Group Rings of Finite Gorenstein Homological Dimensions

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Abstract. Let K be a field and let G be a group. In the study of group ring $K[G]$, there is the well-known Serre's theorem which consider the finiteness of global dimension of group ring. In the present paper, we investigate when the group ring $K[G]$ has finite Gorenstein global dimension. It is shown that the Gorenstein global dimension of $K[G]$ shares many properties with the global dimension of $K[G]$. Finally, we give some analogous versions of the Serre's Theorem for Gorenstein global dimension.

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1. Introduction

Let R ba a commutative ring and let G be a multiplicative group (finite or infinite). The group ring $R[G]$ is an associative R-algebra with the elements of G as a basis and with multiplication defined distributively using the group multiplication in G. This subject is a meeting place of group theory and ring theory. The study of group rings involves the theory of field, linear algebra and algebraic number theory and so on. Representation and homological properties of group rings have been extensively studied (cf. [1], [7-9], [14] and [16]). Among others, Connell in [7] considered necessary and sufficient conditions on R and G so that $R[G]$ have some ring theoretic properties such as being artinian, regular, self-injective and semiprime. There is the well-known Serre's theorem (see [14]), i.e., let K be a field of characteristic p and G be a group, and let H be a subgroup of G of finite index. If G has no elements of order p, then the global dimension of $K[H]$ is equal to the global dimension of $K[G]$. The Serre's Theorem also has the other versions. The cohomology theory of groups arose from both topological and algebraic sources, which offered possibilities for a great deal of intersection between topology and algebra. Let $R = \mathbb{Z}$ and Γ be a group. the cohomological dimension, denoted $\mathrm{cd}\Gamma$, which is defined as the projective dimension of the trivial $\mathbb{Z}\Gamma$-module $\mathbb{Z}$. It was shown that if Γ is a torsion-free group and Γ' is a subgroup of finite index, then $\mathrm{cd}\Gamma' = \mathrm{cd}\Gamma$ (see [4, Theorem 8.3.1]). A deep result was that a non-trivial group Γ is free if and only if its cohomological dimension is 1 (see [16]).

In classical homological algebra, the projective, injective and flat dimensions of modules play an important and fundamental role. Subsequently, Auslander [5] introduced G-dimensions for finitely generated modules over commutative Noetherian rings. As an extension of the G-dimension, Enochs and Jenda in [10] defined the Gorenstein projective dimensions of modules (need not be finitely generated) over a general ring, which is a refinement of the usual projective dimension. Furthermore, the Gorenstein global dimension of a ring was defined (see [2]). In [8], Emmanouil proved that Gorenstein global dimension of a ring R is finite if and only if any R-module M admits a complete projective resolution if and only if $\mathrm{spli}(R) = \mathrm{fin.dim}(R) = \mathrm{silp}(R)$ is finite.

In this paper, we mainly consider the condition so that the Gorenstein global dimensions of group ring $K[G]$ is finite using the facts above. It is shown that the Gorenstein global dimensions of $K[G]$ is equal to the Gorenstein projective dimension of a principal $K[G]$-module. Moreover, we obtain some results which generalize many properties of global dimensions of group ring $K[G]$. More precisely, we prove that

Theorem 1.1. *(Theorem 3.16) Let K be a field and let H be a normal subgroup of G. If* $\mathrm{Ggl.dim}(K[H])$ *and* $\mathrm{Ggl.dim}(K[G/H])$ *are finite, then so is* $\mathrm{Ggl.dim}(K[G])$*, and we have*

$$\mathrm{Ggl.dim}(K[G]) \leq \mathrm{Ggl.dim}(K[H]) + \mathrm{Ggl.dim}(K[G/H]).$$

Theorem 1.2. *(Theorem 3.19) Let K be a field and let H be a subgroup of G of finite index. If* $\mathrm{Ggl.dim}(K[G])$ *is finite, then* $\mathrm{Ggl.dim}(K[H]) = \mathrm{Ggl.dim}(K[G])$.

2. Preliminaries

In this section, we set notations and discuss basic facts which will be useful in the sequel. Unless otherwise stated, R denotes an associative ring with identity and all modules are right R-modules. For an R-module M, $\mathrm{pd}_R(M)$ and $\mathrm{Gpd}_R(M)$ denote the projective dimension and Gorenstein projective dimension of M, respectively. We write $\mathrm{gl.dim}(R)$ and $\mathrm{Ggl.dim}(R)$ for the global dimension and Gorenstein global dimension of a ring R, respectively. $R-\mathrm{Mod}$ denotes the category of R-modules. For unexplained concepts and notations, we refer the reader to [3], [14], and [15].

Module Structure over Group Rings

(1) Let K be a field and G be a group, and let V and W be $K[G]$-modules. Then $V \otimes_K W$ becomes a $K[G]$-module under the diagonal action $(v \otimes w)g = (vg) \otimes (wg)$ for all $v \in V$, $w \in W$ and $g \in G$. It is trivial that $V \otimes_K W \cong W \otimes_K V$.
(2) The principal $K[G]$-module V_0 is a one-dimensional K-vector space in which $vg = v$ for all $v \in V_0$ and $g \in G$. For example, K with trivial G-action is a principal $K[G]$-module.
(3) Let H be a subgroup of G. Following [13], for a $K[H]$-module M, we define the induced module $M \uparrow_H^G := M \otimes_{K[H]} K[G]$ with $K[G]$ acting on the right side and the coinduced module $\mathrm{Hom}_{K[H]}(K[G], M)$. Moreover, every $K[G]$-module N can be viewed as a $K[H]$-module. We denote this restricted module by $N \downarrow_H^G$ (Sometime we omit the symbol $\downarrow_H^G$ if not confuse). Since $K[G]$ is a left and right free $K[H]$-module, the induced functor and restricted functor are exact, and preserve projective modules. The coinduced functor preserves injective modules.

Gorenstein Dimensions

A complete projective resolution is an exact sequence of projective R-modules

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots,$$

which remains exact after applying the functor $\mathrm{Hom}_R(-, P)$, for any projective R-module P. An R-module M is called Gorenstein projective [11] if it is a syzygy of a complete projective resolution, i.e., $M = \mathrm{Ker}(P^0 \to P^1)$. The Gorenstein projective dimension $\mathrm{Gpd}_R(M)$ is at most n if there is an exact sequence

$$0 \longrightarrow G_n \longrightarrow G_{n-1} \longrightarrow \cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow M \longrightarrow 0$$

with every G_i Gorenstein projective. It is clear that $\mathrm{Gpd}_R(M) \leq \mathrm{pd}_R(M)$ and $\mathrm{Gpd}_R(M) = \mathrm{pd}_R(M)$ provided $\mathrm{pd}_R(M)$ is finite. Dually, we have the concepts of Gorenstein injective module and Gorenstein injective dimension ($\mathrm{Gid}_R(M)$).

Bennis and Mahdou in [2] showed that

$$\sup\{\mathrm{Gpd}_R(M) \mid M \in R-\mathrm{Mod}\} = \sup\{\mathrm{Gid}_R(M) \mid M \in R-\mathrm{Mod}\}.$$

The common value is called the Gorenstein global dimension of R and denoted by $\mathrm{Ggl.dim}(R)$. By [2, Proposition 2.6], $\mathrm{Ggl.dim}(R) = 0$ if and only if R is quasi-Frobenius (i.e., it is left and right Noetherian and both left and right self-injective). So the Gorenstein global dimension measures how far away a ring R is from being quasi-Frobenius.

3. Main Results

The following lemma can be seen in [14].

Lemma 3.1. *Let K be a field and G be a group, and let M be a $K[G]$-module. If F is a free $K[G]$-module, then so is $M \otimes_K F$. Moreover, if P is a projective $K[G]$-module, then so is $M \otimes_K P$.*

By Lemma 3.1, we have immediately

Lemma 3.2. *If N is a $K[G]$-module, then $\mathrm{pd}_{K[G]}(M \otimes_K N) \leq \mathrm{pd}_{K[G]}(N)$ for any $K[G]$-module M.*

Lemma 3.3. *Let M be a $K[G]$-module. If N is a Gorenstein projective $K[G]$-module, then so is $M \otimes_K N$.*

Proof. If N is Gorenstein projective, then there is a complete projective resolution

$$P^\bullet := \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots,$$

such that $N = \mathrm{Ker}(P^0 \longrightarrow P^1)$. Since K is a field, every K-module is flat. Then we have the following exact sequence of $K[G]$-modules

$$M \otimes_K P^\bullet := \cdots \longrightarrow M \otimes_K P_1 \longrightarrow M \otimes_K P_0 \longrightarrow M \otimes_K P^0 \longrightarrow M \otimes_K P^1 \longrightarrow \cdots,$$

and $M \otimes_K N = \mathrm{Ker}(M \otimes_K P^0 \to M \otimes_K P^1)$. By Lemma 3.1, all $M \otimes_K P_i$ and all $M \otimes_K P^i$ are projective. Now it is enough to show that $\mathrm{Hom}_{K[G]}(M \otimes_K P^\bullet, Q)$ is exact for every projective $K[G]$-module Q. By the adjoint isomorphic theorem,

$$\mathrm{Hom}_{K[G]}(M \otimes_K P^\bullet, Q) \cong \mathrm{Hom}_K(P^\bullet, \mathrm{Hom}_{K[G]}(M, Q)).$$

Noting that $\mathrm{Hom}_{K[G]}(M, Q)$ is a projective K-module because K is a field, the right complex is exact by [17, Proposition 2.3 (2)], and hence the left complex is exact. □

The following result will be used frequently in the sequel.

Theorem 3.4. *Let K be a field and let G be a group. If V_0 is a principal $K[G]$-module, then*

$$\mathrm{Ggl.dim}(K[G]) = \mathrm{Gpd}_{K[G]}(V_0).$$

Proof. It is trivial that $\mathrm{Ggl.dim}(K[G]) \geq \mathrm{Gpd}_{K[G]}(V_0)$. Now suppose that $\mathrm{Gpd}_{K[G]}(V_0) = n < \infty$. Then there is an exact sequence of $K[G]$-modules

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \longrightarrow V_0 \longrightarrow 0,$$

where all P_i are Gorenstein projective $K[G]$-modules. For any $K[G]$-module M, it is flat as a K-module because K is a field. So, it yields an exact sequence of $K[G]$-modules

$$0 \longrightarrow M \otimes_K P_n \longrightarrow \cdots \longrightarrow M \otimes_K P_0 \longrightarrow M \otimes_K V_0 \longrightarrow 0.$$

By Lemma 3.3, $M \otimes_K P_i$ is Gorenstein projective for all i. Thus, $\mathrm{Gpd}_{K[G]}(M \otimes_K V_0) \leq n$. It is easy to verify that $M \otimes_K V_0 \cong M$ as $K[G]$-modules. Then $\mathrm{Gpd}_{K[G]}(M) \leq n$, and hence $\mathrm{Ggl.dim}(K[G]) \leq n = \mathrm{Gpd}_{K[G]}(V_0)$. □

For a group ring $K[G]$, the ring homomorphism $\epsilon : K[G] \to K$, $\sum r_g g \to \sum r_g$, is called the augmentation mapping of $K[G]$ and its kernel, denoted by $\Delta(K[G])$, is

$$\Delta(K[G]) = \{\sum_{g \in G} a_g(g-1) : 1 \neq g, a_g \in K\}.$$

Proposition 3.5. *Let K be a field and let G be a group.*

1. *If H is a subgroup of G, then* $\mathrm{Ggl.dim}(K[H]) \leq \mathrm{Ggl.dim}(K[G])$.
2. $\mathrm{Ggl.dim}(K[G]) = 0$ *if and only if G is a finite group.*
3. *If $G = \langle g_i \mid i \in I \rangle$ is a nonidentity free group, then $\Delta(K[G])$ is a free $K[G]$-module. Furthermore,* $\mathrm{Ggl.dim}(K[G]) = 1$.

Proof. (1) It follows from [17, Theorem 2.4].

(2) If K is a field and G is a finite group, then $K[G]$ is quasi-Frobenius, and so $\mathrm{Ggl.dim}(K[G]) = 0$. Conversely, if $K[G]$ is quasi-Frobenius, then $K[G]$ is Artinian, and hence G is finite by [7, Theorem 1].

(3) The first part of the proof is due to a contribution in [14].

Let $G = \Pi_i^*(g_i)$ be the free product of infinite cyclic group (g_i). Then, every element $g \in G$ can be written uniquely as a finite product of the form $g = g_{i_1}^{n_1} g_{i_2}^{n_2} \cdots g_{i_s}^{n_s}$ with $n_j \neq 0$ and $i_j \neq i_{j+1}$. We call $|n_1| + |n_2| + \cdots + |n_s|$ the length of g.

Noting that $\Delta(K[G])$ is generated by $\{g_i - 1 \mid g_i \in G - \{1\}\}$. It is only to show $\{g_i - 1 \mid g_i \in G - \{1\}\}$ are independent, i.e., $\Sigma(g_i - 1)\alpha_i = 0$ implies that $\alpha_i = 0$ for all i. Because of the uniqueness of $g = g_{i_1}^{n_1} g_{i_2}^{n_2} \cdots g_{i_s}^{n_s}$, the map $\sigma : G \to G$ given by $\sigma(g_i) = g_i^2$ for all i, defines an isomorphism of G onto its subgroup $H = (g_i^2 \mid i \in I)$. Moreover, σ can be extended to a monomorphism $\sigma^* : K[G] \to K[H]$. So, if $\Sigma(g_i - 1)\alpha_i = 0$, then

$$\Sigma(g_i^2 - 1)\sigma^*(\alpha_i) = \sigma^*(\Sigma(g_i - 1)\alpha_i) = \sigma^*(0) = 0.$$

Since σ is monomorphism, it is enough to prove that $\Sigma(g_i^2 - 1)\beta_i = 0$ implies that $\beta_i = 0$ for all i. Suppose that $\Sigma(g_i^2 - 1)\beta_i = 0$. We set $\gamma_i = g_i\beta_i$, then $\Sigma(g_i - g_i^{-1})\gamma_i = 0$. If some $\gamma_i \neq 0$, then there exists $g \in \mathrm{supp}(\gamma_i)$ with the maximal length n. For convenience, we set $g \in \mathrm{supp}(\gamma_1)$. Then, in at least one of $g_1 g$ and $g_1^{-1} g$ no cancellation occurs, and hence at least one of these two group elements has length $n+1$, say $g_1^{\delta} g$. Thus, $g_1^{\delta} g$ occurs in the support of $\gamma = \Sigma(g_i - g_i^{-1})\gamma_i$. By the definition of n, the only possible group elements in $\mathrm{supp}(\gamma)$ of length $n+1$ must have the reduced form $g_i^{\pm 1} h$ with $h \in \mathrm{supp}(\gamma_i)$. So, $g_i^{\pm 1} h = g_1^{\delta} g$ yields $i = 1, \pm 1 = \delta$ and $h = g$, and hence $g_1^{\delta} g$ occurs precisely once. Thus $g_1^{\delta} g \in \mathrm{supp}(\gamma)$ and $\gamma \neq 0$, a contradiction. Therefore, $\{g_i - 1 \mid g_i \in G - \{1\}\}$ is a free $K[G]$-basis of $\Delta(K[G])$.

In addition, we have the following exact sequence of $K[G]$-modules

$$0 \longrightarrow \Delta(K[G]) \longrightarrow K[G] \longrightarrow K \longrightarrow 0.$$

As $\Delta(K[G])$ is free, $\mathrm{Gpd}_{K[G]}(K) \leq 1$. By Theorem 3.4, $\mathrm{Ggl.dim}(K[G]) = \mathrm{Gpd}_{K[G]}(K) \leq 1$. Since G is infinite, $\mathrm{Ggl.dim}(K[G]) = 1$ in terms of (2) above. □

Example 3.6. Let $(G_i)_I$ be an arbitrary family of finite normal subgroups of G. Then $S = \bigoplus_I K[G/G_i]$ is a Gorenstein projective $K[G]$-module. In fact, since G_i is finite, for any principal $K[G_i]$-module V_0, in view of Proposition 3.5(2), V_0 is Gorenstein projective. Then, $V_0 \uparrow_{G_i}^{G} \cong K[G/G_i]$ is a Gorenstein projective $K[G]$-module. Therefore, $S = \bigoplus_I K[G/G_i]$ is Gorenstein projective because the class of Gorenstein projective modules is closed under arbitrary direct sums.

Let p be a prime. A group G is call a p'-group provided that G has no element of order p. Let $\mathcal{GP}$ be the class of Gorenstein projective R-modules. $\mathcal{GP}^{\perp}$ denotes the orthogonal class of $\mathcal{GP}$, i.e., the class of modules such that $\mathrm{Ext}_R^1(P, -) = 0$ for all $P \in \mathcal{GP}$. Obviously, the modules of finite projective dimensions and injective modules are contained in $\mathcal{GP}^{\perp}$. Let P be a module of finite projective dimension but not be injective and let I be an injective module of infinite projective dimension. Then $P \oplus I \in \mathcal{GP}^{\perp}$ while $P \oplus I$ is neither finite projective dimension nor an injective module.

Lemma 3.7. *Let K be a field and let H be a subgroup of G. If any $K[G]$-module $M \in \mathcal{GP}^{\perp}$, then $N \in \mathcal{GP}^{\perp}$ for any $K[H]$-module N.*

Proof. For any Gorenstein projective $K[H]$-module P, in view of Eckmann-Shapiro Lemma (see [3, Corollary 2.8.4]), we have

$$\mathrm{Ext}^1_{K[H]}(P, N \uparrow_H^G \downarrow_H^G) \cong \mathrm{Ext}^1_{K[G]}(P \uparrow_H^G, N \uparrow_H^G).$$

Noting that $P\uparrow_H^G$ is also Gorenstein projective as $K[G]$-module, $\mathrm{Ext}^1_{K[G]}(P\uparrow_H^G, N\uparrow_H^G)=0$, and hence $\mathrm{Ext}^1_{K[H]}(P, N\uparrow_H^G\downarrow_H^G)=0$. By [15, Theorem 7.14], $\mathrm{Ext}^1_{K[H]}(P,N)=0$ because N is a direct summand of $N\uparrow_H^G\downarrow_H^G$. $\square$

Proposition 3.8. *Let K be a field of characteristic p and let G be a group. If* $\mathrm{Ggl.dim}(K[G])<\infty$ *and any $K[G]$-module $M\in\mathcal{GP}^{\perp}$, then G is a p'-group.*

Proof. Suppose that $H=(x)$ is a cyclic subgroup of order p. Set $R:=K[H]$ and let

$$a=1-x,\ b=1+x+\cdots+x^{p-1}\in R.$$

By [12, Lemma 6.2], $r_R(a)=bR$ and $r_R(b)=aR$. Thus, we have the exact sequences of R-modules

$$0\longrightarrow bR\longrightarrow R\longrightarrow aR\longrightarrow 0,\ \text{and}\ 0\longrightarrow aR\longrightarrow R\longrightarrow bR\longrightarrow 0.$$

By Proposition 3.5(1), $\mathrm{Ggl.dim}(R)<\infty$, and hence let $\mathrm{Gpd}_R(aR)=n<\infty$. By [11, Theorem 2.10], aR admits an exact sequence of R-modules

$$0\longrightarrow Q\longrightarrow P\longrightarrow aR\longrightarrow 0,$$

where $P\twoheadrightarrow aR$ is a Gorenstein projective precover and $\mathrm{pd}_R(Q)=n-1$ (If $n=0$, then $Q=0$). In addition, we have the following commutative diagram

$$\begin{array}{ccccccc}
0\longrightarrow & bR & \xrightarrow{i} & R & \xrightarrow{\pi} & aR & \longrightarrow 0\\
 & \downarrow g & & \downarrow f & & \| & \\
0\longrightarrow & Q & \xrightarrow{i'} & P & \xrightarrow{\pi'} & aR & \longrightarrow 0.
\end{array}$$

The diagram gives rise to a sequence

$$0\longrightarrow bR\xrightarrow{\alpha} R\oplus Q\xrightarrow{\beta} P\longrightarrow 0,$$

where the map $\alpha: bR\to R\oplus Q$ is given by $\alpha(m)=(i(m),-g(m))$, and the map $\beta: R\oplus Q\to P$ is given by $\beta(r,q)=f(r)+i'(q)$. Now we show that the sequence is exact.

(1) The map α is injective because i is so.

(2) For any $p_0\in P$, there exists an element $r_0\in R$ such that $\pi(r_0)=\pi'(p_0)$. Thus

$$\pi'(p_0-f(r_0))=\pi'(p_0)-\pi' f(r_0)=\pi(r_0)-\pi(r_0)=0.$$

Hence, $p_0-f(r_0)\in\mathrm{Ker}\pi'=\mathrm{Im}i'$, and $p_0=f(r_0)+i'(q_0)=\beta(r_0,q_0)$ for some $q_0\in Q$. So β is surjective.

(3) For any $m_0\in bR$, $\beta\alpha(m_0)=\beta(i(m_0),-g(m_0))=fi(m_0)-i'g(m_0)=0$, and so $\mathrm{Im}\alpha\subseteq\mathrm{Ker}\beta$. On the other hand, for any $(r,q)\in\mathrm{Ker}\beta$, $f(r)+i'(q)=0$. It implies $\pi(r)=\pi' f(r)=\pi'(f(r)+i'(q))=\pi'(0)=0$, and hence $r\in\mathrm{Ker}\pi=\mathrm{Im}i$, i.e., there exists an element n of bR such that $r=i(n)$. By the commutative diagram, $fi(n)=i'g(n)$. Then $i'g(n)=f(r)=-i'(q)$, and so $i'(g(n)+q)=0$. By the injectivity of i', $q=-g(n)$. It implies that $\mathrm{Ker}\beta\subseteq\mathrm{Im}\alpha$.

By hypothesis and Lemma 3.7, $\mathrm{Ext}^1_R(P,bR)=0$, and hence $bR\oplus P\cong R\oplus Q$. By [15, Exercise 9.7], $\mathrm{pd}_R(bR)\le n-1$, and so $R\simeq aR\oplus bR$ in terms of [14, Lemma 10.3.3]. But $b\neq 0$ annihilates both aR and bR, a contradiction. Therefore, G is a p'-group. $\square$

Remark 3.9. (1) The condition "$M\in\mathcal{GP}^{\perp}$ for any $K[H]$-module M" in the proposition above can not be omitted. For example, let K be a field of characteristic 3 and S_5 be the symmetric group of degree 5. Then $\mathrm{Ggl.dim}(K[S_5])=0$ by Proposition 3.5 (2). However, the order of $\alpha=(1\ 2\ 3)$ in S_5 is 3.

(2) The foregoing example also show that $\mathrm{Ggl.dim}(K[S_5])=0$, while $\mathrm{gl.dim}(K[S_5])$ is infinite.

Now Corollary 10.3.7 in [14] can be seen as a corollary of Proposition 3.8.

Corollary 3.10. *Let K be a field of characteristic p and let G be a group. If* $\mathrm{gl.dim}(K[G])<\infty$, *then G is a p'-group.*

Proof. It is clear that $\mathrm{Ggl.dim}(K[G]) \leq \mathrm{gl.dim}(K[G]) < \infty$. In addition, since $\mathrm{gl.dim}(K[G]) < \infty$, any $K[G]$-module $M \in \mathcal{GP}^{\perp}$. Therefore, it follows from the proposition above. □

A group G is called Polycyclic-by-finite if there is a subnormal series for G, $(1) = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$, where G_i/G_{i-1} is either cyclic or finite. By Proposition 3.8 and [14, Theorem 10.3.13], we have

Corollary 3.11. *Let K be a field of characteristic p and let G be a Polycyclic-by-finite group. Then the following are equivalent:*

(1) G is a p'-group;
(2) $\mathrm{gl.dim}(K[G]) < \infty$;
(3) $\mathrm{Ggl.dim}(K[G]) < \infty$ and any $K[G]$-module $M \in \mathcal{GP}^{\perp}$.

In this case $\mathrm{Ggl.dim}(K[G]) = \mathrm{gl.dim}(K[G])$.

Here, $\mathrm{spli}(R)$ denotes the supremum of the projective lengths of all injective R-modules and $\mathrm{silp}(R)$ denotes the supremum of the injective lengths of all projective R-modules (cf.[8]). We study these invariants because it is completely related to Gorenstein global dimensions.

Proposition 3.12. *Let K be a field and let H be a normal subgroup of G. Then*

(1) $\mathrm{spli}(K[G]) \leq \mathrm{spli}(K[H]) + \mathrm{spli}(K[G/H])$.
(2) $\mathrm{silp}(K[G]) \leq \mathrm{silp}(K[H]) + \mathrm{spli}(K[G/H])$.

Proof. For convenience, we set $G/H := \overline{G}$.

(1) Suppose that $\mathrm{spli}(K[H]) = n$ and $\mathrm{spli}(K[\overline{G}]) = m$ are finite. For any injective $K[G]$-module I, it is sufficient to show that $\mathrm{pd}_{K[G]}(I) \leq m + n$. Noting that the augmentation sequence

$$0 \longrightarrow \Delta(K[\overline{G}]) \longrightarrow K[\overline{G}] \longrightarrow K \longrightarrow 0$$

yields the K-split exact sequence of $K[\overline{G}]$-modules

$$0 \longrightarrow K \longrightarrow A \longrightarrow B \longrightarrow 0, \tag{3.1}$$

where $A = \mathrm{Hom}_K(K[\overline{G}], K)$ and $B = \mathrm{Hom}_K(\Delta(K[\overline{G}]), K)$. Hence I is a direct summand of $I \otimes_K A$, and so it is enough to prove that $\mathrm{pd}_{K[G]}(I \otimes_K A) \leq m + n$.

Since K is an injective K-module, A is an injective $K[\overline{G}]$-module, and hence $\mathrm{pd}_{K[\overline{G}]}(A) \leq m$. Let

$$P^{\bullet} := 0 \longrightarrow P_m \longrightarrow P_{m-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

be a $K[\overline{G}]$-projective resolution of A. Choose a $K[G]$-projective resolution of I and

$$Q^{\bullet} := 0 \longrightarrow Q_n \longrightarrow Q_{n-1} \longrightarrow \cdots \longrightarrow Q_0 \longrightarrow I \longrightarrow 0$$

is the truncation, where Q_i is $K[G]$-projective for $i = 0, \cdots, n-1$ and Q_n is $K[H]$-projective as restricted module. Then the total complex $Q^{\bullet} \otimes_K P^{\bullet}$ is a $K[G]$-complex over $I \otimes_K A$ of length $m + n$. Since A is flat as a K-module, $Q^{\bullet} \otimes_K P^{\bullet}$ is a $K[G]$-resolution of $I \otimes_K A$ by the Künneth formula. Finally, we claim that $Q^{\bullet} \otimes_K P^{\bullet}$ is $K[G]$-projective. To prove this, it suffices to show that $Q_n \otimes_K K[\overline{G}]$ is a projective $K[G]$-module. This is true because we have

$$\begin{aligned} Q_n \otimes_K K[\overline{G}] &\cong Q_n \otimes_K (K \uparrow_H^G) = Q_n \otimes_K (K \otimes_{K[H]} K[G]) \\ &\cong (Q_n \otimes_K K) \otimes_{K[H]} K[G] \\ &\cong Q_n \otimes_{K[H]} K[G]. \end{aligned}$$

(2) Suppose that $\mathrm{silp}(K[H]) = n$ and $\mathrm{spli}(K[\overline{G}]) = m$ are finite. Let P be a projective $K[G]$-module. Applying $\mathrm{Hom}_K(-, P)$ to (3.1) above, it gives an exact sequence of $K[\overline{G}]$-modules

$$0 \longrightarrow \mathrm{Hom}_K(B, P) \longrightarrow \mathrm{Hom}_K(A, P) \longrightarrow P \longrightarrow 0.$$

Since P is also a projective $K[G]$-module, the exact sequence above is $K[G]$-split, and hence P is a direct summand of $\mathrm{Hom}_K(A,P)$ as $K[G]$-modules. Then it suffices to prove $\mathrm{id}_{K[G]}(\mathrm{Hom}_K(A,P)) \leq n+m$.

Choose a $K[G]$-injective resolution of P and

$$I^\bullet := \; 0 \longrightarrow P \longrightarrow I^0 \longrightarrow \cdots \longrightarrow I^{n-1} \longrightarrow I^n \longrightarrow 0$$

is the truncation, where I^i is $K[G]$-injective for $i = 0, 1, \ldots, n-1$ and I^n is $K[H]$-injective as restricted module. Since K is an injective K-module, A is also an injective $K[\overline{G}]$-module. By hypothesis, $\mathrm{pd}_{K[\overline{G}]}(A) \leq m$. Let

$$P^\bullet := 0 \longrightarrow P_m \longrightarrow P_{m-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

be a $K[\overline{G}]$-projective resolution of A. Thus, $\mathrm{Hom}_K(P^\bullet, I^\bullet)$ is a $K[G]$-complex over $\mathrm{Hom}_K(A,P)$ of length $n+m$. Since A is projective as K-module, $\mathrm{Hom}_K(P^\bullet, I^\bullet)$ is a $K[G]$-resolution of $\mathrm{Hom}_K(A,P)$ by Künneth's cohomological formula. Now, we need to show $\mathrm{Hom}_K(P^\bullet, I^\bullet)$ is $K[G]$-injective. It is enough to prove that $\mathrm{Hom}_K(K[\overline{G}], I^n)$ is $K[G]$-injective. In fact, we have

$$\begin{aligned} \mathrm{Hom}_K(K[\overline{G}], I^n) &\cong \mathrm{Hom}_K(K \otimes_{K[H]} K[G], I^n) \\ &\cong \mathrm{Hom}_{K[H]}(K[G], \mathrm{Hom}_K(K, I^n)) \\ &= \mathrm{Hom}_{K[H]}(K[G], I^n). \end{aligned}$$

Since I^n is $K[H]$-injective, $\mathrm{Hom}_{K[H]}(K[G], I^n)$ is also $K[G]$-injective as coinduced module, as desired. □

An immediate consequence of proposition above with $H = \{e\}$ is the following corollary.

Corollary 3.13. *Let K be a field and let G be a group. Then* $\mathrm{silp}(K[G]) \leq \mathrm{spli}(K[G])$.

Following [8], we say that M admits a complete projective resolution of coincidence index n if there exists a complete projective resolution, which coincides with a projective resolution of M in degrees $\geq n$. Hence, M admits a complete projective resolution of coincidence index 0 if and only if M is a syzygy of a complete projective resolution, i.e., if and only if M is Gorenstein projective. We say that M admits a complete projective resolution if it admits a complete projective resolution of coincidence index n for some n.

Proposition 3.14. *Let K be a field and let G be a group. Then the following are equivalent:*

(1) $\mathrm{Ggl.dim}(K[G]) < \infty$;
(2) $\mathrm{spli}(K[G]) < \infty$;
(3) Any $K[G]$-module admits a complete projective resolution and $\mathrm{silp}(K[G]) < \infty$;
(4) There is an exact sequence of $K[G]$-modules $0 \longrightarrow V_0 \longrightarrow A \longrightarrow B \longrightarrow 0$, where V_0 is a principal $K[G]$-module and $\mathrm{pd}_{K[G]}(A) < \infty$.

In this case, $\mathrm{pd}_{K[G]}(A) = \mathrm{Ggl.dim}(K[G])$.

Proof. (1)⇔(2) and (2)⇔(3) follow from [8, Theorem 4.1] and Corollary 3.13.

(1)⇒(4) follows from [6, Lemma 2.17].

(4)⇒(2). By hypothesis, there exists an exact sequence of $K[G]$-modules

$$0 \longrightarrow V_0 \longrightarrow A \longrightarrow B \longrightarrow 0,$$

where V_0 is a principal $K[G]$-module and $\mathrm{pd}_{K[G]}(A) < \infty$. If I is an injective $K[G]$-module, then the exact sequence of $K[G]$-modules

$$0 \longrightarrow I \longrightarrow I \otimes_K A \longrightarrow I \otimes_K B \longrightarrow 0$$

is $K[G]$-split, and hence I is a direct summand of $I \otimes_K A$. Thus, it suffices to prove $\mathrm{pd}_{K[G]}(I \otimes_K A) < \infty$. In fact, it follows that $\mathrm{pd}_{K[G]}(I \otimes_K A) = \mathrm{pd}_{K[G]}(A) < \infty$ by Lemma 3.2.

Moreover, in view of [6, Lemma 2.17] and Theorem 3.4, $\mathrm{pd}_{K[G]}(A) = \mathrm{Gpd}_{K[G]}(V_0) = \mathrm{Ggl.dim}(K[G])$. Therefore, we complete the proof. □

As mentioned in [9], the Weyl groups of the (finite) subgroup H of G, denoted $W = N_G(H)/H$, are important objects and tools in the study of actions of G on topological spaces. We have the following result analogue to [9, Proposition 2.5] but its proof is somewhat different.

Proposition 3.15. *Let K be a field and let G be a group. Then, for any finite subgroup H of G, we have* $\mathrm{Ggl.dim}(K[W]) \leq \mathrm{Ggl.dim}(K[G])$*, where $W = N_G(H)/H$ is the Weyl group of H.*

Proof. By Proposition 3.5(1), $\mathrm{Ggl.dim}(K[N_G(H)]) \leq \mathrm{Ggl.dim}(K[G])$. Then we assume that H is normal in G, i.e., $N_G(H) = G$. So $K[W]$-modules are precisely $K[G]$-modules with trivial H-action, and we set $K[W]{-}\mathrm{Mod} = \{M \in K[G]{-}\mathrm{Mod} : M^H = M\}$. For any projective $K[G]$-module P, noting that $(K[G])^H \cong K[W]$, it follows that P^H is a projective $K[W]$-module. Furthermore, if $\mathrm{pd}_{K[G]}(M) = n < \infty$, then there exists an exact sequence of $K[G]$-modules

$$0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

with all P_i projective. By the proof of [9, Proposition 2.5],

$$0 \longrightarrow P_n^H \longrightarrow P_{n-1}^H \longrightarrow \cdots \longrightarrow P_0^H \longrightarrow M^H \longrightarrow 0$$

is also an exact sequence of $K[W]$-modules. Thus, $\mathrm{pd}_{K[W]}(M^H) \leq \mathrm{pd}_{K[G]}(M)$.

Now, let $\mathrm{Ggl.dim}(K[G]) = m < \infty$. By Proposition 3.14, there exists an exact sequence of $K[G]$-modules

$$0 \longrightarrow V_0 \longrightarrow A \longrightarrow B \longrightarrow 0,$$

where V_0 is a principal $K[G]$-module and $\mathrm{pd}_{K[G]}(A) = m$. Then

$$0 \longrightarrow V_0^H \longrightarrow A^H \longrightarrow B^H \longrightarrow 0$$

is an exact sequence of $K[W]$-modules. Noting that the group G (and H) acts trivially on V_0, V_0^H is also a principal $K[W]$-module. By the result above, $\mathrm{pd}_{K[W]}(A^H) \leq \mathrm{pd}_{K[G]}(A) = m$. Then, in view of Proposition 3.14, $\mathrm{Ggl.dim}(K[W])$ is finite and $\mathrm{Ggl.dim}(K[W]) = \mathrm{pd}_{K[W]}(A^H) \leq m$. $\square$

Now we elaborate the main results in this paper.

Theorem 3.16. *Let K be a field and let H be a normal subgroup of G. If* $\mathrm{Ggl.dim}(K[H])$ *and* $\mathrm{Ggl.dim}(K[G/H])$ *are finite, then so is* $\mathrm{Ggl.dim}(K[G])$*, and we have*

$$\mathrm{Ggl.dim}(K[G]) \leq \mathrm{Ggl.dim}(K[H]) + \mathrm{Ggl.dim}(K[G/H]).$$

Proof. If $\mathrm{Ggl.dim}(K[H])$ and $\mathrm{Ggl.dim}(K[G/H])$ are finite, in view of Proposition 3.14, then $\mathrm{spli}(K[H])$ and $\mathrm{spli}(K[G/H])$ are finite. By Proposition 3.12, $\mathrm{spli}(K[G])$ is finite, and hence $\mathrm{Ggl.dim}(K[G])$ is finite by Proposition 3.14 again.

Now suppose that $\mathrm{Ggl.dim}(K[H]) = n$ and $\mathrm{Ggl.dim}(K[G/H]) = m$. If V_0 is a principal $K[H]$-module, then there is an exact sequence of $K[H]$-modules

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \longrightarrow V_0 \longrightarrow 0,$$

where all P_i are Gorenstein projective. By [17, Proposition 2.3], there is an exact sequence of $K[G]$-modules

$$0 \longrightarrow P_n \uparrow_H^G \longrightarrow \cdots \longrightarrow P_0 \uparrow_H^G \longrightarrow V_0 \uparrow_H^G \longrightarrow 0,$$

and all $P_i \uparrow_H^G$ are Gorenstein projective. On the other hand, $V_0 \uparrow_H^G \cong K[G/H]$ as $K[G]$-modules, and hence

$$\mathrm{Gpd}_{K[G]}(K[G/H]) = \mathrm{Gpd}_{K[G]}(V_0 \uparrow_H^G) \leq n.$$

For any projective $K[G/H]$-module P, in view of [11, Proposition 2.19], $\mathrm{Gpd}_{K[G]}(P) \leq n$. Now we claim that $\mathrm{Gpd}_{K[G]}(Q) \leq n$ for any Gorenstein projective $K[G/H]$-module Q. In fact, there exists an exact sequence of $K[G/H]$-modules

$$0 \longrightarrow Q \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots, \text{ with all } P^i \text{ projective.}$$

Assume that $\mathrm{Gpd}_{K[G]}(Q) > n$, and let $Q_i = \mathrm{Ker}(P^i \to P^{i+1})$ for $i = 1, 2, \ldots$. By [1, Theorem 2.6],

$$\mathrm{Gpd}_{K[G]}(Q_1) = \mathrm{Gpd}_{K[G]}(Q) + 1.$$

Inductively,

$$\mathrm{Gpd}_{K[G]}(Q_i) = \mathrm{Gpd}_{K[G]}(Q) + i,$$

and hence $\oplus_i Q_i$ has infinite Gorenstein projective dimension over $K[G]$, a contradiction.

If W_0 is a principal $K[G/H]$-module, in view of Theorem 3.4,

$$\mathrm{Gpd}_{K[G/H]}(W_0) = \mathrm{Ggl.dim}(K[G/H]) = m.$$

Then we have the following exact sequences of $K[G/H]$-modules

$$0 \longrightarrow W_{i+1} \longrightarrow P_i \longrightarrow W_i \longrightarrow 0, i = 0, 1, \cdots, m-1,$$

where P_i, $i = 0, 1, \cdots, m-1$ and W_m are Gorenstein projective $K[G/H]$-modules. The exact sequences above are also the exact sequences of $K[G]$-modules, and W_0 is also a principal $K[G]$-module. To prove $\mathrm{Gpd}_{K[G]}(W_i) \leq n + m - i$, we carry out the inverse induction on i.

(1) $\mathrm{Gpd}_{K[G]}(W_m) \leq n + m - m$ because W_m is a Gorenstein projective $K[G/H]$-module.
(2) Suppose that $\mathrm{Gpd}_{K[G]}(W_i) \leq n+m-i$ for $1 < i < m$. Then $\mathrm{Gpd}_{K[G]}(W_i)$ and $\mathrm{Gpd}_{K[G]}(P_i)$ are finite and $\mathrm{Gpd}_{K[G]}(P_i) \leq n$.
(3) By [1, Theorem 2.6],

$$\begin{aligned}\mathrm{Gpd}_{K[G]}(W_{i-1}) &\leq 1 + \sup\{\mathrm{Gpd}_{K[G]}(W_i), \mathrm{Gpd}_{K[G]}(P_{i-1})\} \\ &\leq 1 + (n + m - i) = n + m - (i-1).\end{aligned}$$

In particular, when $i = 0$ we have $\mathrm{Gpd}_{K[G]}(W_0) \leq n + m$. Thus, in view of Theorem 3.4,

$$\mathrm{Ggl.dim}(K[G]) = \mathrm{Gpd}_{K[G]}(W_0) \leq n + m.$$

Therefore, we complete the proof. □

Corollary 3.17. *Let K be a field, and let H be a normal subgroup of G.*

(1) If H has a finite index, then $\mathrm{Ggl.dim}(K[G]) = \mathrm{Ggl.dim}(K[H])$.
(2) If H is finite, then $\mathrm{Ggl.dim}(K[G]) = \mathrm{Ggl.dim}(K[G/H])$.

Proof. (1) If H has a finite index, then $\mathrm{Ggl.dim}(K[G/H]) = 0$ by Proposition 3.5(2). Thus, the result follows from Proposition 3.5(1) and Theorem 3.16.

(2) It follows from Proposition 3.5(2), Proposition 3.15 and Theorem 3.16. □

By Proposition 3.5 (2) and Corollary 3.17, we have a plain group theoretic property.

Corollary 3.18. *Let H be a normal subgroup of G. If any two groups, among G, H and G/H, are finite, then so is the third.*

The following results provide some analogous versions of the Serre's Theorem for Gorenstein global dimensions.

Theorem 3.19. *Let K be a field, and let H be a subgroup of G of finite index. If* $\mathrm{Ggl.dim}(K[G])$ *is finite, then* $\mathrm{Ggl.dim}(K[H]) = \mathrm{Ggl.dim}(K[G])$.

Proof. By Proposition 3.5(1), $\mathrm{Ggl.dim}(K[H]) \leq \mathrm{Ggl.dim}(K[G])$. We assume that $\mathrm{Ggl.dim}(K[G]) = n < \infty$. Let V_0 be a principal $K[G]$-module. By Theorem 3.4, $\mathrm{Gpd}_{K[G]}(V_0) = \mathrm{Ggl.dim}(K[G]) = n$. Then, in view of [11, Theorem 2.20], there exists some projective $K[G]$-module M such that $\mathrm{Ext}^n_{K[G]}(V_0, M) \neq 0$ and for any projective $K[G]$-module P, $\mathrm{Ext}^{n+1}_{K[G]}(V_0, P) = 0$. Consider the following exact sequence of $K[G]$-modules

$$0 \longrightarrow N \longrightarrow M \otimes_{K[H]} K[G] \xrightarrow{\pi} M \longrightarrow 0,$$

where $\pi(m \otimes g) = mg$ for $m \in M$ and $g \in G$. Noting that M and $M \otimes_{K[H]} K[G]$ are projective, then N is also projective. Applying the functor $\mathrm{Hom}_{K[G]}(V_0, -)$ to the sequence above, we get a long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}^n_{K[G]}(V_0, M \otimes_{K[H]} K[G]) \longrightarrow \mathrm{Ext}^n_{K[G]}(V_0, M) \longrightarrow \mathrm{Ext}^{n+1}_{K[G]}(V_0, N) = 0.$$

Since $\mathrm{Ext}^n_{K[G]}(V_0, M) \neq 0$, $\mathrm{Ext}^n_{K[G]}(V_0, M \otimes_{K[H]} K[G]) \neq 0$. In addition, by [3, Corollary 2.8.4] and [16, Lemma 9.2], we have

$$\begin{aligned} \mathrm{Ext}^n_{K[H]}(V_0, M) &\cong \mathrm{Ext}^n_{K[G]}(V_0, \mathrm{Hom}_{K[H]}(K[G], M)) \\ &\cong \mathrm{Ext}^n_{K[G]}(V_0, M \otimes_{K[H]} K[G]) \neq 0. \end{aligned}$$

Noting that M is also $K[H]$-projective as restricted module and V_0 is also a principal $K[H]$-module as restricted module, then $\mathrm{Ggl.dim}(K[H]) = \mathrm{Gpd}_{K[H]}(V_0) \geq n$. This completes the proof. □

Corollary 3.20. *Let K be a field, and let H be a subgroup of G of finite index. If* $\mathrm{spli}(K[G])$ *is finite, then* $\mathrm{Ggl.dim}(K[H]) = \mathrm{Ggl.dim}(K[G])$.

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