Revealing a Binary Pattern Validates 3n+1 Problem for All Positive Integers

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Abstract

This study delves into a unique binary pattern found within the well-known 3n+1 problem, or Collatz conjecture. Through careful analysis of the steps in the 3n+1 sequence, we have discovered a special binary representation that captures the behavior of all positive integers undergoing this transformation. With this new understanding, we have provided a solid proof confirming the validity of the 3n+1 problem for all positive integers. Our method goes beyond the need for extensive computational confirmation, providing a simple and elegant resolution to a long-standing mathematical mystery.

1 Introduction

The Collatz conjecture, also known as the 3n+1 problem, has intrigued mathematicians for many years due to its seemingly simple yet challenging nature. First introduced by Loather Collatz in 1937, the conjecture suggests a basic algorithm for any positive integer: if the number is even, divide it by 2; if it is odd, multiply it by 3 and add 1.

This iterative process eventually converges to the value 1, as boldly claimed by the conjecture. Despite its straightforwardness, the Collatz conjecture remains unproven, making it a longstanding unsolved mystery in number theory. Numerous computational attempts have been made to validate its accuracy for larger numbers, but a comprehensive analytical proof remains elusive.

A new perspective has led to a major discovery in understanding the 3n+1 transformation of integers in binary form. By carefully analyzing the binary patterns in this process, a significant revelation has been made, illuminating the core of the issue.

This research explores a unique viewpoint on the Collatz conjecture. Through studying the binary sequences produced during the 3n+1 transformations, we have uncovered a fundamental pattern that goes beyond individual calculations and captures the behavior of all positive integers affected by this algorithm.

Note that each positive odd integer \( n \), definable as \( n = \sum_{i=0}^{x} 4^i \), for each \( x \in \mathbb{Z}^+ \), needs to be reduced to one by taking one \( 3n + 1 \) step, followed by \( 2(x + 1) \) successive \( \frac{n}{2} \) steps.
The 3n + 1 step that uses an integer in base 2 will demonstrate the veracity of this claim. [1] [2] [3]

2 Example one

Let \( n = \sum_{i=0}^{n}(2)^i = 21 = 10101_2 \), then

\[ 10101_2 \times 10_2 \Rightarrow 101010_2 + 10101_2 \Rightarrow 111111_2 + 12 = 1000000_2 = 2^6 \]

, and

\[ \frac{1000000_2}{10_2} \Rightarrow \frac{100000_2}{10_2} \Rightarrow \frac{10000_2}{10_2} \Rightarrow \frac{100_2}{10_2} \Rightarrow \frac{10_2}{10_2} = 1. \]

Consequently, compared to their base 10 representation, the base 2 representation of positive integers provides further understanding of the 3n + 1 problem.

Proof

Let \( O^+ \) be the set of positive odd integers, then

\[ O^+ = \{ x \in \mathbb{Z} | x = 2y + 1, y \geq 0, y \in \mathbb{Z} \}. \]

3 Theorem one

P will stand for the 3n+1 problem. If P is true for every positive odd integer, then it must also hold true for every positive integer. \( \forall a \in O^+ : P(a) \Rightarrow \forall b \in \mathbb{Z}^+ : P(b) \)

Proof

First Case:

Let \( x \in \mathbb{Z}^+ \), let \( n = 2^x \). In order to reduce \( n \) to 1, \( x \) successive \( \frac{n}{2} \) steps are needed.

Second Case: Multiplication of an odd integer by a power of two

With \( n \in O^+ \) and \( x \in \mathbb{Z}^+ \), let \( y = 2^x \cdot n \). In order to get \( y = n \), then \( x \) consecutive \( \frac{n}{2} \) steps are needed.

If we consider all positive integers \( a \), the 3n + 1 problem encompasses every possible transformation that a positive integer can undergo through iterations. Each step either applies the operation 3n + 1 or removes a factor of 2 through the \( \frac{n}{2} \) step. Ultimately, this process converges for every integer \( n \) to a power of 2, denoted as \( 2^x \), where \( x \) is a non-negative integer.

However, the transformation from any arbitrary integer \( n \) to \( 2^x \) might not be immediately clear due to the interplay between the 3n + 1 and \( \frac{n}{2} \) steps. To elucidate this process, we can focus solely on the 3n+1 step while compensating
for the omission of the $\frac{n}{2}$ step. By adjusting the $3n+1$ operation appropriately, we can still achieve the convergence to $2^x$ for every positive integer, making the iterative nature of the transformation more apparent.

4 Example Two

Let $n = 9 = 1001_2$, then $3n + 2^x$ produces this pattern:

\[
\begin{align*}
1001_2 \times 11_2 &\Rightarrow 110111_2 + 2 = 11100_2 \\
11100_2 \times 11_2 &\Rightarrow 1010100_2 + 100_2 = 101100_2 \\
101100_2 \times 11_2 &\Rightarrow 10001000_2 + 1000_2 = 10001000_2 \\
10001000_2 \times 11_2 &\Rightarrow 110011000_2 + 100000_2 = 1101100000_2 \\
1101100000_2 \times 11_2 &\Rightarrow 101100010000_2 + 10000000_2 = 10100100000000_2 \\
10100100000000_2 \times 11_2 &\Rightarrow 1101000000000000_2 + 1000000000000_2 = 101000000000000000_2 = 2^{13}.
\end{align*}
\]

In example 2, after six $3n+2^x$ steps, the least significant bit exceeds the most significant bit, turning $n$ into a power of two.

Definition

The least significant bit of $s \in \mathbb{Z}^+$, then

\[
\text{LSB} = \left\{ 2^r \mid r \geq 0, r \in \mathbb{Z} \text{ such that } 2^r = \frac{s}{t}, t \in O^+ \right\}.
\]

The least significant bit=LSB

4.1 Theorem Two

The $3n+1$ step is isomorphic to the $3n + \text{LSB}$ step.

Proof

Let $n_0 \in O^+$. Let $n_1 = 3n_0 + 1$ and $n_2 = \frac{n_1}{\text{LSB}}$, then

\[
\frac{3n_1 + \text{LSB}}{3n_2 + 1} = \frac{3n_1 + \text{LSB}}{3\left(\frac{2^r}{t}\right) + 1} = \text{LSB}
\]

Given the congruence $3n + \text{LSB} \equiv 0 \pmod{3n + 1}$, we can establish isomorphism between the $3n + \text{LSB}$ step and the $3n + 1$ step.

Two functions make up the pattern in Example 2. The most significant bit of $n$ or the most significant power of two is increased by the first function, while the least significant bit of $n$ or the least significant power of two is increased by the second function.

Let $m(x)$ be the function for repeated multiplication of $n$ by 3 in terms of $x$, where $x \in \mathbb{Z}^+$. Then $m(x) = 3^{x+\delta}n$.

Let $\text{lsb}(x)$ be the function for repeated multiplication by 4 ($3(\text{LSB}) + \text{LSB}$) of the least significant bit of $n$ in terms of $x$, where $x \in \mathbb{Z}^+$. Then $\text{lsb}(x) = 2^{2(x+\delta)}$.
5 Definition Two

Let \( f(x) \) be the function, in terms of \( x \), \( x \in \mathbb{Z}^+ \), for the \( 3n + \text{LSB} \) step for \( n \in O^+ \). Then

\[
f(x) = m(x) + \text{LSB}(x) = 3^{(x+\delta)}n + 2^{(x+\delta)}.
\]

Let \( T\text{lsb}(x) \) be the function that, for every \( n \in O^+ \), returns the true position of the least significant bit of the \( 3n + \text{LSB} \) step in terms of \( x \in \mathbb{Z}^+ \). Next

\[
\delta = \sum_{x \in \mathbb{Z}^+} (T\text{lsb}(x) - \text{lsb}(x))
\]

Example Three

Assume that multiplying \( n_k \) by 3 produces \ldots001111100\ldots somewhere in the binary representation of the result; and that the rightmost 1 is \( \text{LSB} = 2^x \). Let \( \text{lsb}(x) = T\text{lsb}(x) \). Adding \( \text{LSB} \) to \( n_k \) yields \ldots01000000\ldots

\[
\delta = \sum_x T\text{lsb}(x) - \text{lsb}(x)
\]

\[
\delta = \sum_x (2^{x+5} - 2^{x+2})
\]

\[
\delta = \sum_x (x + 5 - x - 2)
\]

\[
\delta = \sum_x (3) = 3
\]

Example Four

\( T\text{lsb}(x) \leq \text{lsb}(x) \)

Assume that the binary representation of the result, after multiplying \( n_k \) by 3 and adding \( \text{LSB} \), is \ldots001111100\ldots, and that the rightmost 1 is \( \text{LSB} = 2^x \). Assume \( TL\text{sb}(x) = L\text{sb}(x) \). This pattern will be created by multiplying by three again and adding \( \text{LSB} \) after

\ldots001111100\ldots times 3 plus \( 2^x \)
\ldots101111000\ldots times 3 plus \( 2^{x+1} \)
\ldots001110000\ldots times 3 plus \( 2^{x+2} \)
\ldots101100000\ldots times 3 plus \( 2^{x+3} \)
\ldots001000000\ldots , than

\[
\delta = \sum_{x} T\text{lsb}(x) - \text{lsb}(x)
\]

\[
\delta = \sum_{x} (2^{x+1} - 2^{x+2})
\]
\[
\delta = \sum_{x}^{x+3} (x + 1 - x - 2) \\
\delta = \sum_{x}^{x+3} (-1) = -4
\]

Given:
\[
\delta < 0 \lor \delta = 0 \lor \delta > 0
\]

If \( x \) is assumed to be \( x \in Z^+ \), then \( m(x) < \text{lsb}(x) \) indicates that a power of two is greater than the sum of its powers.

Using Example 2 as an illustration:

\[
m(x) - \text{lsb}(x) = 9 \cdot 3^{(x+2)} - 4^{(x+2)} = 0 \text{ for } x \approx 5.6377.
\]

The integer after the root necessitates that \( m(x) < \text{lsb}(x) \). In other words, it requires six \( 3^n + \text{LSB} \) steps for 9 to converge to \( 2^{13} \).

5.1 Theorem Three

There is a positive integer \( x \) such that \( m(x) < \text{lsb}(x) \) for all positive odd integers \( n \).

For every \( n \in O^+ \),
\[
\exists x \in Z^+ (m(x) < \text{lsb}(x))
\]

Proof

Case one

Given: \( \delta \leq -1, \delta \in Z \).
Assume \( n \in O^+ \) and let \( m(x) - \text{lsb}(x) = 3^{x-\delta} n - 4^{x-\delta} = 0 \).
\[
x = \frac{\log(1/n)}{\log(3/4)} + \delta.
\]
Therefore, there exists a unique \( x \in R^+ \) such that \( 3^{x-\delta} n - 4^{x-\delta} = 0 \) and \( \exists \Rightarrow x \in Z^+ \) such that \( m(x) < \text{lsb}(x) \).

Case Two

Given: \( \delta = 0 \).
Assume \( n \in O^+ \) and let \( m(x) - \text{lsb}(x) = 3^x n - 4^x = 0 \).
\[
x = \frac{\log(1/n)}{\log(3/4)}.
\]
Therefore, there exists a unique \( x \in R^+ \) such that \( 3^x n - 4^x = 0 \) and \( \exists \Rightarrow x \in Z^+ \) such that \( m(x) < \text{lsb}(x) \).

Case Three

Given: \( \delta \geq 1, \delta \in Z \).
Assume \( n \in O^+ \) and let \( m(x) - \text{lsb}(x) = 3^{x+\delta} n - 4^{x+\delta} = 0 \).
\[
x = \frac{\log(1/n)}{\log(3/4)} - \delta.
\]
Therefore, there exists a unique \( x \in R^+ \) such that \( 3^{x+\delta}n - 4^{x+\delta} = 0 \) and \( \exists \Rightarrow x \in Z^+ \) such that \( m(x) < \text{lsb}(x) \).

Since these examples are all-inclusive, it demonstrates that

For every \( n \in O^+ \), \( \exists x \in Z^+ \) such that \( m(x) < \text{lsb}(x) \)

For all \( n \in O^+ \), there exists an \( x \in Z^+ \) such that \( m(x) < \text{lsb}(x) \) (Theorem 3), therefore \( f(x) \) converges to \( 2^y \), \( y \in Z^+ \). And since the \( 3n + \) LSB step and the \( 3n + 1 \) step are isomorphic (Theorem 2), it can be concluded that if \( a_0 = n \), \( n \in O^+ \), then...

\[
a_{i+1} = \begin{cases} 
\frac{a_i}{2} & \text{for even } a_i \\
3a_i + 1 & \text{for odd } a_i 
\end{cases}
\]

converges to 1.

Theorem 1 states that the truth applies to all positive integers since the \( 3n + 1 \) issue holds true for all positive odd numbers. As \( n \in Z^+ \), if \( a_0 = n \), then

\[
a_{i+1} = \begin{cases} 
\frac{a_i}{2} & \text{for even } a_i \\
3a_i + 1 & \text{for odd } a_i 
\end{cases}
\]

converges to 1.

6 Conclusion

To wrap up, our research has revealed an interesting alternating pattern in the \( 3n+1 \) problem, providing new insight into how it works. By carefully examining the data, we have not only proven that the theory is true for all whole numbers but have also presented a clear and easy-to-follow explanation, eliminating the need for extensive computer checks. This new finding is a major achievement in the field of math, solving a longstanding puzzle with clarity and accuracy.

References

