Bi- Starlike Function of Complex Order Based on Double Zeta Functions Associated with Crescent Shaped Region

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BI- STARLIKE FUNCTION OF COMPLEX ORDER BASED ON DOUBLE ZETA FUNCTIONS ASSOCIATED WITH CRESCENT SHAPED REGION

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Abstract. In the present paper, we defined two new subclasses of bi-starlike and bi-convex function of complex order involving double zeta functions in the open unit disc and obtained initial coefficients of functions in these classes related with crescent shaped region. Furthermore, we determine the Fekete-Szegö inequalities for function in these classes. Several consequences of the results which are new are also pointed out as corollaries.
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1. Introduction and Definitions

Let $\mathfrak{A}$ represent the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{D} = \{z : |z| < 1\}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Further, let $\mathfrak{S}$ denote the class of all functions in $\mathfrak{A}$ which are univalent in $\mathbb{D}$. Some of the significant and well-investigated subclasses of the univalent function class $\mathfrak{S}$ comprise (for example) the class $\mathfrak{S}^*(\alpha)$ of starlike functions of order α in $\mathbb{D}$ and the class $\mathfrak{K}(\alpha)$ of convex functions of order α $(0 \leq \alpha < 1)$ in $\mathbb{D}$.

It is well recognized that every function $f \in \mathfrak{S}$ has an inverse f^{-1} defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{D})$$
$$\text{and} \quad f(f^{-1}(w)) = w \quad (|w| < r_0(f); r_0(f) \geq 1/4)$$

where

$$f^{-1}(w) = g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{1.2}$$

A function $f(z) \in \mathfrak{A}$ is said to be bi-univalent in $\mathbb{D}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{D}$. Let Σ signify the class of bi-univalent functions in $\mathbb{D}$ given by (1.1). Formerly, Brannan and Taha [5] presented certain subclasses of bi-univalent function class Σ, namely bi-starlike functions of order $\alpha (0 < \alpha \leq 1)$ denoted by $\mathfrak{S}^*_{\Sigma}(\alpha)$ and bi-convex function of order α denoted by $\mathfrak{K}_{\Sigma}(\alpha)$. For each of the function classes $\mathfrak{S}^*_{\Sigma}(\alpha)$ and $\mathfrak{K}_{\Sigma}(\alpha)$, non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ were established [5, 27]. But the coefficient problem for each of the succeeding Taylor-Maclaurin coefficients:

$$|a_n| \qquad (n \in \mathbb{N} \setminus \{1, 2\}; \ \ \mathbb{N} := \{1, 2, 3, \cdots\})$$

is still an open problem (see [5, 4, 12, 15, 27]). Numerous researchers (see [11, 21, 23]) have familiarized and inspected several interesting subclasses of the bi-univalent function class Σ

and they have found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$.

An analytic function f is subordinate to an analytic function g, written $f(z) \prec g(z)$, provided there is an analytic function w defined on $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$ sustaining $f(z) = g(w(z))$.

Definition 1.1. [19] *Let $f \in \mathfrak{A}$ be normalized by $f(0) = f'(0) - 1 = 0$ in the unit disc $\mathbb{D}$. We represent by $\mathrm{S}^*(\varphi)$ the class of analytic functions and satisfying the condition that*

$$\frac{zf'(z)}{f(z)} \prec z + \sqrt{1+z^2} =: \phi(z),$$

where the branch of the square root is chosen to be the principal one, that is $\phi(0) = 1$.

The function $\phi(z) := z + \sqrt{1+z^2}$ maps the unit disc $\mathbb{U}$ onto a shell shaped region on the right half plane, and it is analytic and univalent on $\mathbb{D}$. The range $\phi(\mathbb{D})$ is symmetric respecting the real axis and $\phi(z)$ is a function with positive real part in $\mathbb{D}$, with $\phi(0) = \phi'(0) = 1$. Moreover, it is a starlike domain with respect to the point $\phi(0) = 1$(see Fig1) also see([20])

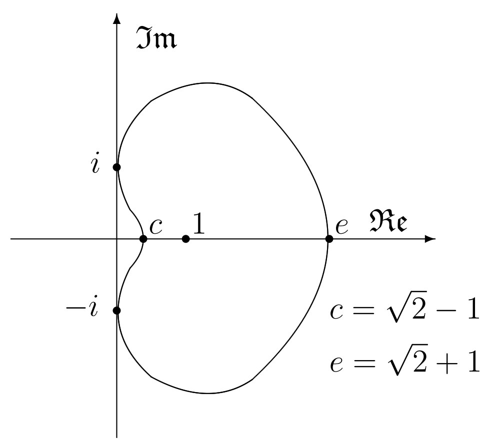

Fig. 1. The boundary of the set $\phi(\mathbb{D})$.

Inspired by the aforementioned works, we define a subclass of bi-univalent functions namely Σ as follows.

The study of operators plays an central role in the geometric function theory and its correlated fields. In the recent years, there has been an collective importance in problems concerning evaluations of various families of series linked with the Riemann zeta function and Hurwitz zeta function and their extensions and generalities such as the Hurwitz-Lerch zeta function. These functions ascend naturally in many branches of analytic function theory and their studies have plentiful important applications in mathematics [1]. As a overview of both Riemann and Hurwitz zeta functions, the so-called Hurwitz-Lerch zeta function is defined in [10]. Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ defined in [22] given by

$$\Phi(z, s, a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \tag{1.3}$$

$(a \in \mathbb{C}\backslash\mathbb{Z}_0^-; s \in \mathbb{C}; \mathfrak{R}(s) > 1$ and $|z| < 1$ where, as usual, $\mathbb{Z}_0^- := \mathbb{Z}\backslash\mathbb{N}$, $(\mathbb{Z} := \{0, \pm1, \pm2, \pm3, ...\})$. It is clear that Φ is an analytic function in both variables s and z in a suitable region and it eases to the ordinary Lerch zeta function when $z = 2\pi\ i\lambda$. Besides, Φ yields the following known result [10]:

$$\Phi(z,1,a) = a^{-1}{}_2F_1(a,1;a+1,z),$$

where ${}_2F_1$ is the Gaussian hypergeometric function. Several interesting properties and characteristics of the Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ can be found in the recent investigations by Choi and Srivastava [6], and (also see [14]) the references stated therein . The double zeta function of Barnes [2] is defined by

$$\zeta(x,a,\tau) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(m+a+\tau)^{-x},$$

where $a \neq 0$ and τ is a non-zero complex number with $|\arg(\tau)| < \pi$. Combining (1-3), Bin-Saad [3] posed a generalized double zeta function of the form

$$\zeta_\tau^\kappa(z,s,a) = \sum_{n=0}^{\infty}(\kappa)_n\Phi(z,s,a+n\tau)\frac{z^n}{n!}$$

where $\tau \in \mathbb{C}\setminus\{0\}; \kappa \in \mathbb{C}\setminus\mathbb{Z}_0^-; a \in \mathbb{C}\setminus\{-(m+\tau n)\}, n,m \in \mathbb{N}_0 := \mathbb{N}\cup\{0\},\ |s| < 1; |z| < 1$ and Φ is the Hurwitz-Lerch zeta function distinct by (1.3) and $(\kappa)_n$ is the Pochhammer symbol defined by

$$(\kappa)_n = \begin{cases} 1, & n = 0 \\ \kappa(\kappa+1)(\kappa+2)\ldots(\kappa+n-1), & n \in \mathbb{N}. \end{cases} \tag{1.4}$$

The convolution or Hadamard product of two functions $f, h \in \mathfrak{A}$ is denoted by $f * h$ and is defined as

$$(f*h)(z) = z + \sum_{n=2}^{\infty}a_nb_nz^n, \tag{1.5}$$

where $f(z)$ is given by (1.1) and $h(z) = z + \sum\limits_{n=2}^{\infty}b_nz^n$. In this work, by using the Hadamard product or the convolution product of generalized Hurwitz- Lerch zeta function in[17] defined a function as follows:

$$\Theta_n(z,s,a) = \frac{\Phi(z,s,a+n\tau)}{\Phi(z,s,a)}, \quad n \in \mathbb{N}_0. \tag{1.6}$$

It is clear that $\Theta_0(z,s,a) = 1$. Now consider the function

$$\Upsilon_\kappa(z,s,a) = \sum_{n=0}^{\infty}\frac{(\kappa)_n}{n!}\Theta_n(z,s,a)z^n, \tag{1.7}$$

which implies

$$z\Upsilon_\kappa(z,s,a) = z + \sum_{n=2}^{\infty}\frac{(\kappa)_{n-1}}{(n-1)!}\Theta_{n-1}(z,s,a)z^n.$$

Thus,

$$z\Upsilon_\kappa(z,s,a) * (z\Upsilon_\kappa(z,s,a))^{-1} = \frac{z}{(1-z)^\delta} = z + \sum_{n=2}^{\infty}\frac{(\delta)_{n-1}}{(n-1)!}z^n,\ \ \delta > -1$$

poses a linear operator

$$\mathfrak{I}_\kappa^\delta(z,s,a)f(z) = (z\Upsilon_\kappa(z,s,a))^{-1} * f(z) = z + \sum_{n=2}^{\infty}\frac{(\delta)_{n-1}}{(\kappa)_{n-1}\Theta_{n-1}(z,s,a)}a_nz^n \tag{1.8}$$

where $\tau \in \mathbb{C}\setminus\{0\}; \kappa \in \mathbb{C}\setminus\mathbb{Z}_0^-; a \in \mathbb{C}\setminus\{-(m+\tau n)\}, n, m \in \mathbb{N}_0,\ |s|<1; |z|<1$ and $\Theta_n(z,s,a)$ is defined in (1.6). It is clear that $I_\kappa^\delta(z,s,a)f(z) \in \mathfrak{A}$.

$$\mathfrak{I}_\kappa^\delta f(z) = \mathfrak{I}_\kappa^\delta(z,s,a)f(z) = z + \sum_{n=2}^{\infty} \Psi_n\, a_n\, z^n, \tag{1.9}$$

where

$$\Psi_n = \frac{(\delta)_{n-1}}{(\kappa)_{n-1}\Theta_{n-1}(z,s,a)}, \tag{1.10}$$

$\tau \in \mathbb{C}\setminus\{0\}; \kappa \in \mathbb{C}\setminus\mathbb{Z}_0^-; a \in \mathbb{C}\setminus\{-(m+\tau n)\}, n, m \in \mathbb{N}_0, |s|<1; |z|<1$ and $\Theta_n(z,s,a)$ is defined in (1.6) .

Inspired by the work of Silverman and Silvia [25](also see[26]) and recent study by Srivastava et al [24],and by the earlier work of Deniz[9] and Huo Tang et al [11], in the present paper we introduce two new subclasses of the function class Σ of complex order $\vartheta \in \mathbb{C}\backslash\{0\}$, involving the linear operator $\mathfrak{I}_\kappa^\delta$ and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the new subclasses of the function class Σ. Several related classes are also considered , and connection to earlier known results are made.

Definition 1.2. *A function $f(z) \in \Sigma$ given by (1.1) is said to be in the class $\mathfrak{S}_{\Sigma,\phi}^{\delta,\kappa}(\vartheta,\lambda)$ if the following conditions are satisfied:*

$$1 + \frac{1}{\vartheta}\left(\frac{z(\mathfrak{I}_\kappa^\delta f(z))'}{\mathfrak{I}_\kappa^\delta f(z)} + \left(\frac{1+e^{i\lambda}}{2}\right)\frac{z^2(\mathfrak{I}_\kappa^\delta f(z))''}{\mathfrak{I}_\kappa^\delta f(z)} - 1\right) \prec \phi(z) \quad (\lambda \in (-\pi,\pi], z \in \mathbb{D}) \tag{1.11}$$

and

$$1 + \frac{1}{\vartheta}\left(\frac{w(\mathfrak{I}_\kappa^\delta g(w))'}{\mathfrak{I}_\kappa^\delta g(w)} + \left(\frac{1+e^{i\lambda}}{2}\right)\frac{w^2(\mathfrak{I}_\kappa^\delta g(w))''}{\mathfrak{I}_\kappa^\delta g(w)} - 1\right) \prec \phi(w) \quad (\lambda \in (-\pi,\pi], w \in \mathbb{D}), \tag{1.12}$$

where $\vartheta \in \mathbb{C}\backslash\{0\}$ and the function g is given by (1.2).

Definition 1.3. *A function $f(z)$ given by (1.1) is said to be in the class $\mathfrak{K}_{\Sigma,\phi}^{\delta,\kappa}(\vartheta,\lambda)$ if the following conditions are satisfied:*

$$1 + \frac{1}{\vartheta}\left(\frac{[z(\mathfrak{I}_\kappa^\delta f(z))' + \left(\frac{1+e^{i\lambda}}{2}\right)z^2(\mathfrak{I}_\kappa^\delta f(z))'']'}{(\mathfrak{I}_\kappa^\delta f(z))'} - 1\right) \prec \phi(z) \quad (\lambda \in (-\pi,\pi], z \in \mathbb{D}) \tag{1.13}$$

and

$$1 + \frac{1}{\vartheta}\left(\frac{[w(\mathfrak{I}_\kappa^\delta g(w))' + \left(\frac{1+e^{i\lambda}}{2}\right)w^2(\mathfrak{I}_\kappa^\delta g(w))'']'}{(\mathfrak{I}_\kappa^\delta g(w))'} - 1\right) \prec \phi(w) \quad (\lambda \in (-\pi,\pi], w \in \mathbb{D}), \tag{1.14}$$

where $\vartheta \in \mathbb{C}\backslash\{0\}$ and the function g is given by (1.2).

Remark 1.1. *A function $f(z) \in \Sigma$ given by (1.1) and for $\lambda = 0$, we note that $\mathfrak{S}_{\Sigma,\phi}^{\delta,\kappa}(\vartheta,\lambda) \equiv \mathfrak{S}_{\Sigma,\phi}^{\delta,\kappa}(\vartheta)$ and $\mathfrak{K}_{\Sigma,\phi}^{\delta,\kappa}(\vartheta,\lambda) \equiv \mathfrak{K}_{\Sigma,\phi}^{\delta,\kappa}(\vartheta)$ satisfies the following conditions respectively:*

$$\left[1 + \frac{1}{\vartheta}\left(\frac{z(\mathfrak{I}_\kappa^\delta f(z))'}{\mathfrak{I}_\kappa^\delta f(z)} - 1\right)\right] \prec \phi(z) \text{ and } \left[1 + \frac{1}{\vartheta}\left(\frac{w(\mathfrak{I}_\kappa^\delta g(w))'}{\mathfrak{I}_\kappa^\delta g(w)} - 1\right)\right] \prec \phi(w)$$

and

$$\left[1 + \frac{1}{\vartheta}\left(\frac{z(\mathfrak{I}_\kappa^\delta f(z))''}{(\mathfrak{I}_\kappa^\delta f(z))'}\right)\right] \prec \phi(z) \text{ and } \left[1 + \frac{1}{\vartheta}\left(\frac{w(\mathfrak{I}_\kappa^\delta g(w))''}{(\mathfrak{I}_\kappa^\delta g(w))'}\right)\right] \prec \phi(w),$$

where $\vartheta \in \mathbb{C}\backslash\{0\}$ $z, w \in \mathbb{D}$ and the function g is given by (1.2).

Remark 1.2. *A function $f(z) \in \Sigma$ given by (1.1) and for $\vartheta = 1$, we note that $\mathfrak{S}^{\delta,\kappa}_{\Sigma,\phi}(\vartheta,\lambda) \equiv \mathfrak{S}^{\delta,\kappa}_{\Sigma,\phi}(\lambda)$ and satisfies the following conditions respectively:*

$$\left(\frac{z(\mathfrak{I}^{\delta}_{\kappa}f(z))'}{\mathfrak{I}^{\delta}_{\kappa}f(z)} + \left(\frac{1+e^{i\lambda}}{2}\right)\frac{z^2(\mathfrak{I}^{\delta}_{\kappa}f(z))''}{\mathfrak{I}^{\delta}_{\kappa}f(z)}\right) \prec \phi(z)$$

and

$$\left(\frac{w(\mathfrak{I}^{\delta}_{\kappa}g(w))'}{\mathfrak{I}^{\delta}_{\kappa}g(w)} + \left(\frac{1+e^{i\lambda}}{2}\right)\frac{w^2(\mathfrak{I}^{\delta}_{\kappa}g(w))''}{\mathfrak{I}^{\delta}_{\kappa}g(w)}\right) \prec \phi(w).$$

Also $\mathfrak{K}^{\delta,\kappa}_{\Sigma,\phi}(\vartheta,\lambda) \equiv \mathfrak{K}^{\delta,\kappa}_{\Sigma,\phi}(\lambda)$ and satisfies the following conditions

$$\left(\frac{[z(\mathfrak{I}^{\delta}_{\kappa}f(z))' + \left(\frac{1+e^{i\lambda}}{2}\right)z^2(\mathfrak{I}^{\delta}_{\kappa}f(z))'']'}{(\mathfrak{I}^{\delta}_{\kappa}f(z))'}\right) \prec \phi(z)$$

and

$$\left(\frac{[w(\mathfrak{I}^{\delta}_{\kappa}g(w))' + \left(\frac{1+e^{i\lambda}}{2}\right)w^2(\mathfrak{I}^{\delta}_{\kappa}g(w))'']'}{(\mathfrak{I}^{\delta}_{\kappa}g(w))'}\right) \prec \phi(w),$$

where $\lambda \in (-\pi,\pi]$, $z, w \in \mathbb{D}$ and the function g is given by (1.2).

2. Coefficient estimates for the function class $\mathfrak{S}^{\delta,\kappa}_{\Sigma,\phi}(\vartheta,\lambda)$ and $\mathfrak{K}^{\delta,\kappa}_{\Sigma,\phi}(\vartheta,\lambda)$

For notational simplicity, in the sequel we let,

$$\mathfrak{I}^{\delta}_{\kappa}f(z) \text{ for } \mathfrak{I}^{\delta}_{\kappa}(z,s,a)f(z)$$

and

$$\Psi_2 = \frac{(\delta)_1}{(\kappa)_1\Theta_1(z,s,a)}, \tag{2.1}$$

$$\Psi_3 = \frac{(\delta)_2}{(\kappa)_2\Theta_2(z,s,a)} \tag{2.2}$$

where $\tau \in \mathbb{C}\setminus\{0\}; \kappa \in \mathbb{C}\setminus\mathbb{Z}^-_0; a \in \mathbb{C}\setminus\{-(m+\tau n)\}, n, m \in \mathbb{N}_0, |s| < 1; |z| < 1$ and $\Theta_n(z,s,a)$ is defined in (1.6) .

$$\phi(z) := z + \sqrt{1+z^2} = 1 + z + \frac{1}{2}z^2 - \frac{1}{8}z^4 + \cdots. \tag{2.3}$$

For deriving our main results, we need the following lemma.

Lemma 2.1. [16] *If $h \in \mathfrak{P}$, then $|c_k| \le 2$ for each k, where $\mathfrak{P}$ is the family of all functions h analytic in $\mathbb{D}$ for which $\Re(h(z)) > 0$ and*

$$h(z) = 1 + c_1z + c_2z^2 + \cdots \quad \textit{for } z \in \mathbb{D}.$$

Define the functions $p(z)$ and $q(z)$ by

$$p(z) := \frac{1+u(z)}{1-u(z)} = 1 + p_1z + p_2z^2 + \cdots$$

and

$$q(z) := \frac{1+v(z)}{1-v(z)} = 1 + q_1z + q_2z^2 + \cdots.$$

It follows that

$$u(z) := \frac{p(z)-1}{p(z)+1} = \frac{1}{2}\left[p_1 z + \left(p_2 - \frac{p_1^2}{2}\right) z^2 + \cdots\right]$$

and

$$v(z) := \frac{q(z)-1}{q(z)+1} = \frac{1}{2}\left[q_1 z + \left(q_2 - \frac{q_1^2}{2}\right) z^2 + \cdots\right].$$

Then $p(z)$ and $q(z)$ are analytic in $\mathbb{D}$ with $p(0) = 1 = q(0)$.

Since $u, v : \mathbb{D} \to \mathbb{D}$, the functions $p(z)$ and $q(z)$ have a positive real part in $\mathbb{D}$, and $|p_i| \leq 2$ and $|q_i| \leq 2$ for each i.

Theorem 2.1. *Let $f(z)$ given by (1.1) be in the class $\mathfrak{S}^{\delta,\kappa}_{\Sigma,\phi}(\vartheta, \lambda)$, $\vartheta \in \mathbb{C}\backslash\{0\}$ and $\lambda \in (-\pi, \pi]$. Then*

$$|a_2| \leq \frac{\sqrt{2}\ |\vartheta|}{\sqrt{2\,|\vartheta[(5+3e^{i\lambda})\Psi_3 - (2+e^{i\lambda})\Psi_2^2] + (2+e^{i\lambda})^2\Psi_2^2|}}. \tag{2.4}$$

and

$$|a_3| \leq \frac{|\vartheta|^2}{|2+e^{i\lambda}|^2\Psi_2^2} + \frac{|\vartheta|}{|5+3e^{i\lambda}|\Psi_3}. \tag{2.5}$$

Proof. It follows from (1.11) and (1.12) that

$$1 + \frac{1}{\vartheta}\left(\frac{z(\mathfrak{I}^\delta_\kappa f(z))'}{\mathfrak{I}^\delta_\kappa f(z)} + \left(\frac{1+e^{i\lambda}}{2}\right)\frac{z^2(\mathfrak{I}^\delta_\kappa f(z))''}{\mathfrak{I}^\delta_\kappa f(z)} - 1\right) = \phi(u(z)) \tag{2.6}$$

and

$$1 + \frac{1}{\vartheta}\left(\frac{w(\mathfrak{I}^\delta_\kappa g(w))'}{\mathfrak{I}^\delta_\kappa g(w)} + \left(\frac{1+e^{i\lambda}}{2}\right)\frac{w^2(\mathfrak{I}^\delta_\kappa g(w))''}{\mathfrak{I}^\delta_\kappa g(w)} - 1\right) = \phi(v(w)), \tag{2.7}$$

where

$$\begin{aligned}\phi(u(z)) &= \sqrt{1 + \left(\frac{p(z)-1}{p(z)+1}\right)^2} + \frac{p(z)-1}{p(z)+1} \\ &= 1 + \frac{p_1}{2}z + \left(\frac{p_2}{2} - \frac{p_1^2}{8}\right)z^2 + \left(\frac{p_3}{2} - \frac{p_1p_2}{4}\right)z^3 + \cdots .\end{aligned} \tag{2.8}$$

and similarly we get

$$\phi(v(w)) = 1 + \frac{q_1}{2}w + \left(\frac{q_2}{2} - \frac{q_1^2}{8}\right)w^2 + \left(\frac{q_3}{2} - \frac{q_1q_2}{4}\right)w^3 + \cdots . \tag{2.9}$$

Now, equating the coefficients in (2.6) and (2.7), we get

$$\frac{1}{\vartheta}(2+e^{i\lambda})\Psi_2 a_2 = \frac{1}{2}p_1, \tag{2.10}$$

$$\frac{1}{\vartheta}\left[(5+3e^{i\lambda})\Psi_3 a_3 - (2+e^{i\lambda})\Psi_2^2 a_2^2\right] = \frac{1}{2}(p_2 - \frac{p_1^2}{2}) + \frac{1}{8}p_1^2, \tag{2.11}$$

$$-\frac{1}{\vartheta}(2+e^{i\lambda})\Psi_2 a_2 = \frac{1}{2}q_1, \tag{2.12}$$

and

$$\frac{1}{\vartheta}\left([2(5+3e^{i\lambda})\Psi_3 - (2+e^{i\lambda})\Psi_2^2]a_2^2 - (5+3e^{i\lambda})\Psi_3 a_3\right) = \frac{1}{2}(q_2 - \frac{q_1^2}{2}) + \frac{1}{8}q_1^2. \tag{2.13}$$

From (2.10) and (2.12), we get

$$p_1 = -q_1 \tag{2.14}$$

and

$$8(2+e^{i\lambda})^2\Psi_2^2 a_2^2 = \vartheta^2(p_1^2+q_1^2). \tag{2.15}$$

Now from (2.11), (2.13) and (2.15), we obtain

$$\left(2\{2\vartheta[(5+3e^{i\lambda})\Psi_3-(2+e^{i\lambda})\Psi_2^2]+(2+e^{i\lambda})^2\Psi_2^2\}\right)a_2^2 = \vartheta^2(p_2+q_2). \tag{2.16}$$

Applying Lemma (2.1) to the coefficients p_2 and q_2, we have

$$|a_2| \leq \frac{\sqrt{2}\,|\vartheta|}{\sqrt{2\,|\vartheta[(5+3e^{i\lambda})\Psi_3-(2+e^{i\lambda})\Psi_2^2]+(2+e^{i\lambda})^2\Psi_2^2|}}.$$

Next, in order to find the bound on $|a_3|$, by subtracting (2.11) from (2.13) and using (2.14), we get

$$\frac{2}{\vartheta}(5+3e^{i\lambda})\Psi_3(a_3-a_2^2) = \frac{1}{4}(p_2-q_2).$$

Upon substituting the value of a_2^2 from (2.15), we get

$$a_3 = \frac{\vartheta^2(p_1^2+q_1^2)}{8(2+e^{i\lambda})^2\Psi^2} + \frac{\vartheta(p_2-q_2)}{4(5+3e^{i\lambda})\Psi_3}.$$

Applying Lemma (2.1) once again to the coefficients p_1, p_2, q_1 and q_2, we get

$$|a_3| \leq \frac{|\vartheta|^2}{|2+e^{i\lambda}|^2\Psi_2^2} + \frac{|\vartheta|}{|5+3e^{i\lambda}|\Psi_3}.$$

□

Theorem 2.2. *Let $f(z)$ given by (1.1) be in the class $\mathfrak{K}_{\Sigma,\phi}^{\delta,\kappa}(\vartheta,\lambda)$, $\vartheta \in \mathbb{C}\backslash\{0\}$ and $\lambda \in (-\pi,\pi]$. Then*

$$|a_2| \leq \frac{|\vartheta|}{\sqrt{|\vartheta[3(5+3e^{i\lambda})\Psi_3-4(2+e^{i\lambda})\Psi_2^2]+2(2+e^{i\lambda})^2\Psi_2^2|}} \tag{2.17}$$

and

$$|a_3| \leq \frac{|\vartheta|^2}{4|2+e^{i\lambda}|^2\Psi_2^2} + \frac{|\vartheta|}{3|5+3e^{i\lambda}|\Psi_3}. \tag{2.18}$$

Proof. We can write the argument inequalities in (1.13) and (1.14) equivalently as follows:

$$1+\frac{1}{\vartheta}\left(\frac{[z(\mathfrak{I}_\kappa^\delta f(z))' + \left(\frac{1+e^{i\lambda}}{2}\right)z^2(\mathfrak{I}_\kappa^\delta f(z))'']'}{(\mathfrak{I}_\kappa^\delta f(z))'} - 1\right) = \phi(u(z)) \tag{2.19}$$

and

$$1+\frac{1}{\vartheta}\left(\frac{[w(\mathfrak{I}_\kappa^\delta g(w))' + \left(\frac{1+e^{i\lambda}}{2}\right)w^2(\mathfrak{I}_\kappa^\delta g(w))'']'}{(\mathfrak{I}_\kappa^\delta g(w))'} - 1\right) = \phi(v(w)), \tag{2.20}$$

and proceeding as in the proof of Theorem 2.1, from (2.19) and (2.20),we can arrive the following relations

$$\frac{2}{\vartheta}(2+e^{i\lambda})\Psi_2 a_2 = \frac{1}{2}p_1, \tag{2.21}$$

$$\frac{1}{\vartheta}[3(5+3e^{i\lambda})\Psi_3 a_3 - 4(2+e^{i\lambda})\Psi_2^2 a_2^2] = \frac{1}{2}(p_2-\frac{p_1^2}{2})+\frac{1}{8}p_1^2, \tag{2.22}$$

and

$$-\frac{2}{\vartheta}(2+e^{i\lambda})\Psi_2 a_2 = \frac{1}{2}q_1, \tag{2.23}$$

$$\frac{1}{\vartheta}[3(5+3e^{i\lambda})(2a_2^2-a_3)\Psi_3-4(2+e^{i\lambda})\Psi_2^2a_2^2]=\frac{1}{2}(q_2-\frac{q_1^2}{2})+\frac{1}{8}q_1^2. \tag{2.24}$$

From (2.21) and (2.23), we get

$$p_1=-q_1 \tag{2.25}$$

and

$$32(2+e^{i\lambda})^2\Psi_2^2a_2^2=\vartheta^2(p_1^2+q_1^2). \tag{2.26}$$

Now from (2.22), (2.24) and (2.26), we obtain

$$a_2^2=\frac{\vartheta^2(p_2+q_2)}{4[\vartheta[3(5+3e^{i\lambda})\Psi_3-4(2+e^{i\lambda})\Psi_2^2]+2(2+e^{i\lambda})^2\Psi_2^2]}. \tag{2.27}$$

Applying Lemma (2.1) to the coefficients p_2 and q_2, we have the desired inequality given in (2.17).

Next, in order to find the bound on $|a_3|$, by subtracting (2.22) from (2.24), and using (2.25), we get

$$\frac{6}{\vartheta}(5+3e^{i\lambda})(a_3-a_2^2)\Psi_3=\frac{1}{2}(p_2-q_2).$$

Upon substituting the value of a_2^2 given (2.26), the above equation leads to

$$a_3=\frac{\vartheta(p_2-q_2)}{12(5+3e^{i\lambda})\Psi_3}+\frac{\vartheta^2(p_1^2+q_1^2)}{32(2+e^{i\lambda})^2\Psi_2^2}. \tag{2.28}$$

Applying the Lemma (2.1) once again to the coefficients p_1, p_2, q_1 and q_2, we get the desired coefficient given in (2.18). □

Putting $\lambda=\pi$ in Theorems (2.1) and (2.2) , we can state the coefficient estimates for the functions in the subclasses $\mathfrak{S}_{\Sigma,\phi}^{\delta,\kappa}(\vartheta)$ and $\mathfrak{K}_{\Sigma,\phi}^{\delta,\kappa}(\vartheta)$ defined in Remark (1.1).

Corollary 2.1. *Let $f(z)$ given by (1.1) be in the class $\mathfrak{S}_{\Sigma,\phi}^{\delta,\kappa}(\vartheta)$. Then*

$$|a_2|\le\frac{\sqrt{2}\ |\vartheta|}{\sqrt{2|\vartheta|(2\Psi_3-\Psi_2^2)+\Psi_2^2}}\quad\text{and}\quad|a_3|\le\frac{|\vartheta|^2}{\Psi_2^2}+\frac{|\vartheta|}{2\Psi_3}.$$

Corollary 2.2. *Let $f(z)$ given by (1.1) be in the class $\mathfrak{K}_{\Sigma,\phi}^{\delta,\kappa}(\vartheta)$. Then*

$$|a_2|\le\frac{|\vartheta|}{\sqrt{2|\vartheta|(3\Psi_3-2\Psi_2^2)+2\Psi_2^2}}\quad\text{and}\quad|a_3|\le\frac{|\vartheta|^2}{4\Psi_2^2}+\frac{|\vartheta|}{6\Psi_3}.$$

Taking $\vartheta=1$ in Theorems (2.1) and (2.2) , we can state the coefficient estimates for the functions in the subclasses $\mathfrak{S}_{\Sigma,\phi}^{\delta,\kappa}(\lambda)$ and $\mathfrak{K}_{\Sigma,\phi}^{\delta,\kappa}(\lambda)$ defined in Remark (1.2).

Corollary 2.3. *Let $f(z)$ given by (1.1) be in the class $\mathfrak{S}_{\Sigma,\phi}^{\delta,\kappa}(\lambda)$. Then*

$$|a_2|\le\frac{\sqrt{2}}{\sqrt{2|(5+3e^{i\lambda})\Psi_3-(2+e^{i\lambda})\Psi_2^2|+|2+e^{i\lambda}|^2\Psi_2^2}}$$

and

$$|a_3|\le\frac{1}{|2+e^{i\lambda}|^2\Psi_2^2}+\frac{1}{|5+3e^{i\lambda}|\Psi_3}.$$

Corollary 2.4. *Let $f(z)$ given by (1.1) be in the class $\mathfrak{K}^{\delta,\kappa}_{\Sigma,\phi}(\lambda)$. Then*

$$|a_2| \leq \frac{1}{\sqrt{\{[3|5+3e^{i\lambda}|\Psi_3 - 4\,|2+e^{i\lambda}|\,\Psi_2^2] + 2|2+e^{i\lambda}|^2\Psi_2^2\}}}$$

and

$$|a_3| \leq \frac{1}{4\,|2+e^{i\lambda}|^2\,\Psi_2^2} + \frac{1}{3\,|5+3e^{i\lambda}|\,\Psi_3}.$$

3. Fekete-Szegö inequality for $f \in \mathfrak{S}^{\delta,\kappa}_{\Sigma,\phi}(\vartheta,\lambda)$

In this section, we prove Fekete-Szegö inequalities for functions in the class $\mathfrak{S}^{\delta,\kappa}_{\Sigma,\phi}(\vartheta,\lambda)$. These inequalities are given in the following theorem.

Theorem 3.1. *Let f given by (1) be in the class $\mathfrak{S}^{\delta,\kappa}_{\Sigma,\phi}(\vartheta,\lambda)$ and $\varrho \in \mathbb{R}$ Then*

$$|\, a_3 - \varrho a_2^2 \,| \leq \begin{cases} \frac{\vartheta}{3|5+3e^{i\lambda}|\Psi_3}, & 0 \leq |\, \phi(\varrho,r)\,| \leq \frac{\vartheta}{3|5+3e^{i\lambda}|\Psi_3} \\ 2|\vartheta||\phi(\varrho,r)|, & |\phi(\varrho,r)| \geq \frac{\vartheta}{3|5+3e^{i\lambda}|\Psi_3}. \end{cases}$$

where

$$\phi(\varrho,r) = \frac{\vartheta^2(1-\varrho)}{4[\vartheta[3(5+3e^{i\lambda})\Psi_3 - 4(2+e^{i\lambda})\Psi_2^2] + 2(2+e^{i\lambda})^2\Psi_2^2]}.$$

Proof. From (2.27) and (2.28)

$$\begin{aligned} a_3 - \varrho a_2^2 &= \frac{(1-\varrho)\vartheta^2(p_2+q_2)}{4[\vartheta[3(5+3e^{i\lambda})\Psi_3 - 4(2+e^{i\lambda})\Psi_2^2] + 2(2+e^{i\lambda})^2\Psi_2^2]} + \frac{\vartheta(p_2-q_2)}{12(5+3e^{i\lambda})\Psi_3} \\ &= \left[\phi(\varrho,r) + \frac{\vartheta}{12(5+3e^{i\lambda})\Psi_3}\right] p_2 + \left[\phi(\varrho,r) - \frac{\vartheta}{12(5+3e^{i\lambda})\Psi_3}\right] q_2 \end{aligned}$$

where

$$\phi(\varrho,r) = \frac{\vartheta^2(1-\varrho)}{4[\vartheta[3(5+3e^{i\lambda})\Psi_3 - 4(2+e^{i\lambda})\Psi_2^2] + 2(2+e^{i\lambda})^2\Psi_2^2]}.$$

Thus by applying Lemma 2.1,we get

$$|\, a_3 - \varrho a_2^2 \,| \leq \begin{cases} \frac{\vartheta}{3|5+3e^{i\lambda}|\Psi_3}, & 0 \leq |\, \phi(\varrho,r)\,| \leq \frac{\vartheta}{3|5+3e^{i\lambda}|\Psi_3} \\ 2|\vartheta||\phi(\varrho,r)|, & |\phi(\varrho,r)| \geq \frac{\vartheta}{3|5+3e^{i\lambda}|\Psi_3}. \end{cases}$$

In particular by taking $\varrho = 1$, we get

$$|\, a_3 - a_2^2 \,| \leq \frac{\vartheta}{3|5+3e^{i\lambda}|\Psi_3}$$

□

4. Concluding remarks

We defined two new subclasses of bi-starlike and bi-convex function of complex order involving double zeta functions in the open unit disc and obtained initial coefficients of functions in these classes related with shell shaped region. Furthermore, we determine the Fekete-Szegö inequalities for function in these classes. Several consequences of the results which are new are also pointed out as corollaries.Also we note that lately various subclasses of starlike functions were introduced see [7, 8, 13] by fixing some particular functions such as functions linked with Bell numbers, shell-like curve connected with Fibonacci numbers, functions associated with conic domains and rational functions instead of ϕ in (2.3) one can determine new results for the subclasses introduced in this paper.

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