Connected Total Monophonic Eccentric Domination in Graphs

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Abstract

For any two vertices x and y in a non-trivial connected graph G, the *monophonic distance* $d_m(x, y)$ is the length of a longest monophonic path joining the vertices x and y in G. The *monophonic eccentricity* of a vertex x is defined as $e_m(x) = \max \{d_m(x, y) : y \in V(G)\}$. A vertex y in G is a *monophonic eccentric vertex* of a vertex x in G if $e_m(x) = d_m(x, y)$. A set $S \subseteq V$ in a graph G is a *total monophonic eccentric dominating set* if every vertex of G has a monophonic eccentric vertex in S. The *total monophonic eccentric domination number* $\gamma_{tme}(G)$ is the cardinality of a minimum total monophonic eccentric dominating set of G. A set $S \subseteq V$ in a graph G is a *connected total monophonic eccentric dominating set* if S is a total monophonic eccentric dominating set and the induced subgraph $\langle S \rangle$ is connected. The *connected total monophonic eccentric domination number* $\gamma_{ctme}(G)$ is the cardinality of a minimum connected total monophonic eccentric dominating set of G. We investigate some properties of connected total monophonic eccentric dominating sets. Also, we determine the bounds of connected total monophonic eccentric domination number and find the same for some standard graphs.

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1 Introduction

Let $G = (V, E)$ be a finite undirected connected graph with $|V| = p$ and $|E| = q$. We refer [1, 4] for basic graph theoritic concepts and notations. The *distance* between any two vertices x and y is the length of a shortest path (geodesic) joining the vertices x and y, and it is denoted by $d(x, y)$. If $d(x, y) = 1$, then x is a *neighbor* of y and vice versa. A subset S of the vertex set V is called a *dominating set* if every vertex in $V - S$ has a neighbor in S. The *domination number* of G is defined as $\gamma(G) = min\{|S| : S \text{ is a dominating set of } G\}$. The idea of domination was introduced in [1] and further studied in [6]. Recently there are some new parameters introduced based on domination, and a text book [5] on domination was published in 1998. Also, total domination in graphs was introduced in [3].

The *detour distance* between any two vertices x and y is the length of a longest path (detour) joining the vertices x and y, and it is denoted by $D(x, y)$. A vertex y is called a *detour neighbor* of a vertex x if $D(x, y) \leq D(x, z)$ for any $z \in V - \{x, y\}$. A subset S of the vertex set V is called a *detour dominating set* if every vertex in $V - S$ has a detour neighbor in S and the *detour domination number* is defined as $\gamma_D(G) = min\{|S| : S \text{ is a detour dominating set of } G\}$. The concept of detour domination was introduced and studied in [2].

A chordless path is also called as a *monophonic path*. The *monophonic distance* between any two vertices x and y is the length of a longest monophonic path joining the vertices x and y, and it is denoted by $d_m(x, y)$. For any vertex x in G, the *monophonic eccentricity* of a vertex x is defined as $e_m(x) = \max\ \{d_m(x, y) : y \in V\}$. A vertex y in G is a *monophonic eccentric vertex* of a vertex x in G if $e_m(x) = d_m(x, y)$. The *monophonic radius* $rad_m(G)$ is the minimum monophonic eccentricity among the vertices of G and the *monophonic diameter* $diam_m(G)$ is the maximum monophonic eccentricity among the vertices of G. In

[7, 8], Santhakumaran and Titus initiated the study of monophonic distance and further related results.

The monophonic eccentric dominating set and the monophonic eccentric domination number of a graph were introduced and studied in [9, 10]. The total monophonic eccentric domination number was introduced and studied in [11]. The parameters monophonic eccentric domination number and total monophonic eccentric domination number have many useful applications in channel assignment problems in radio technologies. Further, these concepts have huge amount of application in molecular problems in theoretical chemistry. The following definitions will be used in the sequal.

Definition 1.1 *[9] A set $S \subseteq V$ in a graph G is a monophonic eccentric dominating set if every vertex in $V - S$ has a monophonic eccentric vertex in S. The monophonic eccentric domination number $\gamma_{me}(G)$ is the cardinality of a minimum monophonic eccentric dominating set of G.*

Definition 1.2 *[11] A set $S \subseteq V$ in a graph G is a total monophonic eccentric dominating set if every vertex in G has a monophonic eccentric vertex in S. The total monophonic eccentric domination number $\gamma_{tme}(G)$ is the cardinality of a minimum total monophonic eccentric dominating set of G.*

2 Connected Total Monophonic Eccentric Domination Number

Definition 2.1 *Let $G = (V, E)$ be a non-trivial connected graph. A set $S \subseteq V$ is a connected total monophonic eccentric dominating set if S is a total monophonic eccentric dominating set and the induced subgraph $\langle S \rangle$ is connected. The connected total monophonic eccentric domination number $\gamma_{ctme}(G)$ is the cardinality of a minimum connected total monophonic eccentric dominating set of G.*

Example 2.2 *Consider the graph G given in Figure 2.1. The set $\{v_2, v_5\}$ is a minimum monophonic eccentric dominating set of G so that $\gamma_{me}(G) = 2$. The set $\{v_2, v_4, v_5\}$ is a* *minimum total monophonic eccentric dominating set of G and so $\gamma_{tme}(G) = 3$. Also, the set $\{v_2, v_3, v_4, v_5\}$ is a minimum connected total monophonic eccentric dominating set of G and so $\gamma_{ctme}(G) = 4$. Thus the parameters $\gamma_{me}(G)$, $\gamma_{tme}(G)$ and $\gamma_{ctme}(G)$ are different.*

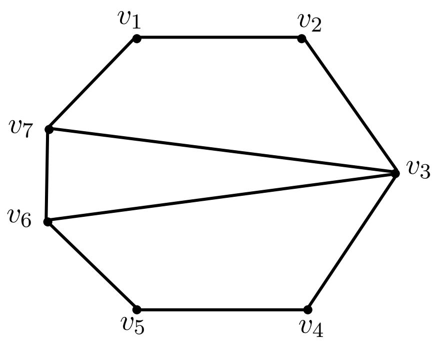

Figure 2.1: G

Next theorem gives the bounds of the connected total monophonic eccentric domination number of a graph.

Theorem 2.3 *For any non-trivial connected graph G of order p, $2 \leq \gamma_{ctme}(G) \leq p$.*

Proof. It is clear that every vertex in G has at least one monophonic eccentric vertex in G. Therefore, connected total monophonic eccentric dominating set of G contains at most p vertices and so $\gamma_{ctme}(G) \leq p$. Also, every connected total monophonic eccentric dominating set of G contains at least two vertices and so $\gamma_{ctme}(G) \geq 2$. Hence $2 \leq \gamma_{ctme}(G) \leq p$. ■

Remark 2.4 *The bounds in Theorem 2.3 are sharp. For the complete graph K_p $(p \geq 2)$, $\gamma_{ctme}(K_p) = 2$ and for the cycle C_4, $\gamma_{ctme}(C_4) = 4$.*

In the following theorem we establish the relationship between the monophonic eccentric domination number, the total monophonic eccentric domination number and the connected total monophonic eccentric domination number of a graph.

Theorem 2.5 *For any connected graph G, $\gamma_{me}(G) \leq \gamma_{tme}(G) \leq \gamma_{ctme}(G)$.*

Proof. Since any total monophonic eccentric dominating set of G is also a monophonic eccentric dominating set of G, it follows that $\gamma_{me}(G) \leq \gamma_{tme}(G)$. Since any connected total monophonic eccentric dominating set is necessarily a total monophonic eccentric dominating set, it follows that $\gamma_{tme}(G) \leq \gamma_{ctme}(G)$. Hence $\gamma_{me}(G) \leq \gamma_{tme}(G) \leq \gamma_{ctme}(G)$. ∎

Remark 2.6 *The bounds in Theorem 2.5 are sharp. For any tree T with $diam_m(T) \geq 3$, any pair of antipodal vertices will form a minimum monophonic eccentric dominating set and also form a minimum total monophonic eccentric dominating set of T. Therefore $\gamma_{me}(T) = \gamma_{tme}(T)$. For the cycle C_4, $\gamma_{tme}(C_4) = \gamma_{ctme}(C_4)$. Also, all the inequalities in Theorem 2.5 are strict. For the path P_3, $\gamma_{me}(P_3) = 1$, $\gamma_{tme}(P_3) = 2$ and $\gamma_{ctme}(P_3) = 3$ so that $\gamma_{me}(P_3) < \gamma_{tme}(P_3) < \gamma_{ctme}(P_3)$.*

Theorem 2.7 *If $G = H + K_n$, where H is any connected graph, then $\gamma_{ctme}(G) = \gamma_{ctme}(H)$.*

Proof. Let S be a minimum connected total monophonic eccentric dominating set of H and let $v_1, v_2, \ldots, v_n$ be the vertices of K_n. Clearly, any vertex in H is a monophonic eccentric vertex of v_i $(1 \leq i \leq n)$. Since all the vertices of K_n are adjacent to every vertex of H, any monophonic eccentric vertex of a vertex, say u, in H is again a monophonic eccentric vertex of u in G. Hence S is also a minimum connected total monophonic eccentric dominating set of G and so $\gamma_{ctme}(G) = \gamma_{ctme}(H)$. ∎

Now we proceed to characterize graphs for which the bounds in Theorem 2.3 are attained. For this, we introduce the following definition.

Definition 2.8 *A graph G is called a unique monophonic eccentric graph if each vertex of G has a unique monophonic eccentric vertex.*

Theorem 2.9 *Let G be a connected monophonic self-centered graph of order p. Then $\gamma_{ctme}(G) = p$ if and only if G is a unique monophonic eccentric graph.*

Proof. Let G be a connected monophonic self-centered graph of order p. Let $v_1, v_2, \ldots, v_p$ be the vertices of G. Assume that $\gamma_{ctme}(G) = p$. Suppose that G is not a unique monophonic eccentric graph. Then there exists a vertex, say v_1, such that v_1 has two monophonic eccentric vertices, say v_l and v_k $(2 \leq l < k \leq p)$. Since G is a monophonic self-centered graph, it is clear that the vertices v_l and v_k are monophonic eccentric dominated by the vertex v_1. Hence $S = V(G) - \{v_k\}$ is a total monophonic eccentric dominating set of G. Since G is a monophonic self-centered graph, no vertex of G is a cut vertex of G and so the induced subgraph $\langle S \rangle$ is connected. Hence S is a connected total monophonic eccentric dominating set of G and so $\gamma_{ctme}(G) \leq p-1$, which is a contradiction. Hence G is a unique monophonic eccentric graph.

Conversely, let G be a unique monophonic eccentric graph. Then every vertex of G is a monophonic eccentric vertex of some vertex in G. Let $x, y \in V(G)$. Now, claim that x is a monophonic eccentric vertex of y if and only if y is a monophonic eccentric vertex of x. Let x be a monophonic eccentric vertex of y. Then $e_m(y) = d_m(x, y)$. Suppose that y is not a monophonic eccentric vertex of x. Then there exists a vertex $w \neq y$ such that w is a monophonic eccentric vertex of x. Then $e_m(x) = d_m(w, x) > d_m(y, x) = e_m(y)$, which is a contradicton to G a connected monophonic self-centered graph. Hence x is a monophonic eccentric vertex of y if and only if y is a monophonic eccentric vertex of x. Therefore, since G is a connected monophonic self-centered graph, all the vertices of G will form a minimum total monophonic eccentric dominating set of G and so $\gamma_{tme}(G) = p$. Thus by Theorem 2.5, $\gamma_{ctme}(G) = p$. ■

Theorem 2.10 *Let G be a connected graph of order $p \geq 2$. Then $\gamma_{ctme}(G) = 2$ if and only if $G = K_p$.*

Proof. Let $\gamma_{ctme}(G) = 2$. Let $S = \{x, y\}$ be a minimum connected total monophonic eccentric dominating set of G. Then x is monophonic eccentric dominated by the vertex y, y is monophonic eccentric dominated by the vertex x, and all the remaining vertices of G are monophonic eccentric dominated by either x or y in G. Since the induced subgraph $\langle S \rangle$ is connected, x and y are adjacent in G and so $d_m(x, y) = 1$. Hence $d_m(x, z) = d_m(y, z) = 1$ for all $z \in V(G) - S$ and so $e_m(x) = e_m(y) = 1$. Now claim that $e_m(u) = 1$ for every vertex u in G. If not, there exists a vertex, say v, in G with $e_m(v) \geq 2$. Then v is not monophonic eccentric dominated by both x and y in G, which is a contradiction. Thus $e_m(u) = 1$ for every vertex u in G and so $G = K_p$.

Conversely, let $G = K_p$. Since every vertex of the complete graph K_p ($p \geq 2$) is a monophonic eccentric vertex of other vertices in K_p, any two vertices will form a minimum total monophonic eccentric dominating set of K_p and its induced subgraph $\langle S \rangle$ is connected. Thus $\gamma_{ctme}(K_p) = 2$. ∎

3 Connected Total Monophonic Eccentric Domination Number of Some Standard Graphs

Theorem 3.1 *For any tree T, $\gamma_{ctme}(T) = diam_m(T) + 1$.*

Proof. Let $r = rad_m(T)$ and $d = diam_m(T)$. Let $P : v_0, v_1, \ldots, v_r, v_{r+1} \ldots, v_d$ be a diametral path of T. It is clear that a vertex u with $d_m(v_0, u) \geq r$ is monophonic eccentric dominated by the vertex v_0 and a vertex u with $d_m(u, v_d) \geq r$ is monophonic eccentric dominated by the vertex v_d. It is clear that $V(P)$ is a minimum connected total monophonic eccentric dominating set of T and so $\gamma_{ctme}(T) = diam_m(T) + 1$. ∎

Corollary 3.2 *Let P be any path of order $p \geq 2$. Then $\gamma_{ctme}(P) = p$.*

Theorem 3.3 *If $G = K_{r,s}$ is a complete bipartite graph of order at least 3, then*

$$\gamma_{ctme}(G) = \begin{cases} 3 & \text{either } r = 1 \text{ or } s = 1 \\ 4 & \text{if } r, s \geq 2. \end{cases}.$$

Proof. Let $V_1 = \{u_1, u_2, ..., u_r\}$ and $V_2 = \{v_1, v_2, ..., v_s\}$ be the partite sets of $K_{r,s}$. We prove this theorem by considering two cases.

Case 1. $r = 1$ or $s = 1$.

Then G is a star and hence by Theorem 3.1, $\gamma_{ctme}(G) = 3$.

Case 2. $r, s \geq 2$.

It is clear that no two element subset of the vertex set of G will form a total monophonic eccentric dominating set of G. Let $S = \{u_i, u_j, v_l, v_m\}$ ($i \neq j, l \neq m$, $1 \leq i, j \leq r$ and $1 \leq l, m \leq s$). It can be easily seen that every vertex in $V_1 - \{u_i\}$ has a monophonic eccentric vertex u_i and the vertex u_i has a monophonic eccentric vertex u_j. Similarly, every vertex in $V_2 - \{v_l\}$ has a monophonic eccentric vertex v_l and the vertex v_l has a monophonic eccentric vertex v_m. Hence S is a total monophonic eccentric dominating set of G. Since the induced subgraph $\langle S \rangle$ is connected, S is a minimum connected total monophonic eccentric dominating set of G and so $\gamma_{ctme}(G) = 4$. ■

Theorem 3.4 *If* $G = K_1 + \cup m_j K_j$, *then* $\gamma_{ctme}(G) = \begin{cases} 2 & \text{if } \sum m_j = 1 \\ 3 & \text{otherwise.} \end{cases}$

Proof. Let $G = K_1 + \cup m_j K_j$ and let u be the vertex of K_1. We prove this theorem by considering two cases.

Case 1. $\sum m_j = 1$.

The graph $G = K_1 + \cup m_j K_j$ is a complete graph. Then by Theorem 2.10, $\gamma_{ctme}(G) = 2$.

Case 2. $\sum m_j \geq 2$.

It is clear that u is the cut-vertex of G and hence u is not a monophonic eccentric vertex of any vertex in G. Since u is the cut-vertex of G, $G - u$ has at least two components. Let $S = \{v, w\}$, where v and w belong to two different components, say G_1 and G_2, respectively. Then every vertex of $G - G_1$ is monophonic eccentric dominated by the vertex v and every vertex of $G - G_2$ is monophonic eccentric dominated by the vertex w. But the induced subgraph $\langle S \rangle$ is not connected. Therefore, we choose a set $S_1 = S \cup \{u\}$. It is clear that the induced subgraph $\langle S_1 \rangle$ is connected. Hence S_1 is a minimum connected total monophonic eccentric dominating set of G and so $\gamma_{ctme}(G) = 3$. ■

Theorem 3.5 *If* $G = C_p$ *is a cycle of order* p, *then* $\gamma_{ctme}(G) = \begin{cases} 2 & \text{if } p = 3 \\ 4 & \text{if } 3 < p < 8 \\ p - 4 & \text{if } p \geq 8\ . \end{cases}$

Proof. Let $G : v_1, v_2, \ldots, v_p, v_1$ be a cycle of order p. If $p = 3$, then G is a complete graph K_3 and so by Theorem 2.10, we have $\gamma_{ctme}(G) = 2$. If $3 < p < 8$, it is clear that any 4 consecutive vertices will form a minimum connected total monophonic eccentric dominating set of G and so $\gamma_{ctme}(G) = 4$. If $p \geq 8$, then any $p - 4$ consecutive vertices will form a minimum connected total monophonic eccentric dominating set of G and so $\gamma_{ctme}(G) = p - 4$. ■

Theorem 3.6 *Let G be a wheel of order $p \leq 13$. Then*

$$\gamma_{ctme}(G) = \begin{cases} 2 & \text{if } p = 4 \\ 4 & \text{if } 4 < p < 9 \\ p-5 & \text{if } 9 \leq p \leq 13. \end{cases}$$

Proof. Let $G = W_p = C_{p-1} + K_1$ be the wheel of order $p \leq 13$. Let $v_1, v_2, \ldots, v_{p-1}$ be the vertices of C_{p-1} and let v_p be the vertex of K_1. It is clear that v_p is not a monophonic eccentric vertex of any vertex in G but any vertex in C_{p-1} is a monophonic eccentric vertex of v_p. If $p = 4$, then G is a complete graph K_4 and so by Theorem 2.10, we have $\gamma_{ctme}(G) = 2$. If $4 < p < 9$, then any four consecutive vertices of C_{p-1} will form a minimum connected total monophonic eccentric dominating set of G and so $\gamma_{ctme}(G) = 4$. If $9 \leq p \leq 13$, then any $p-5$ consecutive vertices of C_{p-1} will form a minimum connected total monophonic eccentric dominating set of G and so $\gamma_{ctme}(G) = p - 5$. ■

Theorem 3.7 *Let G be a wheel of order $p > 13$ and let $p \equiv l(mod\ 8)$. Then*

$$\gamma_{ctme}(G) = \begin{cases} \frac{p}{2}+1 & \text{if } l \text{ is even} \\ \frac{p+l}{2} & \text{if } l = 1, 3 \text{ or } 5 \\ \frac{p+3}{2} & \text{if } l = 7. \end{cases}$$

Proof. Let $G = W_p = C_{p-1} + K_1$ be the wheel of order $p > 13$ and let $p \equiv l(mod\ 8)$. Let $v_1, v_2, \ldots, v_{p-1}$ be the vertices of C_{p-1} and let v_p be the vertex of K_1. We prove this theorem by considering two cases.

Case 1. l is even.

Subcase (i) $l = 0$.

Let $S = \{v_1, v_2, v_3, v_4; v_9, v_{10}, v_{11}, v_{12}; \ldots; v_{p-7}, v_{p-6}, v_{p-5}, v_{p-4}\}$. It is easily verified that the vertices v_3 and v_{p-1} are monophonic eccentric dominated by the vertex v_1, the vertices v_4 and v_p are monophonic eccentric dominated by the vertex v_2, the vertices v_1 and v_5 are monophonic eccentric dominated by the vertex v_3, the vertices v_2 and v_6 are monophonic eccentric dominated by the vertex v_4, $\ldots$, the vertices v_{p-5} and v_{p-9} are monophonic eccentric dominated by the vertex v_{p-7}, the vertices v_{p-4} and v_{p-8} are monophonic eccentric dominated by the vertex v_{p-6}, the vertices v_{p-3} and v_{p-7} are monophonic eccentric dominated by the vertex v_{p-5}, the vertices v_{p-6} and v_{p-2} are monophonic eccentric dominated by the vertex v_{p-4}. It is clear that S is a minimum total monophonic eccentric dominating set of G, but the induced subgraph $\langle S\rangle$ is not connected. Therefore, we consider a set $S_1 = S \cup \{v_p\}$. Clearly, S_1 is a minimum connected total monophonic eccentric dominating set of G and so $\gamma_{ctme}(G) = \dfrac{p}{2} + 1$.

Subcase (ii) $l = 2$.

Let $S = \{v_1, v_2, v_3, v_4; v_9, v_{10}, v_{11}, v_{12}; \ldots; v_{p-9}, v_{p-8}, v_{p-7}, v_{p-6}\} \cup \{v_{p-1}, v_p\}$. By an argument similar to Subcase (i), it can be easily seen that S is a minimum connected total monophonic eccentric dominating set of G and so $\gamma_{ctme}(G) = \dfrac{p}{2} + 1$.

Subcase (iii) $l = 4$.

Let $S = \{v_1, v_2; v_9, v_{10}; \ldots; v_{p-3}, v_{p-2}\} \cup \{v_4, v_{12}, \ldots, v_{p-8}\} \cup \{v_7, v_{15}, \ldots, v_{p-5}\} \cup \{v_p\}$. By an argument similar to Subcase (i), it can be easily seen that S is a minimum connected total monophonic eccentric dominating set of G and so $\gamma_{ctme}(G) = \dfrac{p}{2} + 1$.

Subcase (iv) $l = 6$.

Let $S = \{v_1, v_2; v_9, v_{10}; \ldots; v_{p-5}, v_{p-4}\} \cup \{v_4, v_{12}, \ldots, v_{p-2}\} \cup \{v_7, v_{15}, \ldots, v_{p-7}\} \cup \{v_p\}$. By an argument similar to Subcase (i), it can be easily seen that S is a minimum connected total monophonic eccentric dominating set of G and so $\gamma_{ctme}(G) = \dfrac{p}{2} + 1$.

Case 2. l is odd.

Subcase (i) $l = 1$.

Let $S = \{v_1, v_2, v_3, v_4; v_9, v_{10}, v_{11}, v_{12}; \ldots; v_{p-8}, v_{p-7}, v_{p-6}, v_{p-5}\}$. It is easily verified that the vertices v_3 and v_{p-1} are monophonic eccentric dominated by the vertex v_1, the vertices v_4 and v_p are monophonic eccentric dominated by the vertex v_2, the vertices v_1 and v_5 are monophonic eccentric dominated by the vertex v_3, the vertices v_2 and v_6 are monophonic eccentric dominated by the vertex v_4, $\ldots$, the vertices v_{p-10} and v_{p-6} are monophonic eccentric dominated by the vertex v_{p-8}, the vertices v_{p-9} and v_{p-5} are monophonic eccentric dominated by the vertex v_{p-7}, the vertices v_{p-8} and v_{p-4} are monophonic eccentric dominated by the vertex v_{p-6}, the vertices v_{p-7} and v_{p-3} are monophonic eccentric dominated by the vertex v_{p-5}. It is clear that S is a minimum total monophonic eccentric dominating set of G, but the induced subgraph $\langle S\rangle$ is not connected. Therefore, we consider a set $S_1 = S \cup \{v_p\}$. Clearly, S_1 is a minimum connected total monophonic eccentric dominating set of G and so $\gamma_{ctme}(G) = \dfrac{p+1}{2} = \dfrac{p+l}{2}$.

Subcase (ii) $l = 3$.

Let $S = \{v_1, v_2, v_3, v_4; v_9, v_{10}, v_{11}, v_{12}; \ldots; v_{p-10}, v_{p-9}, v_{p-8}, v_{p-7}\} \cup \{v_{p-2}, v_{p-1}, v_p\}$. By an argument similar to Subcase (i), it is clear that S is a minimum connected total monophonic eccentric dominating set of G and so $\gamma_{ctme}(G) = \dfrac{p+3}{2} = \dfrac{p+l}{2}$.

Subcase (iii) $l = 5$.

Let $S = \{v_1, v_2, v_3, v_4; v_9, v_{10}, v_{11}, v_{12}; \ldots; v_{p-4}, v_{p-3}, v_{p-2}, v_{p-1}\} \cup \{v_p\}$. Then S is a minimum connected total monophonic eccentric dominating set of G and so $\gamma_{ctme}(G) = \dfrac{p+5}{2} = \dfrac{p+l}{2}$.

Subcase (iv) $l = 7$.

Let $S = \{v_1, v_2, v_3, v_4; v_9, v_{10}, v_{11}, v_{12}; \ldots; v_{p-6}, v_{p-5}, v_{p-4}, v_{p-3}\} \cup \{v_p\}$. Then S is a minimum connected total monophonic eccentric dominating set of G and so $\gamma_{ctme}(G) = \dfrac{p+3}{2}$. ■

Theorem 3.8 *For the Petersen graph G, $\gamma_{ctme}(G) = 4$.*

Proof. It can be easily verify that any vertex v in the Petersen graph G has monophonic eccentricity 4 and any non-adjacent vertex of v is a monophonic eccentric vertex of v. Let $S = \{x, y\}$. If x and y are adjacent vertices in G, then $d_m(x, y) = 1$ and so x is not a monophonic eccentric vertex of y and y is not a monophonic eccentric vertex of x. If x and y are non-adjacent vertices in G, then $N(x) \cap N(y) \neq \phi$. Let $z \in N(x) \cap N(y)$. Then $d_m(x, z) = d_m(y, z) = 1$ and so both x and y are not monophonic eccentric vertices of z in G. Hence in both cases S is not a total monophonic eccentric dominating set of G and so $\gamma_{tme}(G) \geq 3$. Let $S^{'} = \{u, v, w\}$, where u and v are adjacent vertices. Suppose $w \in N(u)$. Then u has no monophonic eccentric vertex in $S^{'}$. Similarly, if $w \in N(v)$, then v has no monophonic eccentric vertex in $S^{'}$. Therefore, $w \notin N(u) \cup N(v)$. It is clear that any vertex in $V - S^{'}$ is monophonic eccentric dominated by a vertex in $S^{'}$, u and v are monophonic eccentric dominated by w, and w is monophonic eccentric dominated by both u and v in G. Hence $S^{'}$ is a minimum total monophonic eccentric dominating set of G, but the induced subgraph $\left\langle S^{'} \right\rangle$ is not connected. Therefore, we consider a set $S^{''} = \{p, q, r, s\}$

such that p, q, r, s is an arbitrary path of length three and $s \notin N(p) \cup N(q)$. It is clear that S'' is a minimum connected total monophonic eccentric dominating set of G and so $\gamma_{ctme}(G) = 4$. ∎

4 Realization Results

In view of Theorem 2.3, we have the following realization result.

Theorem 4.1 *If n and p are integers such that $2 \leq n \leq p$ and $p - 5l - 6 \geq 0$, where $l = \left\lfloor \frac{n}{3} \right\rfloor$, then there exists a connected graph G of order p and the connected total monophonic eccentric domination number n.*

Proof. If $n = 2$, let G be a complete graph of order p. Then by Theorem 2.10, $\gamma_{ctme}(G) = 2$. Now, let $n \geq 3$ and let $l = \left\lfloor \frac{n}{3} \right\rfloor$. We construct a graph G with the desired properties as follows:

Case 1. $n = 3l$.

Let $C_i : u_i, v_i, w_i, x_i, y_i, u_i$ $(1 \leq i \leq l)$ be l copies of a cycle of order 5 and let $K_{1,p-5l-1}$ be a star with the cut-vertex x and the set of all end vertices $Z = \{z_1, z_2, \ldots, z_{p-5l-1}\}$. Let G be the graph obtained from the cycles C_i $(1 \leq i \leq l)$ and the star $K_{1,p-5l-1}$ by (i) joining every vertex in C_i $(1 \leq i \leq l)$ with the vertex x in $K_{1,p-5l-1}$, and (ii) joining the vertices v_l and x_l in C_l. Then the graph G has order p and it is shown in Figure 4.1.

It is clear that the monophonic eccentricity of any vertex in C_i $(1 \leq i \leq l-1)$ is 3, the monophonic eccentricity of any vertex in $\{u_l, w_l, y_l\}$ is 3, the monophonic eccentricity of any vertex in $Z \cup \{v_l, x_l\}$ is 2, and the monophonic eccentricity of the vertex x is 1. Therefore, any two non-adjacent vertices in C_i $(1 \leq i \leq l-1)$ are mutual monophonic eccentric vertices, any two non-adjacent vertices in $\{u_l, w_l, y_l\}$ are mutual monophonic eccentric vertices, every vertex in C_i $(1 \leq i \leq l)$ is a monophonic eccentric vertex of any vertex in Z, any vertex $x \in \{v_l, x_l\}$ is monophonic eccentric dominated by any non-adjacent vertex of x in G and every vertex in $V(G) - \{x\}$ is a monophonic eccentric vertex of x in G. Hence it is easy to verify that $S = \left(\bigcup_{i=1}^{l-1} \{u_i, w_i, y_i\} \right) \cup \{u_l, w_l, x\}$ is a minimum connected total monophonic eccentric dominating set of G and so $\gamma_{ctme}(G) = 3l = n$.

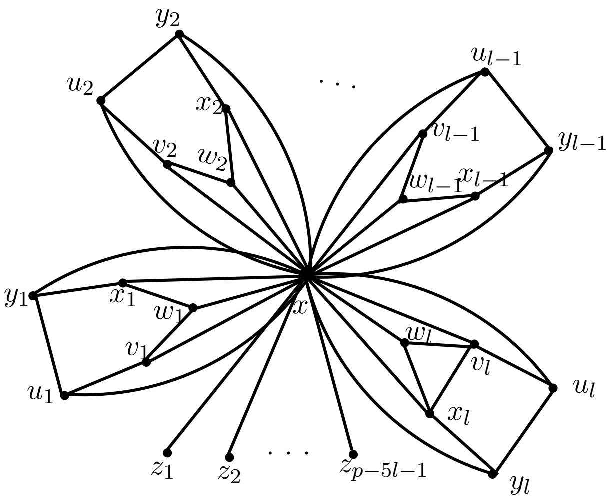

Figure 4.1: G

Case 2. $n = 3l + 1$.

Let $C_i : u_i, v_i, w_i, x_i, y_i, u_i$ $(1 \leq i \leq l)$ be l copies of a cycle of order 5 and let $K_{1,p-5l-1}$ be a star with the cut-vertex x and the set of all end vertices $Z = \{z_1, z_2, \ldots, z_{p-5l-1}\}$. Let G be the graph obtained from the cycles C_i $(1 \leq i \leq l)$ and the star $K_{1,p-5l-1}$ by joining every vertex in C_i $(1 \leq i \leq l)$ with the vertex x in $K_{1,p-5l-1}$. Then the graph G has order p and it is shown in Figure 4.2.

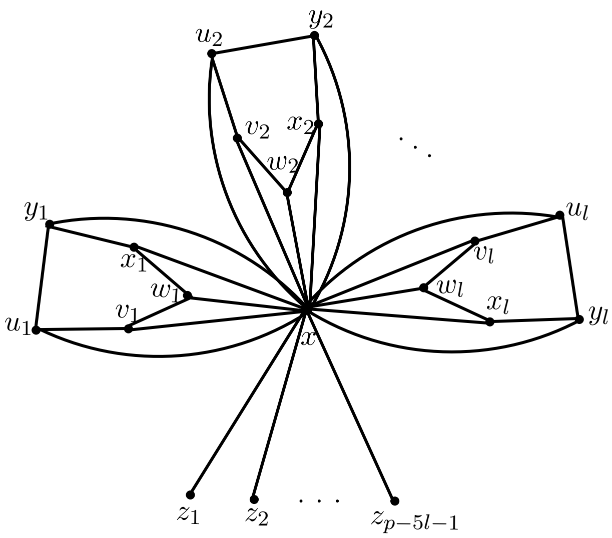

Figure 4.2: G

It is clear that the monophonic eccentricity of any vertex in C_i $(1 \leq i \leq l)$ is 3, the monophonic eccentricity of any vertex in Z is 2 and the monophonic eccentricity of the vertex x is 1. Then by an argument similar to Case 1, $S = \left(\bigcup_{i=1}^{l}\{u_i, w_i, y_i\}\right) \cup \{x\}$ is a minimum connected total monophonic eccentric dominating set of G and so $\gamma_{ctme}(G) = 3l + 1 = n$.

Case 3. $n = 3l + 2$.

Let $C_i : u_i, v_i, w_i, x_i, y_i, u_i$ $(1 \leq i \leq l + 1)$ be $l + 1$ copies of a cycle of order 5 and let $K_{1,p-5l-6}$ be a star with the cut-vertex x and the set of all end vertices $Z = \{z_1, z_2, \ldots, z_{p-5l-6}\}$. Let G be the graph obtained from the cycles C_i $(1 \leq i \leq l + 1)$ and the star $K_{1,p-5l-6}$ by (i) joining every vertex in C_i $(1 \leq i \leq l + 1)$ with the vertex x in $K_{1,p-5l-6}$, (ii) joining the vertices v_l and x_l in C_l, and (iii) joining the vertices v_{l+1} and x_{l+1} in C_{l+1}. Then the graph G has order p and it is shown in Figure 4.3.

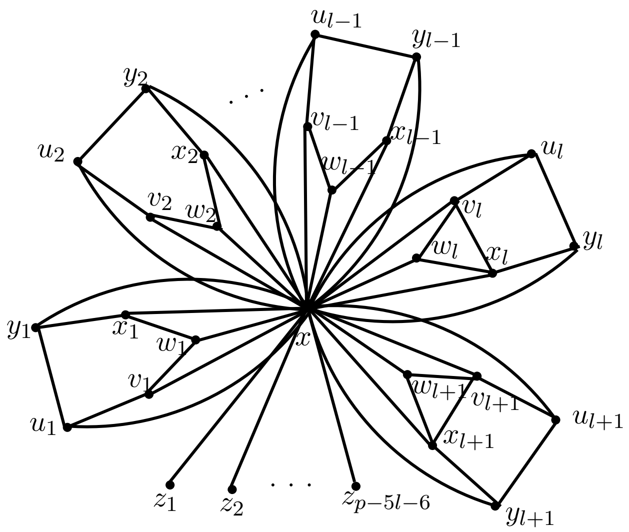

Figure 4.3: G

Then by an argument similar to Case 1, $S = \left(\bigcup_{i=1}^{l-1}\{u_i, w_i, y_i\}\right) \cup \{u_l, w_l, u_{l+1}, w_{l+1}, x\}$ is a minimum connected total monophonic eccentric dominating set of G and so $\gamma_{ctme}(G) = 3l + 2 = n$. ■

For any connected graph G, $rad_m(G) \leq diam_m(G)$. It is shown in [8] that every two positive integers a and b with $a \leq b$ are realizable as the monophonic radius and monophonic diameter, respectively, of some connected graph. This theorem can also be extended so that the connected total monophonic eccentric domination number can be prescribed when $rad_m(G) + 2 < diam_m(G)$.

Theorem 4.2 *For any three positive integers r, d and $n \geq 6$ with $r + 2 < d$, there exists a connected graph G such that $rad_m(G) = r$, $diam_m(G) = d$ and $\gamma_{ctme}(G) = n$.*

Proof. We prove this theorem by considering two cases.

Case 1. $r = 1$.

Subcase 1. n is even.

let $P_i : w_{i,1}, w_{i,2}, w_{i,3}, w_{i,4}$ $(1 \leq i \leq \frac{n-6}{2})$ be $\dfrac{n-6}{2}$ copies of a path of order 4, let $Q : u_1, u_2, \ldots, u_{d+1}$ be a path of order $d+1$, and let $C : v_1, v_2, \ldots, v_6, v_1$ be a cycle of order 6. Let G be the graph obtained from the paths P_i $(1 \leq i \leq \frac{n-6}{2})$, the path Q and the cycle C by (i) joining every vertex in P_i $(1 \leq i \leq \frac{n-6}{2})$ with the vertex v_1 in C, (ii) joining every vertex in Q with the vertex v_1 in C, (iii) joining the vertices v_2 and v_6 in C, and (iv) joining each vertex v_i $(3 \leq i \leq 5)$ in C with the vertex v_1 in C. The graph G is shown in Figure 4.4.

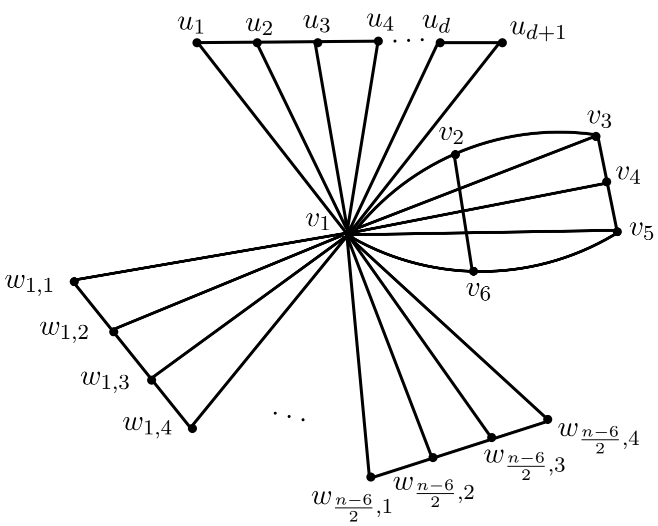

Figure 4.4: G

It is easily verified that $1 \leq e_m(x) \leq d$ for any vertex x in G, $e_m(v_1) = 1$ and $e_m(u_1) = d$. Then $rad_m(G) = 1$ and $diam_m(G) = d$. Also, the vertex u_i $(1 \leq i \leq \lceil \frac{d+1}{2} \rceil)$ is monophonic eccentric dominated by the vertex u_{d+1} and the vertex u_i $(\lceil \frac{d+1}{2} \rceil + 1 \leq i \leq d+1)$ is monophonic eccentric dominated by the vertex u_1, the vertex v_1 is monophonic eccentric dominated by any vertex in $V(G) - \{v_1\}$, the vertices v_4 and v_5 are monophonic eccentric dominated by the vertex v_2, the vertices v_2 and v_3 are monophonic eccentric dominated by the vertex v_5, the vertex v_6 is monophonic eccentric dominated by the vertex v_3, the vertices $w_{i,1}$ and $w_{i,2}$ $(1 \leq i \leq \frac{n-6}{2})$ are monophonic eccentric dominated by the vertex $w_{i,4}$, and the vertices $w_{i,3}$ and $w_{i,4}$ $(1 \leq i \leq \frac{n-6}{2})$ are monophonic eccentric dominated by the vertex $w_{i,1}$. Hence $S = \left(\bigcup_{i=1}^{\frac{n-6}{2}} \{w_{i,1}, w_{i,4}\}\right) \cup \{u_1, u_{d+1}, v_2, v_3, v_5\}$ is a total monophonic eccentric dominating set of G and its induced subgraph $\langle S \rangle$ is not connected. It is clear that $S' = S \cup \{v_1\}$ is a minimum connected total monophonic eccentric dominating set of G and so $\gamma_{ctme}(G) = n$.

Subcase 2. n is odd.

let $P_i : w_{i,1}, w_{i,2}, w_{i,3}, w_{i,4}$ $(1 \leq i \leq \frac{n-3}{2})$ be $\dfrac{n-3}{2}$ copies of a path of order 4 and let $Q : u_1, u_2, \ldots, u_{d+1}$ be a path of order $d+1$. Let G be the graph obtained from the paths P_i $(1 \leq i \leq \frac{n-3}{2})$ and the path Q by (i) joining every vertex in P_i $(1 \leq i \leq \frac{n-3}{2})$ with the new vertex v, and (ii) joining every vertex in Q with the vertex v. The graph G is shown in Figure 4.5.

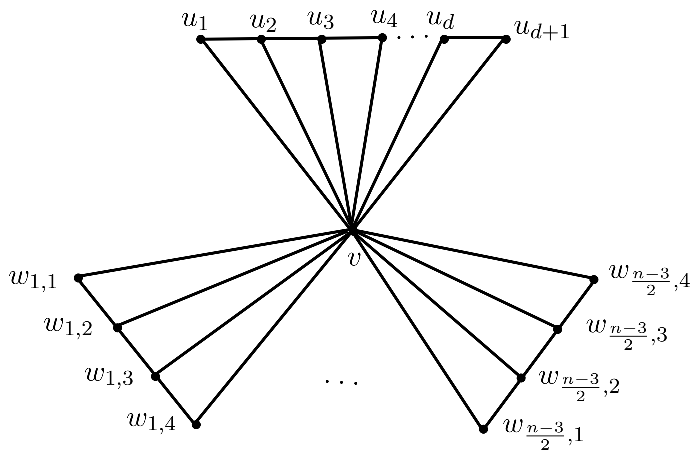

Figure 4.5: G

It is easily verified that $1 \leq e_m(x) \leq d$ for any vertex x in G, $e_m(v) = 1$ and $e_m(u_1) = d$. Then $rad_m(G) = 1$ and $diam_m(G) = d$. Also, the vertex u_i $(1 \leq i \leq \lceil \frac{d+1}{2} \rceil)$ is monophonic eccentric dominated by the vertex u_{d+1} and the vertex u_i $(\lceil \frac{d+1}{2} \rceil + 1 \leq i \leq d+1)$ is monophonic eccentric dominated by the vertex u_1, the vertex v is monophonic eccentric dominated by any vertex in $V(G) - \{v\}$, the vertices $w_{i,1}$ and $w_{i,2}$ $(1 \leq i \leq \frac{n-3}{2})$ are monophonic eccentric dominated by the vertex $w_{i,4}$, and the vertices $w_{i,3}$ and $w_{i,4}$ $(1 \leq i \leq \frac{n-3}{2})$ are monophonic eccentric dominated by the vertex $w_{i,1}$. Hence $S =$

$\left(\bigcup_{i=1}^{\frac{n-3}{2}} \{w_{i,1}, w_{i,4}\}\right) \cup \{u_1, u_{d+1}\}$ is a total monophonic eccentric dominating set of G and its induced subgraph $\langle S \rangle$ is not connected. It is clear that $S' = S \cup \{v\}$ is a minimum connected total monophonic eccentric dominating set of G and so $\gamma_{ctme}(G) = n$.

Case 2. $r > 1$.

Subcase 1. n is even.

Let $P_i : w_{i,1}, w_{i,2}, \ldots, w_{i,d+1}$ $(1 \le i \le \frac{n}{2} - 2)$ be $\dfrac{n}{2} - 2$ copies of a path of order $d+1$, let $P : v_1, v_2, \ldots, v_{r+1}$ be a path of order $r+1$ and let $Q : u_1, u_2, \ldots, u_d$ be a path of order d. Let G be the graph obtained from the paths P_i $(1 \le i \le \frac{n}{2} - 2)$, the path P and the path Q by (i) joining every vertex in P_i $(1 \le i \le \frac{n}{2} - 2)$ with the vertex v_1 in P, (ii) joining every vertex in Q with the vertex v_1 in P, and (iii) joining the vertices v_j $(2 \le j \le r+1)$ in P with the vertex u_d in Q. The graph G is shown in Figure 4.6.

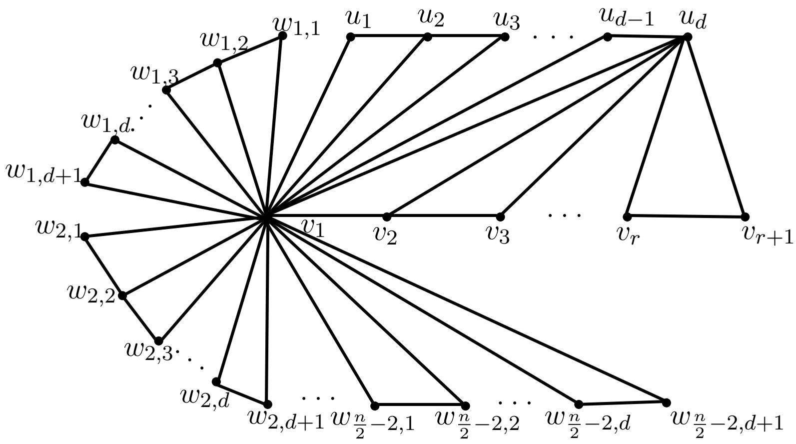

Figure 4.6: G

It is easily verified that $r \le e_m(x) \le d$ for any vertex x in G, $e_m(v_1) = r$ and $e_m(u_1) = d$. Then $rad_m(G) = r$ and $diam_m(G) = d$. Also, the vertex v_1 is monophonic eccentric dominated by the vertex v_{r+1} and the vertex v_j $(2 \le j \le r+1)$ is monophonic eccentric dominated by the vertex u_1. If $r+3 \le d \le 2r$, the vertex u_i $(1 \le i \le r+1)$ is monophonic eccentric dominated by the vertex v_{r+1}, the vertex u_i $(r+2 \le i \le d)$ is monophonic eccentric dominated by the vertex u_1, the vertex $w_{i,j}$ $(1 \le i \le \frac{n}{2} - 2, 1 \le j \le d-r)$ is monophonic eccentric dominated by the vertex $w_{i,d+1}$, the vertex $w_{i,j}$ $(1 \le i \le \frac{n}{2} - 2, d-r+1 \le j \le r+1)$ is monophonic eccentric dominated by the vertex v_{r+1} and the vertex $w_{i,j}$

$(1 \le i \le \frac{n}{2} - 2, r + 2 \le j \le d + 1)$ is monophonic eccentric dominated by the vertex $w_{i,1}$. If $d > 2r$, the vertex u_i $(1 \le i \le \lceil \frac{d}{2} \rceil)$ is monophonic eccentric dominated by the vertex v_{r+1}, the vertex u_i $(\lceil \frac{d}{2} \rceil + 1 \le i \le d)$ is monophonic eccentric dominated by the vertex u_1, the vertex $w_{i,j}$ $(1 \le i \le \frac{n}{2} - 2, 1 \le j \le \lceil \frac{d}{2} \rceil)$ is monophonic eccentric dominated by the vertex $w_{i,d+1}$ and the vertex $w_{i,j}$ $(1 \le i \le \frac{n}{2} - 2, \lceil \frac{d}{2} \rceil + 1 \le j \le d + 1)$ is monophonic eccentric dominated by the vertex $w_{i,1}$. Hence $S = \left(\bigcup_{i=1}^{\frac{n}{2}-2} \{w_{i,1}, w_{i,d+1}\} \right) \cup \{u_1, v_{r+1}\}$ is a total monophonic eccentric dominating set of G and its induced subgraph $\langle S \rangle$ is not connected. It is clear that $S' = S \cup \{v_1, u_d\}$ is a minimum connected total monophonic eccentric dominating set of G and so $\gamma_{ctme}(G) = n$.

Subcase 2. n is odd.

Let $P_i : w_{i,1}, w_{i,2}, \ldots, w_{i,d+1}$ $(1 \le i \le \frac{n-5}{2})$ be $\dfrac{n-5}{2}$ copies of a path of order $d + 1$, let $P : v_1, v_2, \ldots, v_{r+2}$ be a path of order $r + 2$ and let $Q : u_1, u_2, \ldots, u_d$ be a path of order d. Let G be the graph obtained from the paths P_i $(1 \le i \le \frac{n-5}{2})$, the path P and the path Q by (i) joining every vertex in P_i $(1 \le i \le \frac{n-5}{2})$ with the vertex v_1 in P, (ii) joining every vertex in Q with the vertex v_2 in P, and (iii) joining the vertices v_j $(3 \le j \le r + 2)$ in P with the vertex u_d in Q. The graph G is shown in Figure 4.7.

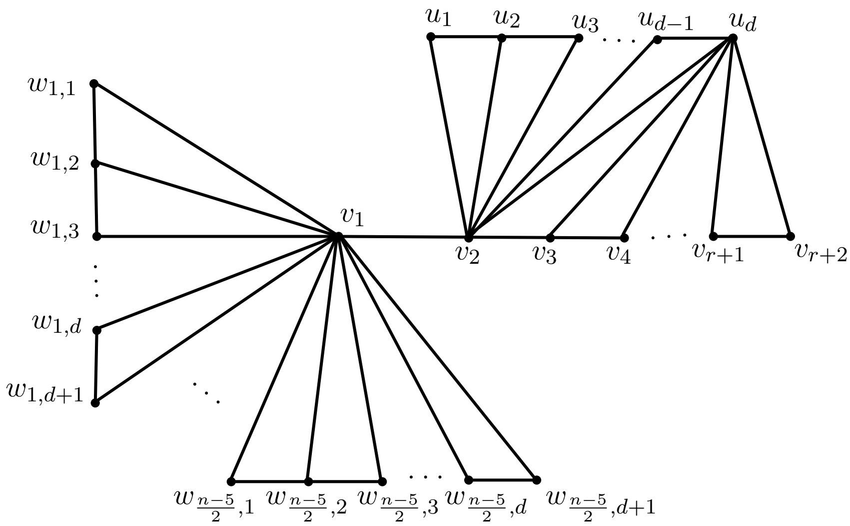

Figure 4.7: G

It is easily verified that $r \le e_m(x) \le d$ for any vertex x in G, $e_m(v_2) = r$ and $e_m(u_1) = d$. Then $rad_m(G) = r$ and $diam_m(G) = d$. Also, the vertex v_j $(3 \le j \le r + 2)$ is monophonic eccentric dominated by the vertex u_1 and the vertex v_j $(j = 1, 2)$ is monophonic eccentric dominated by the vertex v_{r+2}. If $r+3 \leq d \leq 2r$, the vertex u_i $(1 \leq i \leq r+1)$ is monophonic eccentric dominated by the vertex v_{r+2}, the vertex u_i $(r+2 \leq i \leq d)$ is monophonic eccentric dominated by the vertex u_1, the vertex $w_{i,j}$ $(1 \leq i \leq \frac{n-5}{2}, 1 \leq j \leq d-r)$ is monophonic eccentric dominated by the vertex $w_{i,d+1}$, the vertex $w_{i,j}$ $(1 \leq i \leq \frac{n-5}{2}, d-r+1 \leq j \leq r+1)$ is monophonic eccentric dominated by the vertex v_{r+2} and the vertex $w_{i,j}$ $(1 \leq i \leq \frac{n-5}{2}, r+2 \leq j \leq d+1)$ is monophonic eccentric dominated by the vertex $w_{i,1}$. If $d > 2r$, the vertex u_i $(1 \leq i \leq \lceil \frac{d}{2} \rceil)$ is monophonic eccentric dominated by the vertex v_{r+2}, the vertex u_i $(\lceil \frac{d}{2} \rceil + 1 \leq i \leq d)$ is monophonic eccentric dominated by the vertex u_1, the vertex $w_{i,j}$ $(1 \leq i \leq \frac{n-5}{2}, 1 \leq j \leq \lceil \frac{d}{2} \rceil)$ is monophonic eccentric dominated by the vertex $w_{i,d+1}$ and the vertex $w_{i,j}$ $(1 \leq i \leq \frac{n-5}{2}, \lceil \frac{d}{2} \rceil + 1 \leq j \leq d+1)$ is monophonic eccentric dominated by the vertex $w_{i,1}$. It is clear that $S = \left(\bigcup_{i=1}^{\frac{n-5}{2}} \{w_{i,1}, w_{i,d+1}\} \right) \cup \{v_1, v_2, v_{r+2}, u_1, u_d\}$ is a minimum connected total monophonic eccentric dominating set of G and so $\gamma_{ctme}(G) = n$. ■

Problem 4.3 *For any three positive integers r, d and $n \geq 6$ with $d = r, r+1$ or $r+2$, does there exist a connected graph G with $rad_m(G) = r$, $diam_m(G) = d$ and $\gamma_{ctme}(G) = n$?*

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