Subclass of Pseudo-Type Meromorphic Bi-Univalent Functions of Complex Order Associated with Linear Operator

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A SUBCLASS OF PSEUDO-TYPE MEROMORPHIC BI-UNIVALENT FUNCTIONS OF COMPLEX ORDER ASSOCIATED WITH LINEAR OPERATOR

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Abstract

In the present article, we define a new subclass of pseudo-type meromorphic bi-univalent function class of complex order, associated with linear operator and investigate the initial coefficient estimates $|b_0|$; $|b_1|$ and $|b_2|$.Furthermore we mention several new or known consequences of our result.

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1 Introduction and Definitions

Let $\mathcal{A}$ denote the class of all analytic functions of the form

$$f(\xi) = \xi + \sum_{n=2}^{\infty} a_n \xi^n, \tag{1.1}$$

which are univalent in the open unit disk $\mathbb{U} = \{\xi : |\xi| < 1\}$. Also let $\mathcal{S}$, the class of all functions in $\mathcal{A}$, univalent and normalized by the conditions $f(0) = 0, f'(0) = 1$ in $\mathbb{U}$.
An analytic function f_1 is subordinate to an analytic function f_2, written by $f_1(\xi) \prec f_2(\xi)$, provided there is an analytic function ϖ defined on $\mathbb{U}$ with $\varpi(0) = 0$ and $|\varpi(z)| < 1$ satisfying $f_1(\xi) = f_2(\varpi(\xi))$. Ma and Minda[8] consolidated various subclasses of starlike and convex functions for which either

$$\frac{\xi f'(\xi)}{f(\xi)} \quad \text{or} \quad 1 + \frac{\xi f''(\xi)}{f'(\xi)}$$

is subordinate to a more general function. These classes are denoted respectively by $\mathfrak{S}^*_{\Sigma}(\varphi)$ and $\mathfrak{K}_{\Sigma}(\varphi)$. In this article, it is assumed that φ is an analytic function in the unit disk $\mathbb{U}$, satisfying $\varphi(0) = 1$ and $\varphi'(0) > 0$ and $\varphi(\mathbb{U})$ is symmetric with respect to the real axis. This function has a series expansion of the form

$$\varphi(\xi) = 1 + \beta_1\xi + \beta_2\xi^2 + \beta_3\xi^3 + \cdots, (\beta_1 > 0). \tag{1.2}$$

By setting $\phi(\xi)$ as given

$$\varphi(\xi) = \left(\frac{1+\xi}{1-\xi}\right)^{\delta} = 1 + 2\delta\xi + 2\delta^2\xi^2 + \frac{4\delta^2 + 2\delta}{3}\xi^3 + \cdots, \ 0 < \delta \le 1 \tag{1.3}$$

we have $\beta_1 = 2\delta, \beta_2 = 2\delta^2, \beta_3 = \frac{4\delta^2+2\delta}{3}$.
On the other hand if we take

$$\varphi(\xi) = \frac{1 + (1-2\omega)\xi}{1-\xi} = 1 + 2(1-\omega)\xi + 2(1-\omega)\xi^2 + \cdots, \ (0 \le \omega < 1) \tag{1.4}$$

then $\beta_1 = \beta_2 = \beta_3 = 2(1-\omega)$.
Let Σ' denote the class of all meromorphic univalent functions g of the form

$$\mathsf{g}(\xi) = \xi + b_0 + \sum_{n=1}^{\infty} \frac{b_n}{\xi^n}, \tag{1.5}$$

defined on the domain $\mathbb{U}^* = \{\xi : 1 < |\xi| < \infty\}$. Since $\mathsf{g} \in \Sigma'$ is univalent it has an inverse $\mathsf{g}^{-1} = \upsilon$ that satisfy

$$\mathsf{g}^{-1}(\mathsf{g}(\xi) = \xi, \ \xi \in \mathbb{U}^* \text{ and } \mathsf{g}^{-1}(\mathsf{g}(w)) = w, \ M < |w| < \infty, \ M > 0$$

where

$$\mathsf{g}^{-1}(w) = \upsilon(w) = w + \sum_{n=0}^{\infty} \frac{C_n}{w^n}, \ M < |w| < \infty \tag{1.6}$$

Analogous to the bi-univalent analytic functions, $\mathsf{g} \in \Sigma'$ is said to be meromorphic bi-univalent if $\mathsf{g}^{-1} \in \Sigma'$. Denote the class of all meromorphic bi-univalent functions by $\mathfrak{M}_{\Sigma'}$. In literature, the coefficient estimates of meromorphic univalent functions were widely studied, Schiffer[13] obtained the estimate $|b_2| \le \frac{2}{3}$ for meromorphic univalent functions $\mathsf{g} \in \Sigma'$ with $b_0 = 0$ and Duren[3] gave proof $|b_n| \le \frac{2}{(n+1)}$ on the coefficient of meromorphic

univalent functions $\mathsf{g} \in \Sigma'$ with $b_k(0) = 0$ for $1 \leq k < \frac{n}{2}$.. For the coefficient of the inverse of meromorphic univalent functions $h \in \mathfrak{M}_{\Sigma'}$, Springer [15] proved $|C_3| \leq 1; |C_3 + \frac{1}{2}C_1^2| \leq \frac{1}{2}$ and conjectured $|C_{2n-1}| \leq \frac{(2n-1)!}{n!(n-1)!}$, (n=1,2,$\cdots$).
Kubota[7] has proved the Springer's conjecture true for $n = 3, 4, 5$ and Schober[12] obtained the coefficient bounds $C_{2n-1},\ 1 \leq n \leq 7$ for the inverse of meromorphic univalent functions in $\mathbb{U}^*$ and proved the sharpness. Kapoor and Mishra [6] found the coefficient estimates for a class consisting of inverses of meromorphic starlike univalent functions of order δ in $\mathbb{U}^*$.
For $\mathsf{g} \in \Sigma'$ as given in (1.5), linear differential operator is defined as follows[10, 14]:

$$\mathcal{F}_\zeta^0 \mathsf{g}(\xi) = \mathsf{g}(\xi),$$

$$\mathcal{F}_\zeta^1 \mathsf{g}(\xi) = (1-\zeta)\mathsf{g}(\xi) + \zeta\xi\mathsf{g}'(\xi) = \mathcal{F}_\zeta \mathsf{g}(\xi) \qquad (\zeta \geq 0) \tag{1.7}$$

$$\mathcal{F}_\zeta^\nu \mathsf{g}(\xi) = \mathcal{F}_\zeta(\mathcal{F}_\zeta^{\nu-1}\mathsf{g}(\xi)) \qquad (\nu \in \mathfrak{N} = \{1, 2, 3, \cdots\}) \tag{1.8}$$

Then from (1.7) and (1.8) we get,

$$\mathcal{F}_\zeta^\nu \mathsf{g}(\xi) = \xi + (1-\zeta)^\nu b_0 + \sum_{n=1}^{\infty} [1-(n+1)\zeta]^\nu b_n \xi^{-n} \qquad (\nu \in \mathfrak{N} = \{0, 1, 2, 3, \cdots\}). \tag{1.9}$$

Babalola [1] defined a new subclass μ - pseudo starlike function of order ϑ $(0 \leq \vartheta < 1)$ satisfying the analytic conditions

$$Re\left(\frac{\xi(f'(\xi))^\mu}{f(\xi)}\right) > \vartheta,\ \xi \in \mathbb{U},\ \mu \geq 1 \in \mathbb{R} \tag{1.10}$$

and denoted by $\mathcal{L}_\mu(\vartheta)$. Babalola[1] remarked that for $\mu > 1$, these classes of $\mu-$ pseudo starlike functions reperesnt the analytic starlike functions. Also, when $\mu = 1$, we have the class of starlike functions of order ϑ (1-pseudo starlike functions of order ϑ) and for $\mu = 2$, we get the class of functions, which is a product combination of geometric expressions for bounded turning and starlike functions.

Motivated by the earlier works [2, 4, 9, 10, 17, 18], we define a new subclass of pseudo type meromorphic bi-univalent functions class Σ' of complex order $\gamma \in \mathbb{C}\backslash\{0\}$ and the coefficient estimates $|b_0|, |b_1|$ and $|b_2|$ are determined when associated with the linear operator as defined in (1.9). Several new consequences of the new results are discussed.

Definition 1.1. For $0 < \eta \leq 1$ and $\mu \geq 1$, a function $\mathsf{g}(\xi) \in \Sigma'$ given by (1.5) is said to be in the class $\mathfrak{P}^{\gamma}_{\Sigma'}(\eta, \mu, \varphi, \zeta, \nu)$ if the following conditions are satisfied:

$$1 + \frac{1}{\gamma}\left[(1-\eta)\left(\frac{F^{\nu}_{\zeta}\mathsf{g}(\xi)}{\xi}\right)^{\mu} + \eta\left(\frac{\xi(F^{\nu}_{\zeta}\mathsf{g}'(\xi))^{\mu}}{F^{\nu}_{\zeta}\mathsf{g}(\xi)}\right) - 1\right] \prec \varphi(\xi) \tag{1.11}$$

and

$$1 + \frac{1}{\gamma}\left[(1-\eta)\left(\frac{F^{\nu}_{\zeta}\upsilon(w)}{w}\right)^{\mu} + \eta\left(\frac{w(F^{\nu}_{\zeta}\upsilon'(w))^{\mu}}{F^{\nu}_{\zeta}\upsilon(w)}\right) - 1\right] \prec \varphi(w) \tag{1.12}$$

where $\xi, w \in \mathbb{U}^*$, $\gamma \in \mathbb{C}\backslash\{0\}$ and the function υ is given by (1.6).

By suitably specializing the parameter η, we state new subclass of meromorphic pseudo bi-univalent functions of complex order $\mathfrak{P}^{\gamma}_{\Sigma'}(\eta, \mu, \varphi, \zeta, \nu)$ as illustrated in the following Examples.

Example 1.2. For $\eta = 1$, a function $\mathsf{g} \in \Sigma'$ given by (1.5) is said to be in the class $\mathfrak{P}^{1}_{\Sigma'}(1, \mu, \varphi, \zeta, \nu) \equiv \mathfrak{P}_{\Sigma'}(\mu, \varphi, \zeta, \nu)$ if it satisfies the following conditions:

$$1+\frac{1}{\gamma}\left(\frac{\xi(F^{\nu}_{\zeta}\mathsf{g}'(\xi))^{\mu}}{F^{\nu}_{\zeta}\mathsf{g}(\xi)} - 1\right) \prec \varphi(\xi) \quad \text{and} \quad 1+\frac{1}{\gamma}\left(\frac{w(F^{\nu}_{\zeta}\upsilon'(w))^{\mu}}{F^{\nu}_{\zeta}\upsilon(w)} - 1\right) \prec \varphi(w)$$

where $\xi, w \in \mathbb{U}^*, \mu \geq 1$, $\gamma \in \mathbb{C}\backslash\{0\}$ and the function υ is given by (1.6).

Remark 1.3. We note that $\mathfrak{P}^{\gamma}_{\Sigma'}(1, 1, \varphi, \zeta, \nu) \equiv \mathfrak{S}^{\gamma}_{\Sigma'}(\varphi)$

Example 1.4. For $\eta = 1$ and $\gamma = 1$, a function $\mathsf{g} \in \Sigma'$ given by (1.5) is said to be in the class $\mathfrak{P}^{1}_{\Sigma'}(1, \mu, \varphi, \zeta, \nu) \equiv \mathfrak{P}_{\Sigma'}(\mu, \varphi, \zeta, \nu)$ if it satisfies the following conditions :

$$\frac{\xi(F^{\nu}_{\zeta}\mathsf{g}'(\xi))^{\mu}}{F^{\nu}_{\zeta}\mathsf{g}(\xi)} \prec \varphi(\xi) \quad \text{and} \quad \frac{w(F^{\nu}_{\zeta}\upsilon'(w))^{\mu}}{F^{\nu}_{\zeta}\upsilon(w)} \prec \phi(w)$$

where $\xi, w \in \mathbb{U}^*, \mu \geq 1$ and the function υ is given by (1.6).

Example 1.5. For $\eta = 0$ a function $\mathsf{g} \in \Sigma'$ given by (1.5) is said to be in the class $\mathfrak{P}^{\gamma}_{\Sigma'}(1, \mu, \varphi, \zeta, \nu) \equiv \mathfrak{R}^{\gamma}_{\Sigma'}(\mu, \varphi, \zeta, \nu)$ if it satisfies the following conditions:

$$1 + \frac{1}{\gamma}\left[\left(\frac{F^{\nu}_{\zeta}\mathsf{g}(\xi)}{\xi}\right)^{\mu} - 1\right] \prec \varphi(\xi) \quad \text{and} \quad 1 + \frac{1}{\gamma}\left[\left(\frac{F^{\nu}_{\zeta}\upsilon(w)}{w}\right)^{\mu} - 1\right] \prec \varphi(w)$$

where $\xi, w \in \mathbb{U}^*, \mu \geq 1$ and the function υ is given by (1.6).

2 Coefficient Estimates

In this section, we obtain the coefficient estimates $|b_0|$, $|b_1|$ and $|b_2|$ for $\mathfrak{P}^{\gamma}_{\Sigma'}(\eta,\mu,\phi,\zeta,\nu)$, a new subclass of meromorphic pseudo bi-univalent functions class Σ' of complex order $\gamma \in \mathbb{C}\backslash\{0\}$. We recall the following lemma, to prove our result.

Lemma 2.1. *[11] If $\Phi \in \mathfrak{P}$, the class of all functions with $\Re(\Phi(\xi)) > 0$, $(\xi \in \mathbb{U})$ then*

$$|c_k| \leq 2, \text{ for each } k,$$

where

$$\Phi(\xi) = 1 + c_1\xi + c_2\xi^2 + \cdots \quad \textit{for } \xi \in \mathbb{U}.$$

Define the functions p and q in $\mathfrak{P}$ given by

$$p(\xi) = \frac{1+r(\xi)}{1-r(\xi)} = 1 + \frac{p_1}{\xi} + \frac{p_2}{\xi^2} + \cdots$$

and

$$q(w) = \frac{1+s(w)}{1-s(w)} = 1 + \frac{q_1}{w} + \frac{q_2}{w^2} + \cdots.$$

It follows that

$$r(\xi) = \frac{p(\xi)-1}{p(\xi)+1} = \frac{1}{2}\left[\frac{p_1}{\xi} + \left(p_2 - \frac{p_1^2}{2}\right)\frac{1}{\xi^2} + \cdots\right]$$

and

$$s(w) = \frac{q(w)-1}{q(w)+1} = \frac{1}{2}\left[\frac{q_1}{w} + \left(q_2 - \frac{q_1^2}{2}\right)\frac{1}{w^2} + \cdots\right].$$

Note that for the functions $p(\xi), q(\xi) \in \mathfrak{P}$, we have

$$|p_i| \leq 2 \text{ and } |q_i| \leq 2 \text{ for each } i.$$

Theorem 2.2. *Let g be given by (1.5) in the class $\mathfrak{P}^{\gamma}_{\Sigma'}(\eta,\mu,\phi,\zeta,\nu)$. Then*

$$|b_0| \leq \frac{|\gamma||\beta_1|}{|\mu-\mu\eta-\eta||(1-\zeta)^{\nu}|}, \tag{2.1}$$

(2.2)

$$|b_1| \leq \frac{|\gamma|}{2|\mu-\eta-2\mu\eta||(1-2\zeta)^{\nu}|}\Bigg(4|(\beta_1-\beta_2)^2| + 4|\beta_1^2| + 8|\beta_1(\beta_1-\beta_2)| + \frac{|\mu(\mu-1)(1-\eta)+2\eta|^2|\gamma|^2|\beta_1|^4}{|\mu-\mu\eta-\eta|^4}\Bigg)^{\frac{1}{2}}$$

and

(2.3)

$$|b_2| \leq \frac{|\gamma|}{2|\mu-\eta-3\mu\eta||(1-3\zeta)^{\nu}|}\Bigg(2|\beta_1|+4|\beta_2-\beta_1|+2|\beta_1-2\beta_2+\beta_3| + \frac{|\mu(\mu-1)(\mu-2)(1-\eta)-6\eta||\gamma|^2|\beta_1|^3}{3|\eta|^3}\Bigg)$$

where $\gamma \in \mathbb{C}\backslash\{0\}, 0<\eta\leq 1, \mu \geq 1$ *and* $\xi, w \in \mathbb{U}^*$.

Proof. It follows from (1.11) and (1.12) that

$$1+\frac{1}{\gamma}\left[(1-\eta)\left(\frac{F_{\zeta}^{\nu}\mathsf{g}(\xi)}{\xi}\right)^{\mu}+\eta\left(\frac{\xi(F_{\zeta}^{\nu}\mathsf{g}'(\xi))^{\mu}}{F_{\zeta}^{\nu}\mathsf{g}(\xi)}\right)-1\right]=\varphi(r(\xi)) \tag{2.4}$$

and

$$1+\frac{1}{\gamma}\left[(1-\eta)\left(\frac{F_{\zeta}^{\nu}\upsilon(w)}{w}\right)^{\mu}+\eta\left(\frac{w(F_{\zeta}^{\nu}\upsilon'(w))^{\mu}}{F_{\zeta}^{\nu}\upsilon(w)}\right)-1\right]=\varphi(s(w)). \tag{2.5}$$

Using (1.5), (1.6), (1.11) and (1.12), we have

$$\begin{aligned}&1+\frac{1}{\gamma}\left[(1-\eta)\left(\frac{F_{\zeta}^{\nu}\mathsf{g}(\xi)}{\xi}\right)^{\mu}+\eta\left(\frac{\xi(F_{\zeta}^{\nu}\mathsf{g}'(\xi))^{\mu}}{F_{\zeta}^{\nu}\mathsf{g}(\xi)}\right)-1\right]\\ &=1+\beta_1p_1\frac{1}{2\xi}+\left[\frac{1}{2}\beta_1\left(p_2-\frac{p_1^2}{2}\right)+\frac{1}{4}\beta_2p_1^2\right]\frac{1}{\xi^2}\\ &\quad+\left[\frac{\beta_1}{2}\left(p_3-p_1p_2+\frac{p_1^3}{4}\right)+\frac{\beta_2}{2}\left(p_1p_2-\frac{p_1^3}{2}\right)+\beta_3\frac{p_1^3}{8}\right]\frac{1}{\xi^3}\cdots\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}&1+\frac{1}{\gamma}\left[(1-\eta)\left(\frac{F_{\zeta}^{\nu}\upsilon(w)}{w}\right)^{\mu}+\eta\left(\frac{w(F_{\zeta}^{\nu}\upsilon'(w))^{\mu}}{F_{\zeta}^{\nu}\upsilon(w)}\right)-1\right]\\ &=1+\beta_1q_1\frac{1}{2w}+\left[\frac{1}{2}\beta_1\left(q_2-\frac{q_1^2}{2}\right)+\frac{1}{4}\beta_2q_1^2\right]\frac{1}{w^2}\\ &\quad+\left[\frac{\beta_1}{2}\left(q_3-q_1q_2+\frac{q_1^3}{4}\right)+\frac{\beta_2}{2}\left(q_1q_2-\frac{q_1^3}{2}\right)+\beta_3\frac{q_1^3}{8}\right]\frac{1}{w^3}\cdots\quad.\end{aligned} \tag{2.7}$$

Equating the coefficients of $\xi^{-1}, \xi^{-2}, \xi^{-3}, \cdots$ and $w^{-1}, w^{-2}, w^{-3}, \cdots$ in (2.6) and (2.7), we get

$$\frac{(\mu-\mu\eta-\eta)(1-\zeta)^{\nu}}{\gamma}b_0=\frac{1}{2}\beta_1p_1, \tag{2.8}$$

(2.9)
$$\frac{1}{2\gamma}\Big[\big(\mu(\mu-1)(1-\eta)+2\eta\big)(1-\zeta)^{2\nu}b_0^2+2(\mu-\eta-2\eta\mu)(1-2\zeta)^{\nu}b_1\Big]=\frac{1}{2}\beta_1\left(p_2-\frac{p_1^2}{2}\right)+\frac{1}{4}\beta_2p_1^2,$$

(2.10)
$$\frac{1}{6\gamma}\Big[\big(\mu(\mu-1)(\mu-2)(1-\eta)-6\eta\big)(1-\zeta)^{3\nu}b_0^3+6\big(\mu(\mu-1)(1-\eta)+2\eta+\eta\mu\big)(1-\zeta)^{\nu}(1-2\zeta)^{\nu}b_0b_1$$
$$+6(\mu-\eta-3\eta\mu)(1-3\zeta)^{\nu}b_2\Big]=\left[\frac{\beta_1}{2}\left(p_3-p_1p_2+\frac{p_1^3}{4}\right)+\frac{\beta_2}{2}\left(p_1p_2-\frac{p_1^3}{2}\right)+\beta_3\frac{p_1^3}{8}\right],$$

$$\frac{-(\mu-\mu\eta-\eta)}{\gamma}(1-\zeta)^{\nu}b_0=\frac{1}{2}\beta_1q_1, \tag{2.11}$$

(2.12)
$$\frac{1}{2\gamma}\Big[\big(\mu(\mu-1)(1-\eta)+2\eta\big)(1-\zeta)^{2\nu}b_0^2+2(\eta-\mu+2\eta\mu)(1-2\zeta)^{\nu}b_1\Big]=\frac{1}{2}\beta_1\left(q_2-\frac{q_1^2}{2}\right)+\frac{1}{4}\beta_2q_1^2$$

and

$$\frac{1}{6\gamma}\big[\big(6\eta-\mu(\mu-1)(\mu-2)(1-\eta)(1-\zeta)^{3\nu}\big)b_0^3 \tag{2.13}$$
$$+6\big(\mu(\mu-1)(1-\eta)-\mu(1-\eta)+3\eta+3\eta\mu\big)(1-\zeta)^{\nu}(1-2\zeta)^{\nu}b_0b_1+6(\eta-\mu+3\eta\mu)(1-3\zeta)^{\nu}b_2\big]$$
$$=\left[\frac{\beta_1}{2}\left(q_3-q_1q_2+\frac{q_1^3}{4}\right)+\frac{\beta_2}{2}\left(q_1q_2-\frac{q_1^3}{2}\right)+\beta_3\frac{q_1^3}{8}\right].$$

From (2.8) and (2.11), we get

$$p_1=-q_1 \tag{2.14}$$

and

$$b_0^2=\frac{\gamma^2\beta_1^2}{8(\mu-\mu\eta-\eta)^2(1-\zeta)^{2\nu}}(p_1^2+q_1^2). \tag{2.15}$$

Applying Lemma 2.1 for the coefficients p_1 and q_1, we have

$$|b_0|\leq\frac{|\gamma||\beta_1|}{|\mu-\mu\eta-\eta||(1-\zeta)^{\nu}|}.$$

In order to find the bound on $|b_1|$ from (2.9), (2.12), (2.14) and (2.15), we obtain

(2.16)
$$2(\mu-\eta-2\eta\mu)^2(1-2\zeta)^{2\nu}\ \frac{b_1^2}{\gamma^2}+[\mu(\mu-1)(1-\eta)+2\eta]^2(1-\zeta)^{4\nu}\frac{b_0^4}{2\gamma^2}$$
$$=(\beta_1-\beta_2)^2\ \frac{p_1^4}{8}+\frac{\beta_1^2}{4}(p_2^2+q_2^2)+\beta_1(\beta_2-\beta_1)\frac{(p_1^2p_2+q_1^2q_2)}{4}.$$

Using (2.15) and Lemma 2.1 again for the coefficients $p_1,\ p_2$ and q_2, we get

$$|b_1|^2\leq\frac{|\gamma^2|}{4|\mu-\eta-2\eta\mu|^2|(1-2\zeta)^{2\nu}|}\times$$
$$\left(4|(\beta_1-\beta_2)^2|+4|\beta_1|^2+8|\beta_1(\beta_1-\beta_2)|+\frac{|\mu(\mu-1)(1-\eta)+2\eta|^2|\gamma|^2|\beta_1|^4}{|\mu-\mu\eta-\eta|^4}\right).$$

That is,

$$|b_1|\leq\ \frac{|\gamma|}{2|\mu-\eta-2\eta\mu||(1-2\zeta)^{\nu}|}\times$$
$$\sqrt{4|(\beta_1-\beta_2)^2|+4|\beta_1|^2+8|\beta_1(\beta_1-\beta_2)|+\frac{|\mu(\mu-1)(1-\eta)+2\eta|^2|\gamma|^2|\beta_1|^4}{|\mu-\mu\eta-\eta|^4}}.$$

To find the estimate $|b_2|$, consider the sum of (2.10) and (2.13) with $p_1=-q_1$, we have

(2.17)
$$\frac{1}{\gamma}b_0b_1=\frac{\beta_1[p_3+q_3]+(\beta_2-B_1)p_1[p_2-q_2]}{2[2\mu(\mu-1)(1-\eta)-(1-\eta)\mu+5\eta+4\eta\mu](1-\zeta)^{\nu}(1-2\zeta)^{\nu}}.$$

Subtracting (2.13) from (2.10) and using $p_1=-q_1$ we have

$$(2.18)\quad 2(\mu-\eta-3\eta\mu)(1-3\zeta)^{\nu}\frac{b_2}{\gamma}$$
$$=-(\mu-\eta-3\mu\eta)(1-\zeta)^{\nu}(1-2\zeta)^{\nu}\frac{b_0b_1}{\gamma}-[\mu(\mu-1)(\mu-2)(1-\eta)-6\eta](1-\zeta)^{3\nu}\frac{b_0^3}{3\gamma}+\frac{\beta_1}{2}(p_3-q_3)$$
$$+\frac{\beta_2-\beta_1}{2}(p_2+q_2)p_1+\frac{\beta_1-2\beta_2+\beta_3}{4}p_1^3.$$

Substituting for $\frac{b_0 b_1}{\gamma}$ and $\frac{b_0^3}{\gamma}$ in (2.18), further computation yields,

$$\begin{aligned}\frac{b_2}{\gamma} &= \frac{-\beta_1}{2(\mu-\eta-3\eta\mu)(1-3\zeta)^\nu}\left(\frac{\mu-3\eta-4\eta\mu-\mu(\mu-1)(1-\eta)}{2\mu(\mu-1)(1-\eta)-\mu+5\eta+5\eta\mu}p_3\right.\\ &\qquad\left.+\frac{2\eta+\eta\mu+\mu(\mu-1)(1-\eta)}{2\mu(\mu-1)(1-\eta)-\mu+5\eta+5\eta\mu}q_3\right)\\ &\quad-\frac{(\beta_2-\beta_1)p_1}{2(\mu-\eta-3\eta\mu)(1-3\zeta)^\nu}\left(\frac{\mu-3\eta-4\eta\mu-\mu(\mu-1)(1-\eta)}{2\mu(\mu-1)(1-\eta)-\mu+5\eta+5\eta\mu}p_2\right.\\ &\qquad\left.-\frac{2\eta+\eta\mu+\mu(\mu-1)(1-\eta)}{2\mu(\mu-1)(1-\eta)-\mu+5\eta+5\eta\mu}q_2\right)\\ &\quad+\frac{\beta_1-2\beta_2+\beta_3}{8(\mu-\eta-3\eta\mu)(1-3\zeta)^\nu}p_1^3-\frac{(\mu(\mu-1)(\mu-2)(1-\eta)-6\eta)\gamma^2\ \beta_1^3}{48(\mu-\eta-3\eta\mu)(1-3\zeta)^\nu\eta^3}p_1^3.\end{aligned}$$

Applying Lemma 2.1 in the above equation yields,

$$\begin{aligned}|b_2| \le \frac{|\gamma|}{2|\mu-\eta-3\eta\mu||(1-3\zeta)^\nu|}\times &\\ \Big(2|\beta_1|+4|\beta_2-\beta_1|+2|\beta_1-2\beta_2+\beta_3| &\\ +\frac{|\mu(\mu-1)(\mu-2)(1-\eta)-6\eta||\gamma|^2|\beta_1|^3}{3|\eta|^3}\Big).\end{aligned} \tag{2.19}$$

□

By taking $\eta = 1$, we state the following results.

Theorem 2.3. *Let g be given by (1.5) in the class $\mathfrak{P}_{\Sigma'}^{\gamma}(\mu,\varphi,\zeta,\nu)$. Then*

$$|b_0| \le \frac{|\gamma|\ |\beta_1|}{|(1-\zeta)^\nu|}, \tag{2.20}$$

(2.21)

$$|b_1| \le \frac{|\gamma|}{|1+\mu||(1-2\zeta)^\nu|}\sqrt{|(\beta_1-\beta_2)^2|+|\beta_1^2|+2|\beta_1(\beta_1-\beta_2)|+|\gamma|^2\ |\beta_1|^4}$$

and

(2.22)

$$|b_2| \le \frac{|\gamma|}{|1+2\mu||(1-3\zeta)^\nu|}\left(|\beta_1|+2|\beta_2-\beta_1|+|\beta_1-2\beta_2+\beta_3|+|\gamma|^2\ |\beta_1|^3\right)$$

where $\gamma \in \mathbb{C}\backslash\{0\}, \mu \ge 1$ and $\xi, w \in \mathbb{U}^*$.

By taking $\eta = 1$ and $\gamma = 1$, we state the following results.

Theorem 2.4. *Let* $\mathtt{g}$ *be given by (1.5) in the class* $\mathfrak{P}_{\Sigma'}(\mu, \varphi, \zeta, \nu)$. *Then*

$$|b_0| \leq \frac{|\beta_1|}{|(1-\zeta)^\nu|},$$

$$|b_1| \leq \frac{1}{|1+\mu||(1-2\zeta)^\nu|} \sqrt{|(\beta_1 - \beta_2)^2| + |\beta_1^2| + 2|\beta_1(\beta_1 - \beta_2)| + |\beta_1|^4}$$

and

$$|b_2| \leq \frac{1}{|1+2\mu||(1-3\zeta)^\nu|} \left(|\beta_1| + 2|\beta_2 - \beta_1| + |\beta_1 - 2\beta_2 + \beta_3| + \ |\beta_1|^3\right)$$

where $\mu \geq 1,\ \xi, w \in \mathbb{U}^*$.

3 Corollaries and concluding Remarks

For functions $\mathtt{g}$ be given by (1.5) and $\mathtt{g} \in \mathfrak{P}^\gamma_{\Sigma'}\left(\eta, \mu, \left(\frac{1+\xi}{1-\xi}\right)^\delta, \zeta, \nu\right) \equiv \mathfrak{P}^\gamma_{\Sigma'}(\eta, \mu, \delta, \zeta, \nu)$ by setting $\beta_1 = 2\delta$, $\beta_2 = 2\delta^2$ and $\beta_3 = \frac{4\delta^2+2\delta}{3}$ and similarly, for $\mathtt{g} \in \mathfrak{P}^\gamma_{\Sigma'}\left(\eta, \mu, \frac{1+(1-2\omega)\xi}{1-\xi}, \zeta, \nu\right) \equiv \mathfrak{P}^\gamma_{\Sigma'}(\eta, \mu, \omega, \zeta, \nu)$ by setting $\beta_1 = \beta_2 = \beta_3 = 2(1-\omega)$, analogously, we can derive the results of Theorems 2.2, 2.3 and 2.4.

Corollary 3.1. *Let* $\mathtt{g}$ *be given by (1.5) in the class* $\mathfrak{P}^\gamma_{\Sigma'}(\eta, \mu, \delta, \zeta, \nu)$. *Then*

$$|b_0| \leq \frac{2|\gamma|\delta}{|\mu - \mu\eta - \eta||(1-\zeta)^\nu|}, \tag{3.1}$$

(3.2)

$$|b_1| \leq \frac{2|\gamma|\delta}{|\mu - \eta - 2\eta\mu||(1-2\zeta)^\nu|} \sqrt{(\delta-2)^2 + \frac{|\mu(\mu-1)(1-\eta) + 2\eta|^2|\gamma^2|}{|\mu - \mu\eta - \eta|^4}\delta^2}$$

and

$$|b_2| \leq \frac{2|\gamma|\delta}{|\mu - \eta - 3\eta\mu||(1-3\zeta)^\nu|} \left(3 - 2\delta + \left(\frac{4 - 6\delta + 2\delta^2}{3}\right) + \frac{2|\gamma|^2\delta^2|\mu(\mu-1)(\mu-2)(1-\eta) - 6\eta|}{3|\eta|^3}\right) \tag{3.3}$$

where $\gamma \in \mathbb{C}\backslash\{0\}, 0 < \eta \leq 1, \mu \geq 1$ *and* $\xi, w \in \mathbb{U}^*$.

Corollary 3.2. *Let* g *be given by (1.5) in the class* $\mathfrak{P}^{\gamma}_{\Sigma'}(\eta, \mu, \omega, \zeta, \nu)$. *Then*

$$|b_0| \leq \frac{2|\gamma|(1-\omega)}{|\mu - \mu\eta - \eta||(1-\zeta)^{\nu}|}, \tag{3.4}$$

(3.5)

$$|b_1| \leq \frac{2|\gamma|(1-\omega)}{|\mu - \eta - 2\eta\mu||(1-2\zeta)^{\nu}|}\sqrt{1 + \frac{|\mu(\mu-1)(1-\eta) + 2\eta|^2|\gamma^2|}{|\mu - \mu\eta - \eta|^4}(1-\omega)^2}$$

and

(3.6)

$$|b_2| \leq \frac{2|\gamma|(1-\omega)}{|\mu - \eta - 3\eta\mu||(1-3\zeta)^{\nu}|}\left(1 + \frac{2|\gamma|^2(1-\omega)^2|\mu(\mu-1)(\mu-2)(1-\eta) - 6\eta|}{3|\eta|^3}\right)$$

where $\gamma \in \mathbb{C}\backslash\{0\}, 0 < \eta \leq 1, \mu \geq 1$ *and* $\xi, w \in \mathbb{U}^*$.

Concluding Remarks: We remark that, when $\eta = 1$ and $\mu = 1$, we can obtain the coefficient estimates b_0, b_1 and b_2 for $\mathfrak{S}^{\gamma}_{\Sigma'}(\varphi, \zeta, \nu)$, leads to the results discussed in Theorem 2.3 of [9]. Also, we can obtain the initial coefficient estimates for function g given by (1.5) in the subclass $\mathfrak{S}^{\gamma}_{\Sigma'}(\varphi, \zeta, \nu)$ by taking $\varphi(\xi)$ given in (1.3) and (1.4) respectively.

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