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THE DISPERSIONLESS INTEGRABLE SYSTEMS AND RELATED CONFORMAL STRUCTURE GENERATING EQUATIONS OF MATHEMATICAL PHYSICS

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Abstract. Based on the diffeomorphism group vector fields on the complexified torus and the related Lie-algebraic structures, we study multi-dimensional dispersionless integrable systems, describing conformal structure generating equations of mathematical physics. An interesting modification of the devised Lie-algebraic approach subject to the spatial dimensional invariance and meromorphicity of the related differential-geometric structures is described and applied to proving complete integrability of some conformal structure generating equations. As examples, we analyzes the Einstein–Weyl metric equation, the modified Einstein–Weyl metric equation, the Dunajski heavenly equation system, the first and second conformal structure generating equations, the inverse first Shabat reduction heavenly equation, the first and modified Plebański heavenly equations and its multi-dimensional generalizations, the Husain heavenly equation and its multi-dimensional generalizations, the general Monge equation and its multi-dimensional generalizations. We also construct superconformal analogs of the Whitham heavenly equation.

1. Vector fields on the complexified torus and the related Lie-algebraic properties

Consider the loop Lie group $\tilde{G} := \widetilde{Diff}(\mathbb{T}^n_{\mathbb{C}})$, consisting [14] of the set of smooth mappings $\{\mathbb{C}^1 \supset \mathbb{S}^1 \longrightarrow \mathcal{G} = Diff(\mathbb{T}^n\}$, extended, respectively, holomorphically from the circle $\mathbb{S}^1 \subset \mathbb{C}^1$ on the set $\mathbb{D}^1_+$ of the internal points of the circle $\mathbb{S}^1$, and on the set $\mathbb{D}^1_-$ of the external points $\lambda \in \mathbb{C}\backslash\overline{\mathbb{D}}^1_+$. The corresponding diffeomorphism Lie algebra splitting $\tilde{\mathcal{G}} := \tilde{\mathcal{G}}_+ \oplus \tilde{\mathcal{G}}_-$, where $\tilde{\mathcal{G}}_+ := \widetilde{diff}(\mathbb{T}^n)_+ \subset \Gamma(\mathbb{T}^n_{\mathbb{C}}; T(\mathbb{T}^n_{\mathbb{C}}))$ is a Lie subalgebra, consisting of vector fields on the complexified torus $\mathbb{T}^n_{\mathbb{C}} \simeq \mathbb{T}^n \times \mathbb{C}$, suitably holomorphic on the disc $\mathbb{D}^1_+$, $\tilde{\mathcal{G}}_- := \widetilde{diff}(\mathbb{T}^n_{\mathbb{C}})_- \subset \Gamma(\mathbb{T}^n_{\mathbb{C}}; T(\mathbb{T}^n_{\mathbb{C}}))$ is a Lie subalgebra, consisting of vector fields on the complexified torus $\mathbb{T}^n_{\mathbb{C}} \simeq \mathbb{T}^n \times \mathbb{C}$, suitably holomorphic on the set $\mathbb{D}^1_-$. The adjoint space $\tilde{\mathcal{G}}^* := \tilde{\mathcal{G}}^*_+ \oplus \tilde{\mathcal{G}}^*_-$, where the space $\tilde{\mathcal{G}}^*_+ \subset \Gamma(\mathbb{T}^n_{\mathbb{C}}; T^*(\mathbb{T}^n_{\mathbb{C}}))$ consists, respectively, from the differntial forms on the complexified torus $\mathbb{T}^n_{\mathbb{C}}$, suitably holomorphic on the set $\mathbb{C}\backslash\overline{\mathbb{D}}^1_+$, and the adjoint space $\tilde{\mathcal{G}}^*_- \subset \Gamma(\mathbb{T}^n_{\mathbb{C}}; T^*(\mathbb{T}^n_{\mathbb{C}}))$ consists, respectively, from the differntial forms on the complexified torus $\mathbb{T}^n_{\mathbb{C}}$, suitably holomorphic on the set $\mathbb{D}^1_+$, so that the space $\tilde{\mathcal{G}}^*_+$ is dual to $\tilde{\mathcal{G}}_+$ and $\tilde{\mathcal{G}}^*_-$ is dual to $\tilde{\mathcal{G}}_-$ with respect to the following convolution form on the product $\tilde{\mathcal{G}}^* \times \tilde{\mathcal{G}}$:

$$(\tilde{l}|\tilde{a}) := res_\lambda \int_{\mathbb{T}^n} < l, a > dx \tag{1.1}$$

for any vector field $\tilde{a} :=< a(\mathrm{x}), \frac{\partial}{\partial \mathrm{x}} >\in \tilde{\mathcal{G}}$ and differential form $\tilde{l} :=< l(\mathrm{x}), d\mathrm{x} > \in \tilde{\mathcal{G}}^*$ on $\mathbb{T}^n_{\mathbb{C}}$, depending on the coordinate $\mathrm{x} := (\lambda; x) \in \mathbb{T}^n_{\mathbb{C}}$, where, by definition, $< \cdot, \cdot >$ is the usual scalar product on the Euclidean space $\mathbb{E}^{n+1}$ and $\frac{\partial}{\partial \mathrm{x}} := (\frac{\partial}{\partial \lambda}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n})^\top$ is the usual gradient vector. The Lie algebra $\tilde{\mathcal{G}}$ allows the

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direct sum splitting $\tilde{\mathcal{G}} = \tilde{\mathcal{G}}_+ \oplus \tilde{\mathcal{G}}_-$, causing with respect to the convolution (1.1) the direct sum splitting $\tilde{\mathcal{G}}^* = \tilde{\mathcal{G}}_+^* \oplus \tilde{\mathcal{G}}_-^*$. If to define now the set $\mathrm{I}(\tilde{\mathcal{G}}^*)$ of Casimir invariant smooth functionals $h : \tilde{\mathcal{G}}^* \to \mathbb{R}$ on the adjoint space $\tilde{\mathcal{G}}^*$ via the coadjoint Lie algebra $\tilde{\mathcal{G}}$ action

$$ad^*_{\nabla h(\tilde{l})}\tilde{l} = 0 \tag{1.2}$$

at a seed element $\tilde{l} \in \tilde{\mathcal{G}}^*$, by means of the classical Adler-Kostant-Symes scheme [15, 7, 2, 1] one can generate [11, 12, 16, 9] a wide class of multi-dimensional completely integrable dispersionless (heavenly type) commuting to each other Hamiltonian systems

$$d\tilde{l}/dt := -ad^*_{\nabla h_+(\tilde{l})}\tilde{l} \ , \tag{1.3}$$

for all $h \in I(\tilde{\mathcal{G}}^*)$, $\nabla h(\tilde{l}) := \nabla h_+(\tilde{l}) \oplus \nabla h_-(\tilde{l}) \in \tilde{\mathcal{G}}_+ \oplus \tilde{\mathcal{G}}_-$, on suitable functional manifolds. Moreover, these commuting to each other flows (1.3) can be equivalently represented as a commuting system of Lax-Sato type [9] vector field equations on the functional space $C^2(\mathbb{T}^n_{\mathbb{C}};\mathbb{C})$, generating an complete set of first integrals for them. As it was appeared, amongst them there are present important equations for modern studies in physics, hydrodynamics and, in particular, in Riemannian geometry, being related with such interesting conformal structures on Riemannian metric spaces as Einstein and Einstein-Weyl metrics equations, the first and second Plebański conformal metric equations, Dunajski metric equations etc. What was observed, some of them were generated by seed elements $\tilde{l} \in \tilde{\mathcal{G}}^*$, meromorphic at some points of the complex plane $\mathbb{C}$, whose analysis needed some modification of the theoretical backgrounds. Moreover, the general differential-geometric structure of seed elements, related with some conformal metric equations, proved to be invariant subject to the spatial dimension of the Riemannian spaces under regard, that made it possible to describe them analytically. Namely these and related aspects of the integrable conformal metric equations, mentioned above, are studied and presented in the work.

2. The Lie-algebraic structures and integrable Hamiltonian systems

Consider the loop Lie algebra $\tilde{\mathcal{G}}$, determined above. This Lie algebra has elements representable as $a(x;\lambda) :=< a(x;\lambda), \frac{\partial}{\partial \mathrm{x}} >= \sum_{j=1}^{n} a_j(x;\lambda)\frac{\partial}{\partial x_j} + a_0(x;\lambda)\frac{\partial}{\partial \lambda} \in \tilde{\mathcal{G}}$ for some holomorphic in $\lambda \in \mathbb{D}^1_\pm$ vectors $a(x;\lambda) \in \mathbb{E} \times \mathbb{E}^n$ for all $x \in \mathbb{T}^n$, where $\frac{\partial}{\partial \mathrm{x}} := (\frac{\partial}{\partial \lambda}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n})^\top$ is the generalized Euclidean vector gradient with respect to the vector variable $\mathrm{x} := (\lambda, x) \in \mathbb{T}^n_{\mathbb{C}}$. As it was mentioned above, the Lie algebra $\tilde{\mathcal{G}}$ naturally splits into the direct sum of two subalgebras:

$$\tilde{\mathcal{G}} = \tilde{\mathcal{G}}_+ \oplus \tilde{\mathcal{G}}_-, \tag{2.1}$$

allowing to introduce on it the classical $\mathcal{R}$-structure:

$$[\tilde{a}, \tilde{b}]_{\mathcal{R}} := [\mathcal{R}\tilde{a}, \tilde{b}] + [\tilde{a}, \mathcal{R}\tilde{b}] \tag{2.2}$$

for any $\tilde{a}, \tilde{b} \in \tilde{\mathcal{G}}$, where

$$\mathcal{R} := (P_+ - P_-)/2, \tag{2.3}$$

and

$$P_\pm \tilde{\mathcal{G}} := \tilde{\mathcal{G}}_\pm \subset \tilde{\mathcal{G}}. \tag{2.4}$$

The space $\tilde{\mathcal{G}}^* \simeq \tilde{\Lambda}^1(\mathbb{T}^n_{\mathbb{C}})$, adjoint to the Lie algebra $\tilde{\mathcal{G}}$ of vector fields on $\mathbb{T}^n_{\mathbb{C}}$, is functionally identified with $\tilde{\mathcal{G}}$ subject to the metric (1.1). Now for arbitrary $f, g \in \mathrm{D}(\tilde{\mathcal{G}}^*)$, one can determine two Lie–Poisson type brackets

$$\{f, g\} := (\tilde{l}, [\nabla f(\tilde{l}), \nabla g(\tilde{l})]) \tag{2.5}$$

and

$$\{f, g\}_{\mathcal{R}} := (\tilde{l}, [\nabla f(\tilde{l}), \nabla g(\tilde{l})]_{\mathcal{R}}) \ , \tag{2.6}$$

where at any seed element $\bar{l} \in \tilde{\mathcal{G}}^*$ the gradient element $\nabla f(\tilde{l})$ and $\nabla g(\tilde{l}) \in \tilde{\mathcal{G}}$ are calculated with respect to the metric (1.1).

Now let us assume that a smooth function $\gamma \in I(\tilde{\mathcal{G}}^*)$ is a Casimir invariant, that is

$$ad^*_{\nabla\gamma(\tilde{l})}\tilde{l} = 0 \tag{2.7}$$

for a chosen seed element $\tilde{l} \in \tilde{\mathcal{G}}^*$. As the coadjoint mapping $ad^*_{\nabla f(\tilde{l})} : \tilde{\mathcal{G}}^* \to \tilde{\mathcal{G}}^*$ for any $f \in \mathrm{D}(\tilde{\mathcal{G}}^*)$ can be rewritten in the reduced form as

$$ad^*_{\nabla f(\tilde{l})}(\tilde{l}) = \left\langle \frac{\partial}{\partial \mathrm{x}}, \circ\nabla f(l)\right\rangle \bar{l} + \sum_{j=1}^{n}\left\langle\left\langle l,\ \frac{\partial}{\partial \mathrm{x}}\ \nabla f(l)\ \right\rangle, dx\right\rangle, \tag{2.8}$$

where, by definition, $\nabla f(\tilde{l}) :=< \nabla f(l), \frac{\partial}{\partial \mathrm{x}} >$. For the Casimir function $\gamma \in \mathrm{D}(\tilde{\mathcal{G}}^*)$ the condition (2.7) is then equivalent to the equation

$$l\left\langle \frac{\partial}{\partial \mathrm{x}}, \nabla\gamma(l)\right\rangle + \left\langle \nabla\gamma(l), \frac{\partial}{\partial \mathrm{x}}\right\rangle l + \left\langle l, (\frac{\partial}{\partial \mathrm{x}}\nabla\gamma(l))\right\rangle = 0, \tag{2.9}$$

which should be solved analytically. In the case when an element $\tilde{l} \in \tilde{\mathcal{G}}^*$ is singular as $|\lambda| \to \infty$, one can consider the general asymptotic expansion

$$\nabla\gamma := \nabla\gamma^{(p)} \sim \lambda^p \sum_{j\in\mathbb{Z}_+} \nabla\gamma_j^{(p)}\lambda^{-j} \tag{2.10}$$

for some suitably chosen $p \in \mathbb{Z}_+$, and upon substituting (2.10) into the equation (2.9), one can proceed to solving it recurrently.

Now let $h^{(y)}, h^{(t)} \in I(\tilde{\mathcal{G}}^*)$ be such Casimir functions for which the Hamiltonian vector field generators

$$\nabla h_+^{(y)}(l) := (\ \nabla\gamma^{(p_y)}(l))|_+, \qquad \nabla h_+^{(t)}(l) := (\ \nabla h^{(p_t)}(l))|_+ \tag{2.11}$$

are, respectively, defined for special integers $p_y, p_t \in \mathbb{Z}_+$. These invariants generate, owing to the Lie–Poisson bracket (2.6), the following commuting flows:

$$\partial l/\partial t \ = - \left\langle \frac{\partial}{\partial \mathrm{x}}, \circ\nabla h_+^{(t)}(l)\right\rangle l - \left\langle l, (\frac{\partial}{\partial \mathrm{x}}\nabla h_+^{(t)}(l))\right\rangle \tag{2.12}$$

and

$$\partial l/\partial y \ = - \left\langle \frac{\partial}{\partial \mathrm{x}}, \circ\nabla h_+^{(y)}(l)\right\rangle l - \left\langle l, (\frac{\partial}{\partial \mathrm{x}}\nabla h_+^{(y)}(l))\right\rangle, \tag{2.13}$$

where $y, t \in \mathbb{R}$ are the corresponding evolution parameters. Since the invariants $h^{(y)}, h^{(t)} \in I(\tilde{\mathcal{G}}^*)$ commute with respect to the Lie–Poisson bracket (2.6), the flows (2.12) and (2.13) also commute, implying that the corresponding Hamiltonian vector field generators

$$A_{\nabla h_+^{(t)}} := \left\langle \nabla h_+^{(t)}(l), \frac{\partial}{\partial\ \mathrm{x}}\right\rangle, \qquad A_{\nabla h_+^{(y)}} := \left\langle \nabla h_+^{(y)}(l), \frac{\partial}{\partial\ \mathrm{x}}\right\rangle \tag{2.14}$$

satisfy the Lax compatibility condition

$$\frac{\partial}{\partial y}A_{\nabla h_+^{(t)}} - \frac{\partial}{\partial t}A_{\nabla h_+^{(y)}} = [A_{\nabla h_+^{(t)}}, A_{\nabla h_+^{(y)}}] \tag{2.15}$$

for all $y, t \in \mathbb{R}$. On the other hand, the condition (2.15) is equivalent to the compatibility condition of two linear equations

$$\left(\frac{\partial}{\partial t} + A_{\nabla h_+^{(t)}}\right)\psi = 0, \qquad \left(\frac{\partial}{\partial y} + A_{\nabla h_+^{(y)}}\right)\psi = 0 \tag{2.16}$$

for a function $\psi \in C^2(\mathbb{R}^2 \times \mathbb{T}^n_{\mathbb{C}}; \mathbb{C})$ for all $y, t \in \mathbb{R}$ and any $\lambda \in \mathbb{C}$. The above can be formulated as the following key result:

Proposition 2.1. *Let a seed vector field be $\tilde{l} \in \tilde{\mathcal{G}}^*$ and $h^{(y)}, h^{(t)} \in I(\tilde{\mathcal{G}}^*)$ be Casimir functions subject to the metric $(\cdot|\cdot)$ on the loop Lie algebra $\tilde{\mathcal{G}}$ and the natural coadjoint action on the loop co-algebra $\tilde{\mathcal{G}}^*$. Then the following dynamical systems*

$$\partial\tilde{l}/\partial y = -ad^*_{\nabla h_+^{(y)}(\tilde{l})}\tilde{l}, \quad \partial\tilde{l}/\partial t = -ad^*_{\nabla h_+^{(t)}(\tilde{l})}\tilde{l} \tag{2.17}$$

are commuting Hamiltonian flows for all $y, t \in \mathbb{R}$. Moreover, the compatibility condition of these flows is equivalent to the vector fields representation (2.16), where $\psi \in C^2(\mathbb{R}^2 \times \mathbb{T}^n_{\mathbb{C}}; \mathbb{C})$ and the vector fields $A_{\nabla h_+^{(y)}}, A_{\nabla h_+^{(t)}} \in \tilde{\mathcal{G}}$ are given by the expressions (2.14) and (2.11).

Remark 2.2. As mentioned above, the expansion (2.10) is effective if a chosen seed element $\tilde{l} \in \tilde{\mathcal{G}}^*$ is singular as $|\lambda| \to \infty$. In the case when it is singular as $|\lambda| \to 0$, the expression (2.10) should be replaced by the expansion

$$\nabla \gamma^{(p)}(l) \sim \lambda^{-p} \sum_{j \in \mathbb{Z}_+} \nabla \gamma_j^{(p)}(l) \lambda^j \tag{2.18}$$

for suitably chosen integers $p \in \mathbb{Z}_+$, and the reduced Casimir function gradients then are given by the Hamiltonian vector field generators

$$\begin{aligned} \nabla h_-^{(y)}(l) &:= \lambda(\lambda^{-p_y-1} \nabla \gamma^{(p_y)}(l))_-, \\ \nabla h_-^{(t)}(l) &:= \lambda(\lambda^{-p_t-1} \nabla \gamma^{(p_t)}(l))_- \end{aligned} \tag{2.19}$$

for suitably chosen positive integers $p_y, p_t \in \mathbb{Z}_+$ and the corresponding Hamiltonian flows are, respectively, written as $\partial \tilde{l}/\partial t = ad^*_{\nabla h_-^{(t)}(\tilde{l})} \tilde{l}, \ \ \partial \tilde{l}/\partial y = ad^*_{\nabla h_-^{(y)}(\tilde{l})} \tilde{l}.$

It is also worth of mentioning that, following Ovsienko's scheme [11, 12], one can consider a slightly wider class of integrable heavenly equations, realized as compatible Hamiltonian flows on the semidirect product of the holomorphic loop Lie algebra $\tilde{\mathcal{G}}$ of vector fields on the torus $\mathbb{T}^n_{\mathbb{C}}$ and its regular co-adjoint space $\tilde{\mathcal{G}}^*$, supplemented with naturally related cocycles.

3. The Lax-Sato type integrable systems and related conformal structure generating equations

3.1. **Einstein–Weyl metric equation.** Define $\tilde{\mathcal{G}}^* = \widetilde{diff}(\mathbb{T}^1_{\mathbb{C}})^*$ and take the seed element

$$\tilde{l} = (u_x \lambda - 2u_x v_x - u_y)\, dx + \left(\lambda^2 - v_x \lambda + v_y + v_x^2\right) d\lambda,$$

which generates with respect to the metric (1.1) the gradient of the Casimir invariants $h^{(p_t)}, h^{(p_y)} \in I(\tilde{\mathcal{G}}^*)$ in the form

$$\begin{aligned} \nabla h^{(p_t)}(l) &\sim \lambda^2 (0,1)^\top + (-u_x, v_x)^\top \lambda + (u_y, u - v_y)^\top + O(\lambda^{-1}), \\ \nabla h^{(p_y)}(l) &\sim \lambda(0,1)^\top + (-u_x, v_x)^\top + (u_y, -v_y)^\top \lambda^{-1} + O(\lambda^{-2}) \end{aligned} \tag{3.1}$$

as $|\lambda| \to \infty$ at $p_t = 2$, $p_y = 1$. For the gradients of the Casimir functions $h^{(t)}, h^{(y)} \in I(\tilde{\mathcal{G}}^*)$, determined by (2.11) one can easily obtain the corresponding Hamiltonian vector field generators

$$\begin{aligned} A_{\nabla h_+^{(t)}} &= \left\langle \nabla h_+^{(t)}(l), \frac{\partial}{\partial \mathrm{x}} \right\rangle = (\lambda^2 + \lambda v_x + u - v_y) \frac{\partial}{\partial x} + (-\lambda u_x + u_y) \frac{\partial}{\partial \lambda}, \\ A_{\nabla h_+^{(y)}} &= \left\langle \nabla h_+^{(y)}(l), \frac{\partial}{\partial \mathrm{x}} \right\rangle = (\lambda + v_x) \frac{\partial}{\partial x} - u_x \frac{\partial}{\partial \lambda}, \end{aligned} \tag{3.2}$$

satisfying the compatibility condition (2.15), which is equivalent to the set of equations

$$\begin{aligned} u_{xt} + u_{yy} + (u u_x)_x + v_x u_{xy} - v_y u_{xx} &= 0, \\ v_{xt} + v_{yy} + u v_{xx} + v_x v_{xy} - v_y v_{xx} &= 0, \end{aligned} \tag{3.3}$$

describing general integrable Einstein–Weyl metric equations [6].

As is well known [10], the invariant reduction of (3.3) at $v = 0$ gives rise to the famous dispersionless Kadomtsev–Petviashvili equation

$$(u_t + u u_x)_x + u_{yy} = 0, \tag{3.4}$$

for which the reduced vector field representation (2.16) follows from (3.2) and is given by the vector fields

$$A_{\nabla h_+^{(t)}} = (\lambda^2+u)\frac{\partial}{\partial x} + (-\lambda u_x + u_y)\frac{\partial}{\partial \lambda}, \tag{3.5}$$
$$A_{\nabla h_+^{(y)}} = \lambda\frac{\partial}{\partial x} - u_x\frac{\partial}{\partial \lambda},$$

satisfying the compatibility condition (2.15), equivalent to the equation (3.4). In particular, one derives from (2.16) and (3.5) the vector field compatibility relationships

$$\frac{\partial \psi}{\partial t} + (\lambda^2+u)\frac{\partial \psi}{\partial x} + (-\lambda u_x + u_y)\frac{\partial \psi}{\partial \lambda} = 0 \tag{3.6}$$
$$\frac{\partial \psi}{\partial y} + \lambda\frac{\partial \psi}{\partial x} - u_x\frac{\partial \psi}{\partial \lambda} = 0,$$

satisfied for $\psi \in C^2(\mathbb{R}^2 \times \mathbb{T}^1_{\mathbb{C}}; \mathbb{C})$ and any $y, t \in \mathbb{R}, (x,\lambda) \in \mathbb{T}^1_{\mathbb{C}}$.

3.2. **The modified Einstein–Weyl metric equation.** This equation system is

$$u_{xt} = u_{yy} + u_x u_y + u_x^2 w_x + u u_{xy} + u_{xy} w_x + u_{xx} a, \tag{3.7}$$
$$w_{xt} = u w_{xy} + u_y w_x + w_x w_{xy} + a w_{xx} - a_y,$$

where $a_x := u_x w_x - w_{xy}$, and was recently derived in [17]. In this case we take also $\tilde{\mathcal{G}}^* = \widetilde{diff}(\mathbb{T}^1_{\mathbb{C}})$, yet for a seed element $\tilde{l} \in \tilde{\mathcal{G}}$ we choose the form

$$\tilde{l} = [\lambda^2 u_x + (2u_x w_x + u_y + 3uu_x)\lambda + 2u_x\partial_x^{-1} u_x w_x + 2u_x \partial_x^{-1} u_y + \tag{3.8}$$
$$+ 3u_x {w_x}^2 + 2u_y w_x + 6uu_x w_x + 2uu_y + 3u^2 u_x - 2au_x]dx +$$
$$+ [\lambda^2 + (w_x + 3u)\lambda + 2\partial_x^{-1}\ u_x w_x + 2\partial_x^{-1} u_y + {w_x}^2 + 3uw_x + 3u^2 - a]d\lambda,$$

which with respect to the metric (1.1) generates two Casimir invariants $\gamma^{(j)} \in I(\tilde{\mathcal{G}}^*)$, $j = \overline{1,2}$, whose gradients are

$$\nabla\gamma^{(2)}(l) \sim \lambda^2[(u_x, -1)^\top + (uu_x + u_y, -u + w_x)^\top \lambda^{-1} + \tag{3.9}$$
$$+ (0, uw_x - a)^\top \lambda^{-2}] + O(\lambda^{-1})\ ,$$
$$\nabla\gamma^{(1)}(l) \sim \lambda[(u_x, -1)^\top + (0, w_x)^\top\ \lambda^{-1}] + O(\lambda^{-1}),$$

as $|\lambda| \to \infty$ at $p_y = 1, p_t = 2$. The corresponding gradients of the Casimir functions $h^{(t)}, h^{(y)} \in I(\mathcal{G}^*)$, determined by (2.11), generate the Hamiltonian vector field expressions

$$\nabla h_+^{(y)} := \nabla\gamma^{(1)}(l)|_+ = (u_x\lambda, -\lambda + w_x)^\top, \tag{3.10}$$
$$\nabla h_+^{(t)} = \nabla\gamma^{(2)}(l)|_+ = (u_x\lambda^2 + (uu_x + u_y)\lambda, -\lambda^2 + (w_x - u)\lambda + uw_x - a)^\top.$$

Now one easily obtains from (3.10) the compatible Lax system of linear equations

$$\frac{\partial \psi}{\partial y} + (-\lambda + w_x)\frac{\partial \psi}{\partial x} + u_x\lambda\frac{\partial \psi}{\partial \lambda} = 0, \tag{3.11}$$
$$\frac{\partial \psi}{\partial t} +\ (-\lambda^2 + (\ w_x - u)\lambda + uw_x - a)\frac{\partial \psi}{\partial x} + (u_x\lambda^2 + (uu_x + u_y)\lambda)\frac{\partial \psi}{\partial \lambda} = 0,$$

satisfied for $\psi \in C^2(\mathbb{R}^2 \times \mathbb{T}^1_{\mathbb{C}}; \mathbb{C})$ and any $y, t \in \mathbb{R},\ (\lambda, x) \in \mathbb{T}^1_{\mathbb{C}}$.

3.3. **The Dunajski heavenly equation system.** This equation, suggested in [5], generalizes the corresponding anti-self-dual vacuum Einstein equation, which is related to the Plebański metric and the celebrated Plebański [13, 8] second heavenly equation. To study the integrability of the Dunajski equations

$$u_{x_1 t} + u_{y x_2} + u_{x_1 x_1} u_{x_2 x_2} - u^2_{x_1 x_2} - v = 0, \tag{3.12}$$
$$v_{x_1 t} + v_{x_2 y} + u_{x_1 x_1} v_{x_2 x_2} - 2u_{x_1 x_2} v_{x_1 x_2} = 0,$$

where $(u,v) \in C^\infty(\mathbb{R}^2 \times \mathbb{T}^2; \mathbb{R}^2)$, $(y, t; x_1, x_2) \in \mathbb{R}^2 \times \mathbb{T}^2$, we define $\tilde{\mathcal{G}}^* = \widetilde{diff}(\mathbb{T}^2_{\mathbb{C}})^*$ and take the following as a seed element $\tilde{l} \in \tilde{\mathcal{G}}^*$

$$\tilde{l} = (\lambda + v_{x_1} - u_{x_1 x_1} + u_{x_1 x_2})dx_1 + (\lambda + v_{x_2} + u_{x_2 x_2} - u_{x_1 x_2})dx_2 + (\lambda - x_1 - x_2)d\lambda. \tag{3.13}$$

With respect to the metric (1.1), the gradients of two functionally independent Casimir invariants $h^{(p_y)}, h^{(p_y)} \in I(\tilde{\mathcal{G}}^*)$ can be obtained as $|\lambda| \to \infty$ in the asymptotic form as

$$\begin{aligned} \nabla h^{(p_y)}(l) &\sim \lambda(1,0,0)^\top + (-u_{x_1x_2}, u_{x_1x_1}, -v_{x_1})^\top + O(\lambda^{-1}), \\ \nabla h^{(p_t)}(l) &\sim \lambda(0,-1,0)^\top + (u_{x_2x_2}, -u_{x_1x_2}, v_{x_2})^\top + O(\lambda^{-1}), \end{aligned} \tag{3.14}$$

at $p_t = 1 = p_y$. Upon calculating the Hamiltonian vector field generators

$$\begin{aligned} \nabla h_+^{(y)} &:= \nabla h^{(p_y)}(l)|_+ = (\lambda - u_{x_1x_2}, u_{x_1x_1}, -v_{x_1})^\top, \\ \nabla h_+^{(t)} &:= \nabla h^{(p_t)}(l)|_+ = (u_{x_2x_2}, -\lambda - u_{x_1x_2}, v_{x_2})^\top, \end{aligned} \tag{3.15}$$

following from the Casimir functions gradients (3.14), one easily obtains the following vector fields

$$\begin{aligned} A_{\nabla h_+^{(t)}} &=< \nabla h_+^{(t)}, \frac{\partial}{\partial \mathrm{x}} >= u_{x_2x_2}\frac{\partial}{\partial x_1} - (\lambda + u_{x_1x_2})\frac{\partial}{\partial x_2} + v_{x_2}\frac{\partial}{\partial \lambda}, \\ A_{\nabla h_+^{(y)}} &=< \nabla h_+^{(y)}, \frac{\partial}{\partial \mathrm{x}} >= (\lambda - u_{x_1x_2})\frac{\partial}{\partial x_1} + u_{x_1x_1}\frac{\partial}{\partial x_2} - v_{x_1}\frac{\partial}{\partial \lambda}, \end{aligned} \tag{3.16}$$

satisfying the Lax compatibility condition (2.15), which is equivalent to the vector field compatibility relationships

$$\begin{aligned} &\frac{\partial \psi}{\partial t} + u_{x_2x_2}\frac{\partial \psi}{\partial x_1} - (\lambda + u_{x_1x_2})\frac{\partial \psi}{\partial x_2} + v_{x_2}\frac{\partial \psi}{\partial \lambda} = 0, \\ &\frac{\partial \psi}{\partial y} + (\lambda - u_{x_1x_2})\frac{\partial \psi}{\partial x_1} + u_{x_1x_1}\frac{\partial \psi}{\partial x_2} - v_{x_1}\frac{\partial \psi}{\partial \lambda} = 0, \end{aligned} \tag{3.17}$$

satisfied for $\psi \in C^2(\mathbb{R}^2 \times \mathbb{T}^2_{\mathbb{C}}; \mathbb{C})$, any $(y,t) \in \mathbb{R}^2$ and all $(\lambda; x_1, x_2) \in \mathbb{T}^2_{\mathbb{C}}$. As was mentioned in [3], the Dunajski equations (3.12) generalize both the dispersionless Kadomtsev–Petviashvili and Plebański second heavenly equations, and is also a Lax integrable Hamiltonian system.

3.4. **First conformal structure generating equation:** $u_{yt} + u_{xt}u_y - u_tu_{xy} = 0$. The seed element $\tilde{l} \in \tilde{\mathcal{G}}^* = \widetilde{diff}(\mathbb{T}^1)^*$ in the form

$$\tilde{l} = [u_t^{-2}(1-\lambda)\lambda^{-1} + u_y^{-2}\lambda(\lambda-1)^{-1}]dx, \tag{3.18}$$

where $u \in C^2(\mathbb{T}^1 \times \mathbb{R}^2; \mathbb{R})$, $x \in \mathbb{T}^1$, $\lambda \in \mathbb{C}\backslash\{0,1\}$ and "d" denotes the full differential, generates two independent Casimir functionals $\gamma^{(1)}$ and $\gamma^{(2)} \in \mathrm{I}(\tilde{\mathcal{G}}^*)$, whose gradients have the following asymptotic expansions:

$$\nabla \gamma^{(1)}(l) \sim u_y + O(\mu^2),$$

as $|\mu| \to 0$, $\mu := \lambda - 1$, and

$$\nabla \gamma^{(2)}(l) \sim u_t + O(\lambda^2),$$

as $|\lambda| \to 0$. The commutativity condition

$$[X^{(y)}, X^{(t)}] = 0 \tag{3.19}$$

of the vector fields

$$X^{(y)} := \partial/\partial y + \nabla h^{(y)}(\tilde{l}), \qquad X^{(t)} := \partial/\partial t + \nabla h^{(t)}(\tilde{l}), \tag{3.20}$$

where

$$\begin{aligned} \nabla h^{(y)}(\tilde{l}) &:= -(\mu^{-1}\nabla\gamma^{(1)}(\tilde{l}))|_- = -\frac{u_y}{\lambda - 1}\frac{\partial}{\partial x}, \\ \nabla h^{(t)}(\tilde{l}) &:= -(\lambda^{-1}\nabla\gamma^{(2)}(\tilde{l}))|_- = -\frac{u_t}{\lambda}\frac{\partial}{\partial x}, \end{aligned} \tag{3.21}$$

leads to the heavenly type equation

$$u_{yt} + u_{xt}u_y - u_{xy}u_t = 0.$$

Its Lax-Sato representation is the compatibility condition for the first order partial differential equations

$$\begin{aligned} \frac{\partial\psi}{\partial y} - \frac{u_y}{\lambda-1}\frac{\partial\psi}{\partial x} &= 0, \\ \frac{\partial\psi}{\partial t} - \frac{u_t}{\lambda}\frac{\partial\psi}{\partial x} &= 0, \end{aligned} \tag{3.22}$$

where $\psi \in C^2(\mathbb{R}^2 \times \mathbb{T}^1_{\mathbb{C}};\mathbb{C})$.

3.5. **Second conformal structure generating equation:** $u_{xt}+u_xu_{yy}-u_yu_{xy}=0$**.** For a seed element $\tilde{l} \in \tilde{\mathcal{G}}^* = \widetilde{diff}(\mathbb{T}^1)^*$ in the form

$$\tilde{l} = [u_x^2 + 2u_x^2(u_y+\alpha)\lambda^{-1} + u_x^2(3u_y^2+4\alpha u_y+\beta)\lambda^{-2}]dx, \tag{3.23}$$

where $u \in C^2(\mathbb{T}^1\times\mathbb{R}^2;\mathbb{R})$, $x \in \mathbb{T}^1$, $\lambda \in \mathbb{C}\backslash\{0\}$,and $\alpha,\beta \in \mathbb{R}$, there is one independent Casimir functional $\gamma^{(1)} \in \mathrm{I}(\tilde{\mathcal{G}}^*)$ with the following asymptotic as $|\lambda| \to 0$ expansion of its functional gradient:

$$\nabla\gamma^{(1)}(l) \sim c_0u_x^{-1} + (-c_0u_y+c_1)u_x^{-1}\lambda + (-c_1u_y+c_2)u_x^{-1}\lambda^2 + O(\lambda^3),$$

where $c_r \in \mathbb{R}$, $r=\overline{1,2}$. If one assumes that $c_0=1$, $c_1=0$ and $c_2=0$, then we obtain two functionally independent gradient elements

$$\begin{aligned} \nabla h^{(y)}(\tilde{l}) &:= -(\lambda^{-1}\nabla\gamma^{(1)}(\tilde{l}))|_- = -\frac{1}{\lambda u_x}\frac{\partial}{\partial x}, \\ \nabla h^{(t)}(\tilde{l}) &:= (\lambda^{-2}\nabla\gamma^{(1)}(\tilde{l}))|_- = \left(\frac{1}{\lambda^2u_x} - \frac{u_y}{\lambda u_x}\right)\frac{\partial}{\partial x}. \end{aligned} \tag{3.24}$$

The corresponding commutativity condition (3.19) of the vector fields (3.20) give rise to the following heavenly type equation:

$$u_{xt} + u_xu_{yy} - u_yu_{xy} = 0, \tag{3.25}$$

whose linearized Lax-Sato representation is given by the first order system

$$\begin{aligned} &\frac{\partial\psi}{\partial y} - \frac{1}{\lambda u_x}\frac{\partial\psi}{\partial x} = 0, \\ &\frac{\partial\psi}{\partial t} + \left(\frac{1}{\lambda^2u_x} - \frac{u_y}{\lambda u_x}\right)\frac{\partial\psi}{\partial x} = 0 \end{aligned} \tag{3.26}$$

of linear vector field equations on a function $\psi \in C^2(\mathbb{R}^2 \times \mathbb{T}^1_{\mathbb{C}};\mathbb{R})$.

3.6. **Inverse first Shabat reduction heavenly equation.** A seed element $\tilde{l} \in \tilde{\mathcal{G}}^* = \widetilde{diff}(\mathbb{T}^1)^*$ in the form

$$\tilde{l} = (a_0u_y^{-2}u_x^2(\lambda+1)^{-1} + a_1u_x^2 + a_1u_x^2\lambda)dx, \tag{3.27}$$

where $u \in C^2(\mathbb{T}^1 \times \mathbb{R}^2;\mathbb{R})$, $x \in \mathbb{T}^1$, $\lambda \in \mathbb{C}\setminus\{-1\}$, and $a_0,a_1 \in \mathbb{R}$, generates two independent Casimir functionals $\gamma^{(1)}$ and $\gamma^{(2)} \in \mathrm{I}(\tilde{\mathcal{G}}^*)$, whose gradients have the following asymptotic expansions:

$$\nabla\gamma^{(1)}(l) \sim u_yu_x^{-1} - u_yu_x^{-1}\mu + O(\mu^2), \tag{3.28}$$

as $|\mu| \to 0$, $\mu := \lambda+1$, and

$$\nabla\gamma^{(2)}(l) \sim u_x^{-1} + O(\lambda^{-2}), \tag{3.29}$$

as $|\lambda| \to \infty$. If we put, by definition,

$$\begin{aligned} \nabla h^{(y)}(\tilde{l}) &:= (\mu^{-1}\nabla\gamma^{(1)}(\tilde{l}))|_- = -\frac{\lambda}{\lambda+1}\frac{u_y}{u_x}\frac{\partial}{\partial x}, \\ \nabla h^{(t)}(\tilde{l}) &:= (\lambda\nabla\gamma^{(2)}(\tilde{l}))|_+ = \frac{\lambda}{u_x}\frac{\partial}{\partial x}, \end{aligned} \tag{3.30}$$

the commutativity condition (3.19) of the vector fields (3.20) leads to the heavenly equation

$$u_{xy} + u_yu_{tx} - u_{ty}u_x = 0, \tag{3.31}$$

which can be obtained as a result of the simultaneous changing of independent variables $\mathbb{R}\ni x\to t\in\mathbb{R}$, $\mathbb{R}\ni y\to x\in\mathbb{R}$ and $\mathbb{R}\ni t\to y\in\mathbb{R}$ in the first Shabat reduction heavenly equation. The corrersponding Lax-Sato representation is given by the compatibility condition for the first order vector field equations equations

$$\begin{aligned}\frac{\partial\psi}{\partial y}-\frac{\lambda}{\lambda+1}\frac{u_y}{u_x}\frac{\partial\psi}{\partial x}&=0,\\ \frac{\partial\psi}{\partial t}+\frac{\lambda}{u_x}\frac{\partial\psi}{\partial x}&=0,\end{aligned}\tag{3.32}$$

where $\psi\in C^2(\mathbb{R}^2\times\mathbb{T}^1_{\mathbb{C}};\mathbb{R})$.

3.7. **First Plebański heavenly equation and its generalizations.** The seed element $\tilde{l}\in\tilde{\mathcal{G}}^*=\widetilde{diff}(\mathbb{T}^2)^*$ in the form

$$\tilde{l}=\lambda^{-1}(u_{yx_1}dx_1+u_{yx_2}dx_2)=\lambda^{-1}du_y,\tag{3.33}$$

where $u\in C^2(\mathbb{T}^2\times\mathbb{R}^2;\mathbb{R})$, $(x_1,x_2)\in\mathbb{T}^2$, $\lambda\in\mathbb{C}\backslash\{0\}$ and "d" designates a full differential, generates two independent Casimir functionals $\gamma^{(1)}$ and $\gamma^{(2)}\in I(\tilde{\mathcal{G}}^*)$, whose gradients have the following asymptotic expansions:

$$\begin{aligned}\nabla\gamma^{(1)}(l)&\sim(-u_{yx_2},u_{yx_1},)^\top+O(\lambda),\\ \nabla\gamma^{(2)}(l)&\sim(-u_{tx_2},u_{tx_1})^\top+O(\lambda),\end{aligned}\tag{3.34}$$

as $|\lambda|\to 0$. The commutativity condition (3.19) of the vector fields (3.20), where

$$\begin{aligned}\nabla h^{(y)}(\tilde{l})&:=(\lambda^{-1}\nabla\gamma^{(1)}(\tilde{l}))|_-=-\frac{u_{yx_2}}{\lambda}\frac{\partial}{\partial x_1}+\frac{u_{yx_1}}{\lambda}\frac{\partial}{\partial x_2},\\ \nabla h^{(t)}(\tilde{l})&:=(\lambda^{-1}\nabla\gamma^{(2)}(\tilde{l}))|_-=-\frac{u_{tx_2}}{\lambda}\frac{\partial}{\partial x_1}+\frac{u_{tx_1}}{\lambda}\frac{\partial}{\partial x_2},\end{aligned}\tag{3.35}$$

leads to the first Plebański heavenly equation [4]:

$$u_{yx_1}u_{tx_2}-u_{yx_2}u_{tx_1}=1.\tag{3.36}$$

Its Lax-Sato representation entails the compatibility condition for the first order partial differential equations

$$\begin{aligned}\frac{\partial\psi}{\partial y}-\frac{u_{yx_2}}{\lambda}\frac{\partial\psi}{\partial x_1}+\frac{u_{yx_1}}{\lambda}\frac{\partial\psi}{\partial x_2}&=0,\\ \frac{\partial\psi}{\partial t}-\frac{u_{tx_2}}{\lambda}\frac{\partial\psi}{\partial x_1}+\frac{u_{tx_1}}{\lambda}\frac{\partial\psi}{\partial x_2}&=0,\end{aligned}$$

where $\psi\in C^\infty(\mathbb{R}^2\times\mathbb{T}^2_{\mathbb{C}};\mathbb{C})$.

Taking into account that the determining condition for Casimir invariants is symmetric and equivalent to the system of nonhomogeneous linear first order partial differential equations for the covector function $l=(l_1,l_2)^\top$, the corresponding seed element can be also chosen in another forms. Moreover, the form (3.33) is invariant subject to the spatial dimension of the underlying torus $\mathbb{T}^n$, what makes it possible to describe the related generalized conformal metric equations for arbitrary dimension.

In particular, one easily observes that the asymptotic expansions (3.34) are also true for such invariant seed elements as

$$\tilde{l}=\lambda^{-1}du_t,$$

and

$$\tilde{l}=\lambda^{-1}(du_y+du_t).$$

The above described Lie-algebraic scheme can be easily generalized for any dimension $n=2k$, where $k\in\mathbb{N}$, and $n>2$. In this case one has $2k$ independent Casimir functionals $\gamma^{(j)}\in I(\tilde{\mathcal{G}}^*)$, where $\tilde{\mathcal{G}}^*=\widetilde{diff}(\mathbb{T}^{2k})^*$, $j=\overline{1,2k}$, with the following asymptotic expansions for their gradients:

$$\begin{aligned}
&\nabla\gamma^{(1)}(l) \sim (-u_{yx_2}, u_{yx_1}, \underbrace{0,\dots,0}_{2k-2})^\top + O(\lambda),\\
&\nabla\gamma^{(2)}(l) \sim (-u_{tx_2}, u_{tx_1}, \underbrace{0,\dots,0}_{2k-2})^\top + O(\lambda),\\
&\nabla\gamma^{(3)}(l) \sim (0,0,-u_{yx_4}, u_{yx_3}, \underbrace{0,\dots,0}_{2k-4})^\top + O(\lambda),\\
&\nabla\gamma^{(4)}(l) \sim (0,0,-u_{tx_4}, u_{tx_3}, \underbrace{0,\dots,0}_{2k-4})^\top + O(\lambda),\\
&\dots,\\
&\nabla\gamma^{(2k-1)}(l) \sim (\underbrace{0,\dots,0}_{2k-2}, -u_{yx_{2k}}, u_{yx_{2k-1}})^\top + O(\lambda),\\
&\nabla\gamma^{(2k)}(l) \sim (\underbrace{0,\dots,0}_{2k-2}, -u_{tx_{2k}}, u_{tx_{2k-1}})^\top + O(\lambda).
\end{aligned}$$

If we put

$$\begin{aligned}
&\nabla h^{(y)}(\tilde{l}) := (\lambda^{-1}(\nabla\gamma^{(1)}(\tilde{l}) + \dots + \nabla\gamma^{(2k-1)}(\tilde{l})))|_- =\\
&= -\sum_{m=1}^{k}\left(\frac{u_{yx_{2m}}}{\lambda}\frac{\partial}{\partial x_{2m-1}} - \frac{u_{yx_{2m-1}}}{\lambda}\frac{\partial}{\partial x_{2m}}\right),\\
&\nabla h^{(t)}(\tilde{l}) := (\lambda^{-1}(\nabla\gamma^{(2)}(\tilde{l}) + \dots + \nabla\gamma^{(2k)}(\tilde{l})))|_- =\\
&= -\sum_{m=1}^{k}\left(\frac{u_{tx_{2m}}}{\lambda}\frac{\partial}{\partial x_{2m-1}} - \frac{u_{tx_{2m-1}}}{\lambda}\frac{\partial}{\partial x_{2m}}\right),
\end{aligned}$$

the commutativity condition (3.19) of the vector fields (3.20) leads to the following multi-dimensional analogs of the first Plebański heavenly equation:

$$\sum_{m=1}^{k}(u_{yx_{2m-1}}u_{tx_{2m}} - u_{yx_{2m}}u_{tx_{2m-1}}) = 1.$$

3.8. **Modified Plebański heavenly equation and its generalizations.** For the seed element $\tilde{l} \in \tilde{\mathcal{G}}^* = \widetilde{diff}(\mathbb{T}^2)^*$ in the form

$$\begin{aligned}
\tilde{l} &= (\lambda^{-1}u_{x_1y} + u_{x_1x_1} - u_{x_1x_2} + \lambda)dx_1 +\\
&+(\lambda^{-1}u_{x_2y} + u_{x_1x_2} - u_{x_2x_2} + \lambda)dx_2 =\\
&= d(\lambda^{-1}u_y + u_{x_1} - u_{x_2} + \lambda x_1 + \lambda x_2).
\end{aligned} \tag{3.37}$$

where $d\lambda = 0$, $u \in C^2(\mathbb{T}^2 \times \mathbb{R}^2;\mathbb{R})$, $(x_1,x_2) \in \mathbb{T}^2, \lambda \in \mathbb{C}\backslash\{0\}$, there exist two independent Casimir functionals $\gamma^{(1)}$ and $\gamma^{(2)} \in I(\tilde{\mathcal{G}}^*)$ with the following gradient asymptotic expansions:

$$\nabla\gamma^{(1)}(l) \sim (u_{yx_2}, -u_{yx_1})^\top + O(\lambda),$$

as $|\lambda| \to 0$, and

$$\nabla\gamma^{(2)}(l) \sim (0,-1)^\top + (-u_{x_2x_2}, u_{x_1x_2})^\top\lambda^{-1} + O(\lambda^{-2}),$$

as $|\lambda| \to \infty$. In the case, when

$$\begin{aligned}
&\nabla h^{(y)}(\tilde{l}) := (\lambda^{-1}\nabla\gamma^{(1)}(\tilde{l}))|_- = \frac{u_{yx_2}}{\lambda}\frac{\partial}{\partial x_1} - \frac{u_{yx_1}}{\lambda}\frac{\partial}{\partial x_2},\\
&\nabla h^{(t)}(\tilde{l}) := (\lambda\nabla\gamma^{(2)}(\tilde{l}))|_+ = -u_{x_2x_2}\frac{\partial}{\partial x_1} + (u_{x_1x_2} - \lambda)\frac{\partial}{\partial x_2},
\end{aligned}$$

the commutativity condition (3.19) of the vector fields (3.20) leads to the modified Plebański heavenly equation [4]:

$$u_{yt} - u_{yx_1}u_{x_2x_2} + u_{yx_2}u_{x_1x_2} = 0, \tag{3.38}$$

with the Lax-Sato representation given by the first order partial differential equations

$$\frac{\partial \psi}{\partial y} - \frac{u_{yx_2}}{\lambda}\frac{\partial \psi}{\partial x_1} + \frac{u_{yx_1}}{\lambda}\frac{\partial \psi}{\partial x_2} = 0,$$
$$\frac{\partial \psi}{\partial t} - u_{x_2x_2}\frac{\partial \psi}{\partial x_1} + (u_{x_1x_2} - \lambda)\frac{\partial \psi}{\partial x_2} = 0$$

for functions $\psi \in C^2(\mathbb{R}^2 \times \mathbb{T}^2_{\mathbb{C}}; \mathbb{C})$.

The differential-geometric form of the seed element (3.37) is also dimension invariant subject to additional spatial variables of the torus $\mathbb{T}^n$, $n > 2$, what poses a natural question of finding the corresponding multi-dimensional generalizations of the modified Plebański heavenly equation (3.38).

If a seed element $\tilde{l} \in \tilde{\mathcal{G}}^* = \widetilde{diff}(\mathbb{T}^{2k})^*$ is chosen in the form (3.37), where $u \in C^2(\mathbb{T}^{2k} \times \mathbb{R}^2; \mathbb{R})$, we have the following asymptotic expansions for gradients of $2k \in \mathbb{N}$ independent Casimir functionals $\gamma^{(j)} \in I(\tilde{\mathcal{G}}^*)$, where $\tilde{\mathcal{G}}^* = \widetilde{diff}(\mathbb{T}^{2k})^*$, $j = \overline{1, 2k}$:

$$\nabla \gamma^{(1)}(l) \sim (-u_{yx_2}, u_{yx_1}, \underbrace{0, \ldots, 0}_{2k-2})^\top + O(\lambda),$$
$$\nabla \gamma^{(3)}(l) \sim (0, 0, -u_{yx_4}, u_{yx_3}, \underbrace{0, \ldots, 0}_{2k-4})^\top + O(\lambda),$$
$$\ldots ,$$
$$\nabla \gamma^{(2k-1)}(l) \sim (\underbrace{0, \ldots, 0}_{2k-2}, -u_{yx_{2k}}, u_{yx_{2k-1}})^\top + O(\lambda),$$

as $|\lambda| \to 0$, and

$$\nabla \gamma^{(2)}(l) \sim (0, -1, \underbrace{0, \ldots, 0}_{2k-2})^\top + (-u_{x_2x_2}, u_{x_1x_2}, \underbrace{0, \ldots, 0}_{2k-2})^\top \lambda^{-1} + O(\lambda^{-2}),$$
$$\nabla \gamma^{(4)}(l) \sim (0, 0, -u_{x_4x_2}, u_{x_3x_2}, \underbrace{0, \ldots, 0}_{2k-4})^\top \lambda^{-1} + O(\lambda^{-2}),$$
$$\ldots ,$$
$$\nabla \gamma^{(2k)}(l) \sim (\underbrace{0, \ldots, 0}_{2k-2}, -u_{x_{2k}x_2}, u_{x_{2k-1}x_2})^\top \lambda^{-1} + O(\lambda^{-2}),$$

as $|\lambda| \to \infty$. In the case, when

$$\nabla h^{(y)}(\tilde{l}) := -(\lambda^{-1}(\nabla \gamma^{(1)}(\tilde{l}) + \ldots + \nabla \gamma^{(2k-1)}(\tilde{l})))|_- =$$
$$= \sum_{m=1}^{k} \left(\frac{u_{yx_{2m}}}{\lambda} \frac{\partial}{\partial x_{2m-1}} - \frac{u_{yx_{2m-1}}}{\lambda} \frac{\partial}{\partial x_{2m}} \right),$$
$$\nabla h^{(t)}(\tilde{l}) := (\lambda(\nabla \gamma^{(2)}(\tilde{l}) + \ldots + \nabla \gamma^{(2k)}(\tilde{l})))|_+ =$$
$$= -u_{x_2x_2}\frac{\partial}{\partial x_1} + (u_{x_1x_2} - \lambda)\frac{\partial}{\partial x_2} - \sum_{m=2}^{k} \left(u_{x_{2m}x_2} \frac{\partial}{\partial x_{2m-1}} - u_{x_{2m-1}x_2} \frac{\partial}{\partial x_{2m}} \right),$$

the commutability condition (3.19) of the vector fields (3.20) leads to the following multi-dimensional analogs of the modified Plebański heavenly equation:

$$u_{yt} - \sum_{m=1}^{k} (u_{yx_{2m}} u_{x_2x_{2m-1}} - u_{yx_{2m-1}} u_{x_2x_{2m}}) = 0.$$

3.9. **Husain heavenly equation and its generalizations.** A seed element $\tilde{l} \in \tilde{\mathcal{G}}^* = \widetilde{diff}(\mathbb{T}^2)^*$ in the form

$$\tilde{l} = \frac{d(u_y + iu_t)}{\lambda - i} + \frac{d(u_y - iu_t)}{\lambda + i} = \frac{2(\lambda du_y - du_t)}{\lambda^2 + 1}, \tag{3.39}$$

where $i^2 = -1$, $d\lambda = 0$, $u \in C^2(\mathbb{T}^2 \times \mathbb{R}^2; \mathbb{R})$, $(x_1, x_2) \in \mathbb{T}^2$, $\lambda \in \mathbb{C}\backslash\{-i; i\}$, generates two independent Casimir functionals $\gamma^{(1)}$ and $\gamma^{(2)} \in I(\tilde{\mathcal{G}}^*)$, with the following gradient asymptotic expansions:

$$\nabla\gamma^{(1)}(l) \sim \frac{1}{2}(-u_{yx_2} - iu_{tx_2}, u_{yx_1} + iu_{tx_1})^\top + O(\mu), \qquad \mu := \lambda - i,$$

as $|\mu| \to 0$, and

$$\nabla\gamma^{(2)}(l) \sim \frac{1}{2}(-u_{yx_2} + iu_{tx_2}, u_{yx_1} - iu_{tx_1})^\top + O(\xi), \qquad \xi := \lambda + i,$$

as $|\xi| \to 0$. In the case, when

$$\begin{aligned}
\nabla h^{(y)}(\tilde{l}) &:= (\mu^{-1}\nabla\gamma^{(1)}(\tilde{l}) + \xi^{-1}\nabla\gamma^{(2)}(\tilde{l}))|_- = \\
&= \frac{1}{2\mu}\left((-u_{yx_2} - iu_{tx_2})\frac{\partial}{\partial x_1} + (u_{yx_1} + iu_{tx_1})\frac{\partial}{\partial x_2}\right) + \\
&+ \frac{1}{2\xi}\left((-u_{yx_2} + iu_{tx_2})\frac{\partial}{\partial x_1} + (u_{yx_1} - iu_{tx_1})\frac{\partial}{\partial x_2}\right) = \\
&= \frac{u_{tx_2} - \lambda u_{yx_2}}{\lambda^2 + 1}\frac{\partial}{\partial x_1} + \frac{\lambda u_{yx_1} - u_{tx_1}}{\lambda^2 + 1}\frac{\partial}{\partial x_2},
\end{aligned}$$

$$\begin{aligned}
\nabla h^{(t)}(\tilde{l}) &:= (-\mu^{-1}i\nabla\gamma^{(1)}(\tilde{l}) + \xi^{-1}i\nabla\gamma^{(2)}(\tilde{l}))|_- = \\
&= \frac{1}{2\mu}\left((-u_{tx_2} + iu_{yx_2})\frac{\partial}{\partial x_1} + (u_{tx_1} - iu_{yx_1})\frac{\partial}{\partial x_2}\right) + \\
&+ \frac{1}{2\xi}\left(-(u_{tx_2} + iu_{yx_2})\frac{\partial}{\partial x_1} + (u_{tx_1} + iu_{yx_1})\frac{\partial}{\partial x_2}\right) = \\
&= -\frac{u_{yx_2} + \lambda u_{tx_2}}{\lambda^2 + 1}\frac{\partial}{\partial x_1} + \frac{u_{yx_1} + \lambda u_{tx_1}}{\lambda^2 + 1}\frac{\partial}{\partial x_2},
\end{aligned}$$

the commutativity condition (3.19) of the vector fields (3.20) leads to the Husain heavenly equation [4]:

$$u_{yy} + u_{tt} + u_{yx_1}u_{tx_2} - u_{yx_2}u_{tx_1} = 0, \tag{3.40}$$

with the Lax-Sato representation given by the first order partial differential equations

$$\begin{aligned}
\frac{\partial\psi}{\partial y} + \frac{u_{tx_2} - \lambda u_{yx_2}}{\lambda^2 + 1}\frac{\partial\psi}{\partial x_1} + \frac{\lambda u_{yx_1} - u_{tx_1}}{\lambda^2 + 1}\frac{\partial\ \psi}{\partial x_2} = 0, \\
\frac{\partial\psi}{\partial t} - \frac{u_{yx_2} + \lambda u_{tx_2}}{\lambda^2 + 1}\frac{\partial\psi}{\partial x_1} + \frac{u_{yx_1} + \lambda u_{tx_1}}{\lambda^2 + 1}\frac{\partial\ \psi}{\partial x_2} = 0,
\end{aligned}$$

where $\psi \in C^2(\mathbb{R}^2 \times \mathbb{T}^2_{\mathbb{C}}; \mathbb{C})$.

The differential-geometric form of the seed element (3.39) is also dimension invariant subject to additional spatial variables of the torus $\mathbb{T}^n$, $n > 2$, what poses a natural question of finding the corresponding multi-dimensional generalizations of the Husain heavenly equation (3.40).

If a seed element $\tilde{l} \in \tilde{\mathcal{G}}^* = \widetilde{diff}(\mathbb{T}^{2k})^*$ is chosen in the form (3.39), where $u \in C^2(\mathbb{T}^{2k} \times \mathbb{R}^2; \mathbb{R})$, we have the following asymptotic expansions for gradients of $2k \in \mathbb{N}$ independent Casimir functionals $\gamma^{(j)} \in I(\tilde{\mathcal{G}}^*)$, where $\tilde{\mathcal{G}}^* = \widetilde{diff}(\mathbb{T}^{2k})^*$, $j = \overline{1, 2k}$:

$$\begin{aligned}
&\nabla\gamma^{(1)}(l) \sim \frac{1}{2}(-u_{yx_2} - iu_{tx_2}, u_{yx_1} + iu_{tx_1}, \underbrace{0, \ldots, 0}_{2k-2})^\top + O(\mu), \\
&\nabla\gamma^{(3)}(l) \sim \frac{1}{2}(0, 0, -u_{yx_4} - iu_{tx_4}, u_{yx_3} + iu_{tx_3}, \underbrace{0, \ldots, 0}_{2k-4})^\top + O(\mu), \\
&\quad \cdots, \\
&\nabla\gamma^{(2k-1)}(l) \sim \frac{1}{2}(\underbrace{0, \ldots, 0}_{2k-2}, -u_{yx_{2k}} - iu_{tx_{2k}}, u_{yx_{2k-1}} + iu_{tx_{2k-1}})^\top + O(\mu),
\end{aligned}$$

as $|\mu|\to 0$, and

$$\begin{aligned}
\nabla\gamma^{(2)}(l) &\sim \frac{1}{2}(-u_{yx_2}+iu_{tx_2}, u_{yx_1}-iu_{tx_1}, \underbrace{0,\dots,0}_{2k-2})^\top + O(\xi),\\
\nabla\gamma^{(4)}(l) &\sim \frac{1}{2}(0,0,-u_{yx_4}+iu_{tx_4}, u_{yx_3}-iu_{tx_3}, \underbrace{0,\dots,0}_{2k-4})^\top + O(\xi),\\
&\dots,\\
\nabla\gamma^{(2k)}(l) &\sim \frac{1}{2}(\underbrace{0,\dots,0}_{2k-2}, -u_{yx_{2k}}+iu_{tx_{2k}}, u_{yx_{2k-1}}-iu_{tx_{2k-1}})^\top + O(\xi),
\end{aligned}$$

as $|\xi|\to 0$. In the case, when

$$\begin{aligned}
\nabla h^{(y)}(\tilde{l}) &:= \sum_{m=1}^{k}(\mu^{-1}\nabla\gamma^{(2m-1)}(\tilde{l})+\xi^{-1}\nabla\gamma^{(2m)}(\tilde{l}))|_- =\\
&= \sum_{m=1}^{k}\left(\frac{u_{tx_{2m}}-\lambda u_{yx_{2m}}}{\lambda^2+1}\frac{\partial}{\partial x_{2m-1}}+\frac{\lambda u_{yx_{2m-1}}-u_{tx_{2m-1}}}{\lambda^2+1}\frac{\partial}{\partial x_{2m}}\right),\\
\nabla h^{(t)}(\tilde{l}) &:= \sum_{m=1}^{k} i(-\mu^{-1}\nabla\gamma^{(2m-1)}(\tilde{l})+\xi^{-1}\nabla\gamma^{(2m)}(\tilde{l}))|_- =\\
&= \sum_{m=1}^{k}\left(-\frac{u_{yx_{2m}}+\lambda u_{tx_{2m}}}{\lambda^2+1}\frac{\partial}{\partial x_{2m-1}}+\frac{u_{yx_{2m-1}}+\lambda u_{tx_{2m-1}}}{\lambda^2+1}\frac{\partial}{\partial x_{2m}}\right),
\end{aligned}$$

the commutability condition (3.19) of the vector fields (3.20) leads to the following multi-dimensional analogs of the Husain heavenly equation:

$$u_{yy}+u_{tt}+\sum_{m=1}^{k}(u_{yx_{2m-1}}u_{tx_{2m}}-u_{yx_{2m}}u_{x_2x_{2m-1}})=0.$$

3.10. **The general Monge heavenly equation and its generalizations.** A seed element $\tilde{l}\in\tilde{\mathcal{G}}^*=\widetilde{diff}(\mathbb{T}^4)^*$, taken in the form

$$\tilde{l}=du_y+\lambda^{-1}(dx_1+dx_2), \tag{3.41}$$

where $u\in C^2(\mathbb{T}^4\times\mathbb{R}^2;\mathbb{R})$, $(x_1,x_2,x_3,x_4)\in\mathbb{T}^4$, $\lambda\in\mathbb{C}\setminus\{0\}$, generates four independent Casimir functionals $\gamma^{(1)}$, $\gamma^{(2)}$, $\gamma^{(3)}$ and $\gamma^{(4)}\in\tilde{I}(\tilde{\mathcal{G}}^*)$, whose gradients have the following asymptotic expansions:

$$\begin{aligned}
(3.42)\quad \nabla\gamma^{(1)}(l) &\sim (0,1,0,0)^\top +\\
&+(-u_{yx_2}-(\partial_{x_2}-\partial_{x_1})^{-1}u_{yx_2x_1}, (\partial_{x_2}-\partial_{x_1})^{-1}u_{yx_2x_1},0,0)^\top\lambda+O(\lambda^2),\\
\nabla\gamma^{(2)}(l) &\sim (1,0,0,0)^\top +\\
&+(\partial_{x_1}-\partial_{x_2})^{-1}u_{yx_1x_2}, -u_{yx_1}-(\partial_{x_1}-\partial_{x_2})^{-1}u_{yx_1x_2},0,0)^\top\lambda+O(\lambda^2),\\
\nabla\gamma^{(3)}(l) &\sim (0,0,-u_{yx_4},u_{yx_3})^\top+O(\lambda^2),\\
\nabla\gamma^{(4)}(l) &\sim (0,0,-u_{tx_4},u_{tx_3})^\top+(u_{yx_3}u_{tx_4}-u_{yx_4}u_{tx_3},0,\\
&u_{yx_4}u_{tx_1}-u_{yx_1}u_{tx_4}, u_{yx_1}u_{tx_3}-u_{yx_3}u_{tx_1})^\top\lambda+O(\lambda^2),
\end{aligned}$$

as $|\lambda|\to 0$. In the case, when

$$\begin{aligned}
(3.43)\quad \nabla h^{(y)}(\tilde{l}) &:= (\lambda^{-1}(\nabla\gamma^{(1)}(\tilde{l})+\nabla\gamma^{(3)}(\tilde{l})))|_- =\\
&= 0\frac{\partial}{\partial x_1}+\frac{1}{\lambda}\frac{\partial}{\partial x_2}-\frac{u_{yx_4}}{\lambda}\frac{\partial}{\partial x_3}+\frac{u_{yx_3}}{\lambda}\frac{\partial}{\partial x_4},\\
\nabla h^{(t)}(\tilde{l}) &:= (\lambda^{-1}(-\nabla\gamma^{(2)}(\tilde{l})+\nabla\gamma^{(4)}(\tilde{l})))|_- =\\
&= -\frac{1}{\lambda}\frac{\partial}{\partial x_1}+0\frac{\partial}{\partial x_2}-\frac{u_{tx_4}}{\lambda}\frac{\partial}{\partial x_3}+\frac{u_{tx_3}}{\lambda}\frac{\partial}{\partial x_4},
\end{aligned}$$

the commutability condition (3.19) of the vector fields (3.20) leads to the general Monge heavenly equation [20]:

$$u_{yx_1} + u_{tx_2} + u_{yx_3}u_{tx_4} - u_{yx_4}u_{tx_3} = 0, \tag{3.44}$$

with the Lax-Sato representation given by the first order partial differential equations

$$\begin{aligned}
&\frac{\partial\psi}{\partial y} + \frac{1}{\lambda}\frac{\partial\psi}{\partial x_2} - \frac{u_{yx_4}}{\lambda}\frac{\partial\psi}{\partial x_3} + \frac{u_{yx_3}}{\lambda}\frac{\partial\psi}{\partial x_4} = 0,\\
&\frac{\partial\psi}{\partial t} - \frac{1}{\lambda}\frac{\partial\psi}{\partial x_1} - \frac{u_{tx_4}}{\lambda}\frac{\partial\psi}{\partial x_3} + \frac{u_{tx_3}}{\lambda}\frac{\partial\psi}{\partial x_4} = 0,
\end{aligned}$$

where $\psi \in C^2(\mathbb{T}^4 \times \mathbb{R}^2; \mathbb{R})$ and $\lambda \in \mathbb{C}\backslash\{0\}$.

Taking into account that the condition for Casimir invariants is equivalent to a system of homogeneous linear first order partial differential equations for a covector function $l = (l_1, l_2, l_3, l_4)$, the corresponding seed element can be chosen in different forms. For example, if the expression

$$\tilde{l} = du_t + \lambda^{-1}(dx_1 + dx_2)$$

is considered as a seed element, one obtains that it generates four independent Casimir functionals $\gamma^{(1)}$, $\gamma^{(2)}$, $\gamma^{(3)}$ and $\gamma^{(4)} \in I(\tilde{\mathcal{G}}^*)$, whose gradients have the following asymptotic expansions:

$$\begin{aligned}
&\nabla\gamma^{(1)}(l) \sim (0,1,0,0)^\top +\\
&\quad +(-u_{tx_2} - (\partial_{x_2} - \partial_{x_1})^{-1}u_{tx_2x_1}, (\partial_{x_2} - \partial_{x_1})^{-1}u_{tx_2x_1}, 0, 0)^\top\lambda + O(\lambda^2),\\
&\nabla\gamma^{(2)}(l) \sim (1,0,0,0)^\top +\\
&\quad +((\partial_{x_1} - \partial_{x_2})^{-1}u_{tx_1x_2}, -u_{tx_1} - (\partial_{x_1} - \partial_{x_2})^{-1}u_{tx_1x_2}, 0, 0)^\top\lambda + O(\lambda^2),\\
&\nabla\gamma^{(3)}(l) \sim (0,0,-u_{tx_4}, u_{tx_3})^\top + (0, u_{tx_3}u_{yx_4} - u_{tx_4}u_{yx_3},\\
&\quad u_{tx_4}u_{yx_2} - u_{tx_2}u_{yx_4}, u_{tx_2}u_{yx_3} - u_{tx_3}u_{yx_2})^\top\lambda + O(\lambda^2),\\
&\nabla\gamma^{(4)}(l) \sim (0,0,-u_{yx_4}, u_{yx_3})^\top + O(\lambda^2),
\end{aligned}$$

as $|\lambda| \to 0$. If a seed element has the form

$$\tilde{l} = du_y + du_t + \lambda^{-1}(dx_1 + dx_2), \tag{3.45}$$

the asymptotic expansions for gradients of four independent Casimir functionals $\gamma^{(1)}, \gamma^{(2)}, \gamma^{(3)}$ and $\gamma^{(4)} \in I(\tilde{\mathcal{G}}^*)$ are written as

$$\begin{aligned}
&\nabla\gamma^{(1)}(l) \sim (0,1,0,0)^\top + (-(u_{yx_2} + u_{tx_2}) -\\
&\quad -(\partial_{x_2} - \partial_{x_1})^{-1}(u_{yx_2x_1} + u_{tx_2x_1}),\\
&\quad (\partial_{x_2} - \partial_{x_1})^{-1}(u_{yx_2x_1} + u_{tx_2x_1}), 0, 0)^\top\lambda + O(\lambda^2),\\
&\nabla\gamma^{(2)}(l) \sim (1,0,0,0)^\top + ((\partial_{x_1} - \partial_{x_2})^{-1}(u_{yx_1x_2} + u_{tx_1x_2}),\\
&\quad -(u_{yx_1} + u_{tx_1}) - (\partial_{x_1} - \partial_{x_2})^{-1}(u_{yx_1x_2} + u_{tx_1x_2}), 0, 0)^\top\lambda + O(\lambda^2),\\
&\nabla\gamma^{(3)}(l) \sim (0,0,-u_{yx_4}, u_{yx_3})^\top + (0, u_{tx_3}u_{yx_4} - u_{tx_4}u_{yx_3},\\
&\qquad u_{tx_4}u_{yx_2} - u_{tx_2}u_{yx_4}, u_{tx_2}u_{yx_3} - u_{tx_3}u_{yx_2})^\top\lambda + O(\lambda^2),\\
&\nabla\gamma^{(4)}(l) \sim (0,0,-u_{tx_4}, u_{tx_3})^\top + (u_{yx_3}u_{tx_4} - u_{yx_4}u_{tx_3}, 0,\\
&\qquad u_{yx_4}u_{tx_1} - u_{yx_1}u_{tx_4}, u_{yx_1}u_{tx_3} - u_{yx_3}u_{tx_1})^\top\lambda + O(\lambda^2),
\end{aligned}$$

as $|\lambda| \to 0$.

The above described scheme is generalized for all $n = 2k$, where $k \in \mathbb{N}$, and $n > 2$. In this case one has $2k$ independent Casimir functionals $\gamma^{(j)} \in \tilde{I}(\tilde{\mathcal{G}}^*)$, where $\tilde{\mathcal{G}}^* = \widetilde{diff}(\mathbb{T}^{2k})^*$, $j = \overline{1, 2k}$, whose gradient asymptotic expansions are equal to the following expressions:

$$
\begin{aligned}
&\nabla\gamma^{(1)}(l) \sim (0,1,\underbrace{0,\ldots,0}_{2k-2})^\top + (-(u_{yx_2}+u_{tx_2}) - \\
&-(\partial_{x_2}-\partial_{x_1})^{-1}(u_{yx_2x_1}+u_{tx_2x_1}), (\partial_{x_2}-\partial_{x_1})^{-1}(u_{yx_2x_1}+u_{tx_2x_1}), \\
&\quad \underbrace{0,\ldots,0}_{2k-2})^\top\lambda + O(\lambda^2), \\
&\nabla\gamma^{(2)}(l) \sim (1,0,\underbrace{0,\ldots,0}_{2k-2})^\top + ((\partial_{x_1}-\partial_{x_2})^{-1}(u_{yx_1x_2}+u_{tx_1x_2}), \\
&\quad -(u_{yx_1}+u_{tx_1}) - (\partial_{x_1}-\partial_{x_2})^{-1}(u_{yx_1x_2}+u_{tx_1x_2}), 0, 0)^\top\lambda + O(\lambda^2), \\
&\nabla\gamma^{(3)}(l) \sim (0,0,-u_{yx_4}, u_{yx_3}, \underbrace{0,\ldots,0}_{2k-4})^\top + (0, u_{tx_3}u_{yx_4} - u_{tx_4}u_{yx_3}, \\
&\quad u_{tx_4}u_{yx_2} - u_{tx_2}u_{yx_4}, u_{tx_2}u_{yx_3} - u_{tx_3}u_{yx_2}, \underbrace{0,\ldots,0}_{2k-4})^\top\lambda + O(\lambda^2), \\
&\nabla\gamma^{(4)}(l) \sim (0,0,-u_{tx_4}, u_{tx_3}, \underbrace{0,\ldots,0}_{2k-4})^\top + (u_{yx_3}u_{tx_4} - u_{yx_4}u_{tx_3}, 0, \\
&\quad u_{yx_4}u_{tx_1} - u_{yx_1}u_{tx_4}, u_{yx_1}u_{tx_3} - u_{yx_3}u_{tx_1}, \underbrace{0,\ldots,0}_{2k-4})^\top\lambda + O(\lambda^2), \\
&\nabla\gamma^{(2k-1)}(l) \sim (\underbrace{0,\ldots,0}_{2k-4}, 0, 0, -u_{yx_{2k}}, u_{yx_{2k-1}})^\top + \\
&\quad +(\underbrace{0,\ldots,0}_{2k-4}, 0, u_{tx_{2k-1}}u_{yx_{2k}} - u_{tx_{2k}}u_{yx_{2k-1}}, \\
&\quad u_{tx_{2k}}u_{yx_2} - u_{tx_2}u_{yx_{2k}}, u_{tx_2}u_{yx_{2k-1}} - u_{tx_{2k-1}}u_{yx_2})^\top\lambda + O(\lambda^2), \\
&\nabla\gamma^{(2k)}(l) \sim (\underbrace{0,\ldots,0}_{2k-4}, 0, 0, -u_{tx_{2k}}, u_{tx_{2k-1}})^\top + \\
&\quad +(\underbrace{0,\ldots,0}_{2k-4}, u_{yx_{2k-1}}u_{tx_{2k}} - u_{yx_{2k}}u_{tx_{2k-1}}, 0, \\
&\quad u_{yx_{2k}}u_{tx_1} - u_{yx_1}u_{tx_{2k}}, u_{yx_1}u_{tx_{2k-1}} - u_{yx_{2k-1}}u_{tx_1})^\top\lambda + O(\lambda^2),
\end{aligned}
$$

when a seed element $\tilde{l} \in \tilde{\mathcal{G}}^*$ is chosen as in (3.45). If

$$
\begin{aligned}
&\nabla h^{(y)}(\tilde{l}) := (\lambda^{-1}(\nabla\gamma^{(1)}(\tilde{l}) + \nabla\gamma^{(3)}(\tilde{l}) + \ldots + \nabla\gamma^{(2k-1)}(\tilde{l})))|_- = \\
&= 0\frac{\partial}{\partial x_1} + \frac{1}{\lambda}\frac{\partial}{\partial x_2} - \frac{u_{yx_4}}{\lambda}\frac{\partial}{\partial x_3} + \frac{u_{yx_3}}{\lambda}\frac{\partial}{\partial x_4} + \ldots - \\
&-\frac{u_{yx_{2k}}}{\lambda}\frac{\partial}{\partial x_{2k-1}} + \frac{u_{yx_{2k-1}}}{\lambda}\frac{\partial}{\partial x_{2k}} = \\
&= 0\frac{\partial}{\partial x_1} + \frac{1}{\lambda}\frac{\partial}{\partial x_2} - \sum_{j=2}^{k}\left(\frac{u_{yx_{2j}}}{\lambda}\frac{\partial}{\partial x_{2j-1}} - \frac{u_{yx_{2j-1}}}{\lambda}\frac{\partial}{\partial x_{2j}}\right), \\
&\nabla h^{(t)}(\tilde{l}) := (\lambda^{-1}(-\nabla\gamma^{(2)}(\tilde{l}) + \nabla\gamma^{(4)}(\tilde{l}) + \ldots + \nabla\gamma^{(2k)}(\tilde{l})))|_- = \\
&= -\frac{1}{\lambda}\frac{\partial}{\partial x_1} + 0\frac{\partial}{\partial x_2} - \frac{u_{tx_4}}{\lambda}\frac{\partial}{\partial x_3} + \frac{u_{tx_3}}{\lambda}\frac{\partial}{\partial x_4} + \ldots - \\
&-\frac{u_{tx_{2k}}}{\lambda}\frac{\partial}{\partial x_{2k-1}} + \frac{u_{tx_{2k-1}}}{\lambda}\frac{\partial}{\partial x_{2k}} = \\
&= -\frac{1}{\lambda}\frac{\partial}{\partial x_1} + 0\frac{\partial}{\partial x_2} - \sum_{j=2}^{k}\left(\frac{u_{tx_{2k}}}{\lambda}\frac{\partial}{\partial x_{2k-1}} - \frac{u_{tx_{2k-1}}}{\lambda}\frac{\partial}{\partial x_{2k}}\right),
\end{aligned}
$$

the commutability condition (3.19) of the vector fields (3.20) leads to the following multi-dimensional analogs of the general Monge heavenly equation:

$$u_{yx_1} + u_{tx_2} + \sum_{j=2}^{k} (u_{yx_{2j-1}} u_{tx_{2j}} - u_{yx_{2j}} u_{tx_{2j-1}}) = 0.$$

3.11. **Superanalogs of the Whitham heavenly equation.** Assume now that an element $\tilde{l} \in \tilde{\mathcal{G}}^*$, where $\tilde{\mathcal{G}} := di\tilde{f}f(\mathbb{T}^{1|N}) = di\tilde{f}f_+(\mathbb{T}^{1|N}) \oplus di\tilde{f}f_-(\mathbb{T}^{1|N})$ is the loop Lie algebra of the superconformal diffeomorphism group $Di\tilde{f}f(\mathbb{T}^{1|N})$ of vector fields on the $1|N$-dimensional supertorus $\mathbb{T}^{1|N} := \mathbb{S}^1 \times \Lambda_1^N$ (see [23]), imbedded into a finite-dimensional Grassmann algebra $\Lambda := \Lambda_0 \oplus \Lambda_1$ over $\mathbb{C}$, $\Lambda_0 \supset \mathbb{R}$, admits the following asymptotic expansions for gradients of the Casimir invariants $h^{(1)}, h^{(2)} \in I(\tilde{\mathcal{G}}^*)$:

$$\nabla h^{(1)}(l) \sim w_y + O(\lambda) \tag{3.46}$$

as $|\lambda| \to 0$, and

$$\nabla h^{(2)}(l) \sim 1 - w_x \lambda^{-1} + O(\lambda^{-2}) \tag{3.47}$$

as $|\lambda| \to \infty$. Then the commutability condition for the Hamiltonian flows

$$\begin{aligned} d\tilde{l}/dy &= ad^*_{\nabla h_-^{(y)}(\tilde{l})} \tilde{l}, \quad \nabla h_-^{(y)}(l) = -(\lambda^{-1} \nabla h^1(l))_- = -w_y \lambda^{-1}, \\ d\tilde{l}/dt &= -ad^*_{\nabla h_+^{(t)}(\tilde{l})} \tilde{l}, \quad \nabla h_+^{(t)}(l) = -(\lambda \nabla h^{(2)}(l))_+ = -\lambda + w_x, \end{aligned} \tag{3.48}$$

naturally leads to the heavenly type equation

$$w_{yt} = w_x w_{yx} - w_y w_{xx} - \frac{1}{2} \sum_{i=1}^{N} (D_{\vartheta_i} w_x)(D_{\vartheta_i} w_y), \tag{3.49}$$

where $w \in C^\infty(\mathbb{R}^2 \times \mathbb{T}^{1|N}; \Lambda_0)$ and $D_{\vartheta_i} := \partial/\partial\vartheta_i + \vartheta_i \partial/\partial x, i = \overline{1, N}$, are superderivatives with respect to the anticommuting variables $\vartheta_i \in \Lambda_1$, $i = \overline{1, N}$.

This equation can be considered as a supergeneralization of the Whitham heavenly one [21, 22, 9] for arbitrary $N \in \mathbb{N}$. The compatibility condition for the first order partial differential equations

$$\begin{aligned} \psi_y + \frac{1}{\lambda} \left(w_y \psi_x + \frac{1}{2} \sum_{i=1}^{N} (D_{\vartheta_i} w_y)(D_{\vartheta_i} \psi) \right) &= 0, \\ \psi_t + (-\lambda + w_x)\psi_x + \frac{1}{2} \sum_{i=1}^{N} (D_{\vartheta_i} w_x)(D_{\vartheta_i} \psi) &= 0, \end{aligned}$$

where $\psi \in C^2(\mathbb{R}^2 \times \mathbb{T}^{1|N}; \Lambda_0)$ and $\lambda \in \mathbb{C}\backslash\{0\}$, give rise to the corresponding Lax-Sato representation of the heavenly type equation (3.49).

Moreover, based on easy calculations, one can obtain from the Casimir invariant equation the corresponding seed element $\tilde{l} := ldx \in \tilde{\mathcal{G}}^*$, which can be written in the following form for an arbitrary $N \in \mathbb{N}$:

$$l = Ca^{-\frac{4-N}{2}}, \quad a := \nabla h(l),$$

where a scalar function $C = C(x; \vartheta)$ satisfies a linear homogeneous ordinary differential equation

$$C_x = < DC, Q >,$$

$Q = (Q_1, \ldots, Q_N)$, $Q_i = \frac{(-1)^N}{2}(D_{\vartheta_i} \ln a)$, in the superspace $\mathbb{R}^{2^{N-1}|2^{N-1}} \simeq \Lambda_0^{2^{N-1}} \times \Lambda_1^{2^{N-1}}$. Moreover, $C \in C^\infty(\mathbb{T}^{1|N}; \Lambda_1)$, if N is an odd natural number, and suitably $C \in C^\infty(\mathbb{T}^{1|N}; \Lambda_0)$, if N is an even integer. In the case of $N = 1$ one has

$$l = C_1(\partial_x^{-1} D_{\theta_1} a^{-\frac{1}{2}}) a^{-\frac{3}{2}},$$

where $C_1 \in \mathbb{R}$ is some real constant.

If $N = 1$ and $C_1 = 1$, the corresponding seed-element $\tilde{l} \in \tilde{\mathcal{G}}^*$, related to the asymptotic expansions (3.46) and (3.47), can be reduced to

$$\tilde{l} = [\lambda^{-1}(\partial_x^{-1} D_{\theta_1} w_y^{-\frac{1}{2}}) w_y^{-\frac{3}{2}} + \xi_x/2 + \theta_1(2u_x + \lambda)]dx,$$

where $w := u + \theta_1 \xi$, $u \in C^\infty(\mathbb{R}^2 \times \mathbb{S}^1; \Lambda_0)$ and $\xi \in C^\infty(\mathbb{R}^2 \times \mathbb{S}^1; \Lambda_1)$.

4. Conclusion

We succeeded in applying the Lie-algebraic approach to studying vector fields on the complexified n-dimensional torus and the related Lie-algebraic structures, which made it possible to construct a wide class of multi-dimensional dispersionless integrable systems, describing conformal structure generating equations of modern mathematical physics. There was described a modification of the approach subject to the spatial dimensional invariance and meromorphicity of the related differential-geometric structures, giving rise to new generalized multi-dimensional conformal metric equations. There have been analyzed in detail the related differential-geometric structures of the Einstein–Weyl conformal metric equation, the modified Einstein–Weyl metric equation, the Dunajski heavenly equation system, the first and second conformal structure generating equations, the inverse first Shabat reduction heavenly equation, the first and modified Plebański heavenly equations and its multi-dimensional generalizations, the Husain heavenly equation and its multi-dimensional generalizations, the general Monge equation and its multi-dimensional generalizations and superconformal analogs of the Whitham heavenly equation

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