(iii) If $f$ is $AB_{I_{S}}$-continuous, then $f$ is $B_{I_{S}}$-continuous.

Proof: The proof is obvious from Proposition 5.2

Theorem 6.5. For a function $f : (X, \tau, I) \to (Y, \sigma)$, the following property hold.
If $f$ is $AB_{Is}$–continuous, then $f$ is semi $-I_s$–continuous.

Proof: The proof is obvious from Proposition 5.3.

Theorem 6.6. For a function $f : (X, \tau, I) \to (Y, \sigma)$ the following are equivalent.

(i) $f$ is continuous

(ii) $f$ is $\alpha - I_s$–continuous and $AB_{Is}$–continuous.

(iii) $f$ is pre $-I_s$–continuous and $AB_{Is}$–continuous

Proof: This proof follows from Proposition 5.4.

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