Some Remarks on Skula Spaces

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https://github.com/kourgeorge/arxiv-style

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This lecture is a survey of a joint work with Taras Banack and Wiesław Kubiś entitled


This paper is the continuous of the work well-generated Boolean algebras, in a topological way, started in [3].

Stone duality. If $X$ is a compact and 0-dimensional space then the set $\text{Clop}(X)$ of closed and open (clopen) subsets of $X$ is a Boolean algebra generating the topology of $X$. Conversely any Boolean algebra $B$ is the algebra of clopen subsets of the compact and 0-dimensional space $\text{Ult}(B) \subseteq \{0,1\}^B$. By duality, we have the following result.

**Theorem 1.** (§2.3 in [3]) The space $\text{Ult}(B)$ of a Boolean algebra $B$ is Skula if and only if $B$ is well-generated. That is, by the definition: $B$ has a well-founded sublattice generating $B$. □

1. Skula spaces.

For a topological space $X$, we say that a family $\mathcal{U} := \{U_x : x \in X\}$ is a clopen selector if each $U_x$ is a closed and open (clopen) subset of $X$ and if $\mathcal{U}$ satisfies:

1. $x \in U_x$ for every $x \in X$ and
2. the relation “$x \leq^\mathcal{U} y$ if and only if $x \neq y$ and $x \in U_y$” is irreflexive and transitive. ■

Therefore a clopen selector $\mathcal{U} := \{U_x : x \in X\}$ for $X$ induces a partial order relation $\leq^\mathcal{U}$ on $X$, defined by $x \leq^\mathcal{U} y$ if and only if $U_x \subseteq U_y$.

Hence $U_x := \{y \in X : y \leq^\mathcal{U} x\}$ (also denoted by $\downarrow x$) is a clopen principal ideal of $X$ for any $x \in X$ for the order $\leq^\mathcal{U}$.

Remark. The set of $U_x$’s and their complements generate the topology whenever $X$ is compact.

A space $X$ is Skula if $X$ is a Hausdorff compact space and has a clopen selector. ■

**Theorem 2.** [2] Let $\mathcal{U} := \{U_x : x \in X\}$ be a clopen selector for a Skula space $X$. Then

- Every (nonempty) closed initial subset of $X$ is a finite union of $U_x$’s (notice that $X$ and the $U_x$’s are compact clopen sets).
- In particular for distinct $U_x$ and $U_y$ in $\mathcal{U}$, $U_x \cap U_y$ is a finite union of $U_z$’s.

1. $\mathcal{U}$ is well-founded. Therefore $\langle X, \subseteq \rangle$ has a well-founded rank: $\rk\mathcal{U}(x) = \sup\{\rk\mathcal{U}(y) : y < x\}$. So $\rk\mathcal{U}(x) = 0$ if and only if $x$ is minimal, i.e. $U_x = \{x\}$. Moreover $\rk\mathcal{U}(X) := \sup_{x \in X} \rk\mathcal{U}(x)$.

2. $X$ is scattered, i.e. every nonempty subset of $\text{Ult}(B)$ has an isolated point (for the induced topology). Therefore we can define the Cantor-Bendixson height (htcb$X$) of $x \in X$. For instance $\text{htcb}_X(x) = 0$ if and only if $x$ is isolated in $X$. Moreover $\text{htcb}_X(X) := \sup_{x \in X} \text{htcb}_X(x)$. □
Since \( U_x = \{ y \in X : \forall x \leq y \} \) is an initial and clopen subset of \( X \), we have \( \text{ht}_{CB}(U_x) = \text{ht}_{CB}(x) \leq \text{rk}_{WF}(x) = \text{rk}_{WF}(U_x) \) for any \( x \in X \), and so \( \text{ht}_{CB}(X) \leq \text{rk}_{WF}(X) \).

To a Skula space \( X \) we can associate its Vietoris hyperspace \( H(X) \), that is a “free join-semilattice over \( X \) in the category of continuous join semilattice spaces”.

We define the **Vietoris hyperspace** \( H(X) \) over \( X \) as follows:

- \( H(X) \) is the set of all nonempty closed initial subsets of \( \langle X, \leq \rangle \).
- For \( F, G \in H(X) \), we set \( F \leq G \) if and only if \( F \subseteq G \).
- The topology \( \tau \) on \( H(X) \) is the topology generated by the sets \( U^+ := \{ K \in H(X) : K \subseteq U \} \) and \( V^- := \{ K \in H(X) : K \cap V \neq \emptyset \} \)

where \( U \) and \( V \) are any clopen initial subsets and clopen final subsets in \( X \), respectively.

**Theorem 3.** [2] Let \( X \) be a Skula space. Then \( H(X) \) is a Skula space and

1. \( (A, B) \mapsto A \lor B := A \cup B \) is a continuous semilattice operation on \( H(X) \).
2. \( X \) is topologically embeddable in \( H(X) \) by the increasing continuous map \( \eta : x \mapsto \downarrow x := U_x \).
3. The join semilattice generated by \( \eta(X) \) in \( H(X) \) is topologically dense in \( H(X) \).

**Theorem 4.** [2] Let \( X \) be a Skula space and let \( \mathcal{U} \) be a clopen selector for \( X \). Then \( \text{ht}_{CB}(X) \leq \text{rk}_{WF}(X) < \omega^{\text{ht}_{CB}(X)+1} \) and \( \text{rk}_{WF}(H(X)) \leq \omega^{\text{rk}_{WF}(X)} \).

2. **Canonical Skula spaces.**

A space \( X \) is a **canonical Skula space** if \( X \) has a clopen selector \( \mathcal{U} := \{ U_x : x \in X \} \) satisfying one of the following equivalent properties for each \( U_x \in \mathcal{U} \):

1. There is an ordinal \( \alpha \) such that the \( \alpha^{\text{th}} \)-Cantor-Bendixson derivative \( D^\alpha(U_x) \) of \( U_x \) is the singleton \( \{ x \} \).
2. \( \text{rk}_{WF}(U_x) = \text{ht}_{CB}(U_x) \) and \( U_x \) is unitary (meaning that \( D^\beta(U_x) \) is a singleton for some \( \beta \)).

**Examples.** Every continuous image of a compact ordinal space \( \alpha + 1 \) (with the order topology) is canonically Skula. The class of canonically Skula spaces is closed under finite product.

**Theorem 5.** [2] Let \( X \) be a canonical Skula space. Then \( H(X) \) is a canonical Skula space.

Moreover we can compute \( \text{ht}_{CB}(X)(V) = \text{rk}_{WF}(X)(V) \) for every \( V \in H(X) \).

**Remark.**
1. There is a compact and 0-dimensional space which is not Skula [3].
2. There is a Skula space which is not canonically Skula [3].

3. **Poset spaces.**

For a partially ordered set (poset) \( P \) we denote by \( IS(P) \) the set of initial subsets of \( P \) endowed with the pointwise topology. So \( IS(P) \), as compact subspace of \( \{0, 1\}^P \), is compact and 0-dimensional, and we can see \( H(P) := IS(P) \) as the “Vietoris hyperspace” of the poset \( P \).

**Proposition 6.** [1] Theorems 1,3] Let \( P \) be a poset. The space IS\((P)\) is Skula if and only if

1. \( P \) is a narrow, i.e. any antichain is finite, and
2. \( P \) is order-scattered, i.e. does not contain a copy of the rationals chain \( \mathbb{Q} \).

Recall that a well-quasi ordering (wqo) is a narrow and well-founded poset. From the above result, M. Pouzet asks for the following question.

**Question** (M. Pouzet). Let \( P \) be a well-quasi ordering. Is IS\((P)\) canonically Skula?

We do not know the answer of this question even if \( P \) is covered by finitely many well-orderings.

**References**