



Riemann Hypothesis on Grönwall's Function

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Abstract

Grönwall's function \mathbf{G} is defined for all natural numbers $n > 1$ by $\mathbf{G}(n) = \frac{\sigma(n)}{n \cdot \log \log n}$ where $\sigma(n)$ is the sum of the divisors of n and \log is the natural logarithm. We require the properties of colossally abundant numbers in relation to the Grönwall's function \mathbf{G} . There are several statements equivalent to the famous Riemann hypothesis. We state that the Riemann hypothesis is true if and only if there exist infinitely many pairs (N, N') of consecutive colossally abundant numbers $N < N'$ such that $\mathbf{G}(N) < \mathbf{G}(N')$. Using this new criterion, we prove that the Riemann hypothesis is true.

Keywords: Riemann hypothesis, Colossally abundant numbers, Extremely abundant numbers, Arithmetic functions

MSC Classification: 11M26 , 11A25

1 Introduction

As usual $\sigma(n)$ is the sum-of-divisors function of n

$$\sum_{d|n} d,$$

where $d | n$ means the integer d divides n . In 1997, Ramanujan's old notes were published where it was defined the generalized highly composite numbers, which include the superabundant and colossally abundant numbers [1]. A natural number n is called superabundant precisely when, for all natural

A Millennium Prize Problem

2 The Riemann hypothesis

numbers $m < n$

$$\frac{\sigma(m)}{m} < \frac{\sigma(n)}{n}.$$

A number n is said to be colossally abundant if, for some $\epsilon > 0$,

$$\frac{\sigma(n)}{n^{1+\epsilon}} \geq \frac{\sigma(m)}{m^{1+\epsilon}} \text{ for } (m > 1).$$

Every colossally abundant number is superabundant [2]. In 1913, Grönwall studied the function $G(n) = \frac{\sigma(n)}{n \cdot \log \log n}$ for all natural numbers $n > 1$, [3]. We have the Grönwall's Theorem:

Proposition 1

$$\limsup_{n \rightarrow \infty} G(n) = e^\gamma$$

where $\gamma \approx 0.57721$ is the Euler-Mascheroni constant [3].

Next, we have the following Robin's results:

Proposition 2 Let $3 \leq N < N'$ be two consecutive colossally abundant numbers, then

$$G(n) \leq \text{Max}(G(N), G(N'))$$

when satisfying $N < n < N'$ [4, Proposition 1 pp. 192].

Proposition 3 There are infinitely many colossally abundant numbers N such that $G(N) > e^\gamma$ when the Riemann hypothesis is false [4, Proposition 1 pp. 204]. There exist infinitely many colossally abundant numbers N such that $G(N) < e^\gamma$ [4, Theorem 1 pp. 188], [4, Proposition 1 pp. 204].

There are champion numbers (i.e. left to right maxima) of the function $n \mapsto G(n)$:

$$G(m) < G(n)$$

for all natural numbers $10080 \leq m < n$. A positive integer n is extremely abundant if either $n = 10080$, or $n > 10080$ is a champion number of the function $n \mapsto G(n)$. In 1859, Bernhard Riemann proposed his hypothesis [5]. Several analogues of the Riemann hypothesis have already been proved [5].

Proposition 4 The Riemann hypothesis is true if and only if there exist infinitely many extremely abundant numbers [6, Theorem 7 pp. 6].

We use the following property for the extremely abundant numbers:

Proposition 5 *Let $N < N'$ be two consecutive colossally abundant numbers and $n > 10080$ is some extremely abundant number, then N' is also extremely abundant when satisfying $N < n < N'$ [6, Lemma 21 pp. 12].*

This is our main theorem

Theorem 1 *The Riemann hypothesis is true if and only if there exist infinitely many pairs (N, N') of consecutive colossally abundant numbers $N < N'$ such that $G(N) < G(N')$.*

Putting all together yields a new criterion for the Riemann hypothesis. Now, we can conclude with the following result:

Theorem 2 *The Riemann hypothesis is true.*

Proof Note that, for all $u > 0$ [7, pp. 254]:

$$\lim_{n \rightarrow \infty} \frac{\sigma(n)}{n \cdot (\log n)^u} = 0$$

and so, there are infinitely many pairs (N, N') of large enough consecutive colossally abundant numbers $N < N'$ such that

$$\left(\frac{\log N}{\log N'} \right)^u \geq \frac{\left(\frac{\sigma(N)}{N} \right)}{\left(\frac{\sigma(N')}{N'} \right)}$$

for some $u \geq 1$. The inequality $G(N) < G(N')$ is equivalent to say

$$\frac{\log \log N}{\log \log N'} > \frac{\left(\frac{\sigma(N)}{N} \right)}{\left(\frac{\sigma(N')}{N'} \right)}.$$

In addition, we have

$$\frac{\log \log N}{\log \log N'} > \frac{\log N}{\log N'}$$

since $e < \log N < \log N'$ by Lemma 1. So, the following inequality

$$\left(\frac{\log \log N}{\log \log N'} \right)^u > \left(\frac{\log N}{\log N'} \right)^u \geq \frac{\left(\frac{\sigma(N)}{N} \right)}{\left(\frac{\sigma(N')}{N'} \right)}$$

holds. Hence, it is enough to show that

$$\frac{\log \log N}{\log \log N'} \geq \left(\frac{\log \log N}{\log \log N'} \right)^u$$

which is satisfied since $u \geq 1$. Consequently, we obtain that $G(N) < G(N')$ holds anyway. In this way, there are infinitely many pairs (N, N') of consecutive colossally abundant numbers $N < N'$ such that $G(N) < G(N')$. Finally, the proof is complete by Theorem 1. \square

2 Central Lemma

Lemma 1 For two real numbers $y > x > e$:

$$\frac{y}{x} > \frac{\log y}{\log x}.$$

Proof We have $y = x + \varepsilon$ for $\varepsilon > 0$. We obtain that

$$\begin{aligned} \frac{\log y}{\log x} &= \frac{\log(x + \varepsilon)}{\log x} \\ &= \frac{\log\left(x \cdot \left(1 + \frac{\varepsilon}{x}\right)\right)}{\log x} \\ &= \frac{\log x + \log\left(1 + \frac{\varepsilon}{x}\right)}{\log x} \\ &= 1 + \frac{\log\left(1 + \frac{\varepsilon}{x}\right)}{\log x} \end{aligned}$$

and

$$\begin{aligned} \frac{y}{x} &= \frac{x + \varepsilon}{x} \\ &= 1 + \frac{\varepsilon}{x}. \end{aligned}$$

We need to show that

$$\left(1 + \frac{\log\left(1 + \frac{\varepsilon}{x}\right)}{\log x}\right) < \left(1 + \frac{\varepsilon}{x}\right)$$

which is equivalent to

$$\left(1 + \frac{\varepsilon}{x \cdot \log x}\right) < \left(1 + \frac{\varepsilon}{x}\right)$$

using the well-known inequality $\log(1 + x) \leq x$ for $x > 0$. For $x > e$, we have

$$\frac{\varepsilon}{x} > \frac{\varepsilon}{x \cdot \log x}.$$

In conclusion, the inequality

$$\frac{y}{x} > \frac{\log y}{\log x}$$

holds on condition that $y > x > e$. □

3 Proof of Theorem 1

Proof Suppose there are not infinitely many pairs (N, N') of consecutive colossally abundant numbers $N < N'$ such that $G(N) < G(N')$. This implies that the inequality $G(N) \geq G(N')$ always holds for a sufficiently large N when $N < N'$ is a pair of consecutive colossally abundant numbers. That would mean the existence of a single colossally abundant number $N'' \geq 10080$ such that $G(n) \leq G(N'')$ for all natural numbers $n > N''$ according to Proposition 2. Certainly, the existence of such single colossally abundant number N'' is because of the Grönwall's function G would become decreasing on colossally abundant numbers starting from some single value. We use the Proposition 5 to reveal that under these preconditions, then there are not infinitely many extremely abundant numbers. This implies that the Riemann

hypothesis is false as a consequence of Proposition 4. By contraposition, if the Riemann hypothesis is true, then there exist infinitely many pairs (N, N') of consecutive colossally abundant numbers $N < N'$ such that $G(N) < G(N')$.

Suppose that there exist infinitely many pairs (N, N') of consecutive colossally abundant numbers $N < N'$ such that $G(N) < G(N')$. On the one hand, let's assume from these infinitely many pairs (N, N') of consecutive colossally abundant numbers $N < N'$ such that $G(N) < G(N')$, then there could be only a finite amount of these N' such that $e^\gamma < G(N')$. Thus, we deduce there could be only a finite amount of colossally abundant numbers N'' such that $e^\gamma < G(N'')$. However, when the Riemann hypothesis is false, then there are infinitely many colossally abundant numbers N'' such that $e^\gamma < G(N'')$ by Proposition 3. On the other hand, let's assume from these infinitely many pairs (N, N') of consecutive colossally abundant numbers $N < N'$ such that $G(N) < G(N')$, then there could be an infinite amount of these N' such that $e^\gamma < G(N')$.

Based on this opposite assumption, it could appear the possible scenarios:

- there would be an infinite increasing subsequence of colossally abundant numbers N_i such that $e^\gamma < G(N_i)$ and $G(N_i) < G(N_{i+1})$,
- or there would be a colossally abundant number N'' such that for all colossally abundant numbers $N > N''$ we have $e^\gamma \leq G(N)$,
- or there would be infinitely many pairs (N, N') of consecutive colossally abundant numbers $N < N'$ such that $G(N) < e^\gamma < G(N')$.

However, it cannot exist an infinite increasing subsequence of colossally abundant numbers N_i such that $e^\gamma < G(N_i)$ and $G(N_i) < G(N_{i+1})$, by Proposition 1 and the properties of limit superior. Moreover, there cannot be a colossally abundant number N'' such that for all colossally abundant numbers $N > N''$ we have $e^\gamma \leq G(N)$, since this implies that there are not infinitely many colossally abundant numbers N''' such that $G(N''') < e^\gamma$ which is a contradiction by Proposition 3.

Furthermore, there are not infinitely many pairs (N, N') of consecutive colossally abundant numbers $N < N'$ such that $G(N) < e^\gamma < G(N')$. Certainly, we deduce that

$$\frac{G(N')}{G(N)} = \left(1 + \frac{G(N') - G(N)}{G(N)}\right) > \left(1 + \frac{G(N') - G(N)}{e^\gamma}\right) > \left(2 - \frac{G(N)}{e^\gamma}\right).$$

We obtain a contradiction since the inequality $\frac{G(N')}{G(N)} + \frac{G(N)}{e^\gamma} > 2$ tends to be unsatisfied as long as N goes to infinity when $G(N) < e^\gamma < G(N')$ by Proposition 1. Therefore, the Riemann hypothesis would be true when there exist infinitely many pairs (N, N') of consecutive colossally abundant numbers $N < N'$ such that $G(N) < G(N')$. □

4 Conclusions

Practical uses of the Riemann hypothesis include many propositions that are known to be true under the Riemann hypothesis and some that can be shown to be equivalent to the Riemann hypothesis. Indeed, the Riemann hypothesis is closely related to various mathematical topics such as the distribution of primes, the growth of arithmetic functions, the Lindelöf hypothesis, the

Large Prime Gap Conjecture, etc. Certainly, a proof of the Riemann hypothesis could spur considerable advances in many mathematical areas, such as number theory and pure mathematics in general.

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