Plücker'S Conoid, Hyperboloids of Revolution, and Orthogonal Hyperbolic Paraboloids

Hellmuth Stachel

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Hellmuth Stachel
Institute of Discrete Mathematics and Geometry, Vienna University of Technology
Professor emeritus, stachel@dmg.tuwien.ac.at

ABSTRACT

Plücker's conoid $\mathcal{C}$, also known under the name cylindroid, is a ruled surface of degree three with a finite double line and a director line at infinity. The following two properties of $\mathcal{C}$ play a major role in the geometric literature:

The bisector of two skew lines ℓ_1, ℓ_2 in the Euclidean 3-space, i.e., the locus of points at equal distance to ℓ_1 and ℓ_2, is an orthogonal hyperbolic paraboloid $\mathcal{P}$. All generators of $\mathcal{P}$ are axes of one-sheeted hyperboloids of revolution $\mathcal{H}$ which pass through ℓ_1 and ℓ_2. Conversely, the locus of pairs of skew lines ℓ_1, ℓ_2 for which a given orthogonal hyperbolic paraboloid $\mathcal{P}$ is the bisector, is a Plücker conoid $\mathcal{C}$.

In spatial kinematics, Plücker's conoid $\mathcal{C}$ is well-known as the locus of axes ℓ_{12} of the relative screw motion for two wheels which rotate about fixed skew axes ℓ_1 and ℓ_2 with constant velocities. The axodes of the relative screw motion are one-sheeted hyperboloids of revolution $\mathcal{H}_1$, $\mathcal{H}_2$ with mutual contact along ℓ_{12}. The common surface normals along ℓ_{12} form an orthogonal hyperboloid paraboloid $\mathcal{P}$ passing through the axes ℓ_1 and ℓ_2.

The underlying paper aims to discuss these two main properties. It seems that there is no close relation between them though both deal with Plücker's conoid, orthogonal hyperbolic paraboloids, and hyperboloids of revolution – however in different ways.

Keywords: Plücker's conoid; cylindroids; bisector; one-sheeted hyperboloid of revolution; orthogonal hyperbolic paraboloid

1. PLÜCKER'S CONOID

Plücker's conoid $\mathcal{C}$, also known under the name *cylindroid*, is a ruled surface of degree three with a finite double line and a director line at infinity. Using cylinder coordinates (r, φ, z), the conoid can be given by

$$z = c \sin 2\varphi \tag{1}$$

with a constant $c \in \mathbb{R}_{>0}$. All generators of $\mathcal{C}$ are parallel to the $[x,y]$-plane. The z-axis is the double line of $\mathcal{C}$ and an axis of symmetry. The conoid passes through the x- and y-axis. These two lines can be called *central generators* of $\mathcal{C}$ since both are axes of symmetry of $\mathcal{C}$, too. The Plücker conoid $\mathcal{C}$ is the trajectory of the x-axis under a motion composed from a rotation about the z-axis and a harmonic oscillation with double frequency along the z-axis (Wunderlich, 1967, p. 37).

The substitution $x = r \cos \varphi$ and $y = r \sin \varphi$ in (1) yields the Cartesian equation

$$(x^2 + y^2)\, z - 2cxy = 0, \tag{2}$$

which reveals that reflections in the planes $x \pm y = 0$ map $\mathcal{C}$ onto itself. The origin O is called the *center* of $\mathcal{C}$.

The right cylinder $x^2 + y^2 = R^2$ intersects the Plücker conoid $\mathcal{C}$ along a curve c_{cyl} of degree 4[1] (see Fig. 1, left), which in the cylinder's development (in the $[\xi,\eta]$-plane with $\xi = R\varphi$ and $\eta = z$) appears as the Sine-curve

$$\eta = c \sin\frac{2\xi}{R}, \quad 0 \le \xi \le 2R\pi$$

with amplitude c and wavelength $R\pi$. The generators of $\mathcal{C}$ connect points c_{cyl} which are symmetric with respect to (henceforth abbreviated as w.r.t.) the z-axis. The conoid is bounded by the planes $z = \pm c$, which contact $\mathcal{C}$ along the torsal generators t_1 and t_2 in the planes $x \pm y = 0$. We call $2c$ the *width* of the conoid.

The tangent plane $\tau_{X|\mathcal{C}}$ at any point $X \in \mathcal{C}$, $X \notin t_1, t_2$, with position vector

$$\mathbf{x}(r,\varphi) = (r\cos\varphi,\ r\sin\varphi,\ c\sin 2\varphi), \ \text{where}\ r > 0, \tag{3}$$

is orthogonal to the vector product $\mathbf{x}_r \times \mathbf{x}_\varphi$ of the partial derivates

$$\mathbf{x}_r = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}r} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_\varphi = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\varphi} = \begin{pmatrix} -r\sin\varphi \\ r\cos\varphi \\ 2c\cos 2\varphi \end{pmatrix}.$$

This yields the equation

$$\tau_{X|\mathcal{C}}:\ \ 2c\cos 2\varphi\,(x\sin\varphi - y\cos\varphi) + rz = rc\sin 2\varphi. \tag{4}$$

The tangent plane $\tau_{X|\mathcal{C}}$ has a 45°-inclination against the $[x,y]$-plane if r or $-r$ equals the distribution parameter

$$\delta := \frac{\mathrm{d}z}{\mathrm{d}\varphi} = 2c\cos 2\varphi$$

of the generator through X.

For points $X \in \mathcal{C}$ outside the torsal generators, the intersection $\tau_{X|\mathcal{C}} \cap \mathcal{C}$ splits into the generator g_X through X and an ellipse e_X with principal vertices on the torsal generators and the minor axis in the $[x,y]$-plane (Fig. 1, left). After orthogonal projection into the $[x,y]$-plane, the ellipse appears as the circle $e_X{}'$ (see Fig. 1, right) satisfying

$$\cos 2\varphi\,(x^2 + y^2) + r\,(y\sin\varphi - x\cos\varphi) = 0,$$

hence

$$\left(x - \frac{r\cos\varphi}{2\cos 2\varphi}\right)^2 + \left(y + \frac{r\sin\varphi}{2\cos 2\varphi}\right)^2 = \frac{r^2}{2\cos^2 2\varphi} \quad \text{if} \quad \cos 2\varphi \ne 0.$$

All ellipses $e_X \subset \mathcal{C}$ have the same excentricity c, since it equals the difference of the z-coordinates of the respective principal and secondary vertices on the vertical cylinder (Müller and Krames, 1931, p. 208).

For all points P in space with a top view $P' \in e_X{}'$ opposite to the top view of the double line (see Fig. 1, right), the *pedal curve* on $\mathcal{C}$, i.e., the locus of pedal points of P on the generators of $\mathcal{C}$, coincides with e_X . This holds since right angles enclosed with generators of $\mathcal{C}$ appear in the top view again as right angles, provided that the

[1]The remaining part of the curve of intersection consists of two complex conjugate lines at infinity in the plane $x \pm iy = 0$.

spanned plane is not parallel to the z-axis. Thus, all pedal curves of a Plücker conoid are planar. Furthermore, all surface normals of $\mathcal{C}$ at points of e_X meet the vertical line through P'.

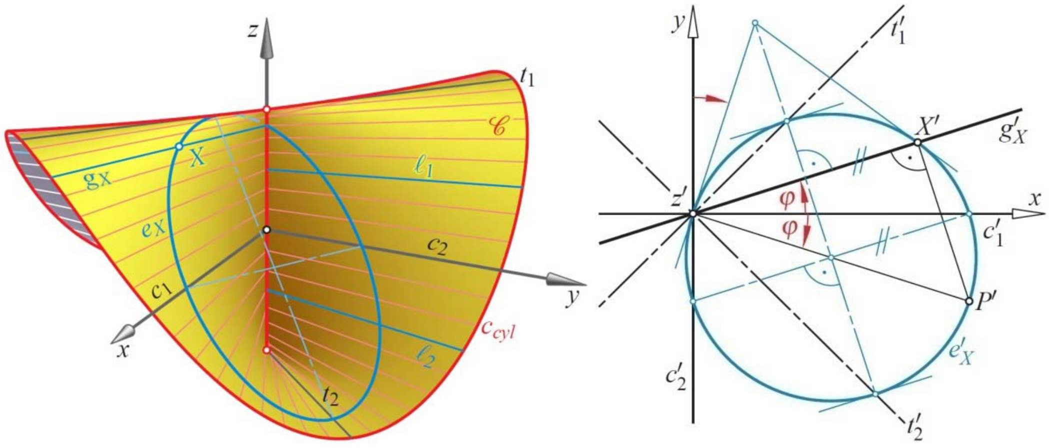

Figure 1. Plücker's conoid $\mathcal{C}$ (left: axonometric view, right: top view) with central generators c_1 and c_2, torsal generators t_1 and t_2, the generator g_X through X, and the ellipse $e_X \subset \mathcal{C} \cap \tau_{X|\mathcal{C}}$.

Remark 1: Another remarkable property of the cylindroid is reported in Stachel (1995): Let four generators g_1, ..., $g_4 \subset \mathcal{C}$ be called *cyclic* if their points of intersection with any fixed tangent plane $\tau_{X|\mathcal{C}}$ are concyclic, i.e., located on a circle (and on the ellipse e_X). Then, in each tangent plane their points of intersection are located on a circle. Moreover, there is an infinite set of spheres which contact these four lines, and, apart from four generators of a one-sheeted hyperboloid of revolution, this is the only choice of four lines in space with this property.

2. BISECTOR OF TWO SKEW LINES

For two given point sets S_1, S_2 in the Euclidean plane $\mathbb{E}^2$ or three-space $\mathbb{E}^3$, the set of points X being equidistant to S_1 and S_2 is called the *bisector* of S_1 and S_2.

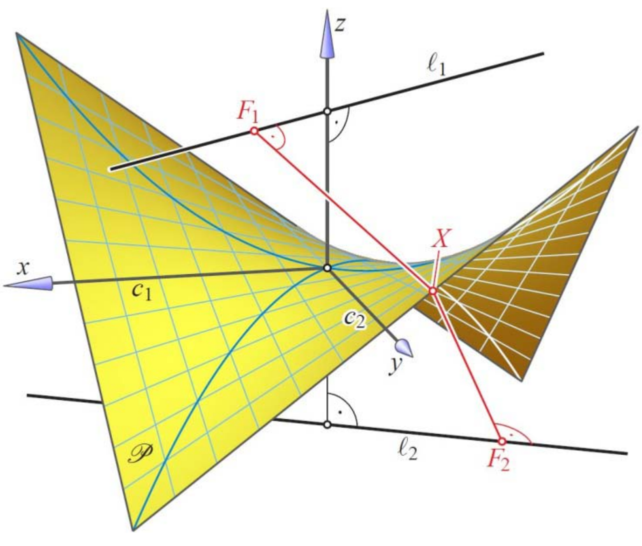

Figure 2. Points X of the bisector $\mathcal{P}$ of the two lines ℓ_1 and ℓ_2 satisfy $\overline{X\ell_1} = \overline{XF_1} = \overline{XF_2} = \overline{X\ell_2}$.

In the case of two given points $P,Q \in \mathbb{E}^3$, the bisector is the orthogonal bisector plane σ_{PQ} of P and Q. The standard definition of a parabola in $\mathbb{E}^2$ as the bisector of its focal point and directrix reveals that each paraboloid of revolution in $\mathbb{E}^3$ is the bisector of a point F and a plane not passing through F. However, also the equilateral hyperbolic paraboloid is a bisector, as reported, e.g., in Salmon and Fiedler, 1863, p. 154, and stated in the theorem below.

Theorem 1: *Let ℓ_1 and ℓ_2 be two skew lines in $\mathbb{E}^3$ with $2\varphi := \measuredangle \ell_1\ell_2$ and shortest distance $2d := \overline{\ell_1\ell_2}$.*

1. *The bisector of ℓ_1 and ℓ_2 is an orthogonal hyperbolic paraboloid $\mathcal{P}$ (Fig. 2). If ℓ_1 and ℓ_2 are given by $z = \pm d$ and $x \sin\varphi = \pm y \cos\varphi$, then*

$$\mathcal{P}:\quad z + \frac{\sin 2\varphi}{2d}xy = 0. \tag{5}$$

2. *The axes of symmetry c_1 and c_2 of the two skew lines ℓ_1, ℓ_2, which coincide with the x- and y-axis of our coordinate frame, are the vertex generators of $\mathcal{P}$; the common perpendicular of ℓ_1 and ℓ_2 is the paraboloid's axis. The lines ℓ_1 and ℓ_2 are polar w.r.t. $\mathcal{P}$, i.e., each point $X_1 \in \ell_1$ is conjugate w.r.t. $\mathcal{P}$ to all points $X_2 \in \ell_2$, and vice versa.*
3. *At any point $X \in \mathcal{P}$, the tangent plane $\tau_{X|\mathcal{P}}$ to $\mathcal{P}$ is the orthogonal bisector plane $\sigma_{F_1F_2}$ of the pedal points F_1, F_2 of X on the lines ℓ_1 and ℓ_2, respectively. Hence, $\mathcal{P}$ is the envelope of the bisecting planes $\sigma_{F_1F_2}$ for all points $F_1 \in \ell_1$ and $F_2 \in \ell_2$.*
4. *The generators of $\mathcal{P}$ are the axes of rotations in $\mathbb{E}^3$ which send the line ℓ_1 to the line ℓ_2. Therefore, the generators of $\mathcal{P}$ are axes of one-sheeted hyperboloids of revolution passing through the given pair of skew lines (ℓ_1, ℓ_2). These hyperboloids are centered on the vertex generators c_1, c_2 of $\mathcal{P}$ and share the secondary semiaxis $b = d\cot\varphi$ (Fig. 4 and 5).*

Proof: 1. Let any line ℓ be given in vector form as $\mathbf{p} + \mathbb{R}\mathbf{v}$ with $\|\mathbf{v}\| = 1$. Then, its distance to any point X with position vector $\mathbf{x}$ satisfies

$$\overline{X\ell}^{\,2} = \|\mathbf{x}-\mathbf{p}\|^2 - \langle \mathbf{x}-\mathbf{p}, \mathbf{v}\rangle^2, \tag{6}$$

where $\langle\, , \rangle$ denotes the standard dot product. If ℓ is replaced with one of the given lines ℓ_1, ℓ_2 with

$$\mathbf{p} = (0, 0, \pm d) \text{ and } \mathbf{v} = (\cos\varphi, \pm\sin\varphi, 0) \text{ for } 0 < \varphi < \pi/2 \text{ and } d > 0,$$

then $\overline{X\ell_1} = \overline{X\ell_2}$ is equivalent to

$$x^2 + y^2 + (z-d)^2 - (x\cos\varphi + y\sin\varphi)^2 = x^2 + y^2 + (z+d)^2 - (x\cos\varphi - y\sin\varphi)^2,$$

and consecutively, to

$$\mathcal{P}:\ 2dz + xy\sin 2\varphi = 0. \tag{7}$$

This is the equation of an orthogonal hyperbolic paraboloid (Fig. 2). The rotation $(x, y, z) \mapsto (x', y', z')$ about the z-axis through $\pi/4$ with

$$x = \frac{1}{\sqrt{2}}(x' - y'),\qquad y = \frac{1}{\sqrt{2}}(x' + y'),\qquad z = z',$$

yields the standard equation

$$2z' + \frac{\sin 2\varphi}{2d}(x'^2 - y'^2) = 0.$$

2. Two points $X_1 = (x_1, y_1, z_1)$ and $X_2 = (x_2, y_2, z_2)$ are conjugate w.r.t. the paraboloid $\mathcal{P}$ (7) if and only if

$$\frac{\sin 2\varphi}{2d}(x_1 y_2 + x_2 y_1) + (z_1 + z_2) = 0.$$

This is satisfied by each $X_1 \in \ell_1$ and $X_2 \in \ell_2$ since

$$X_1 = (r_1 \cos\varphi, r_1 \sin\varphi, d) \text{ and } X_2 = (r_2 \cos\varphi, -r_2 \sin\varphi, -d).$$

The origin is the vertex of the paraboloid $\mathcal{P}$; the *x*- and *y*-axis are the two vertex generators c_1 and c_2.

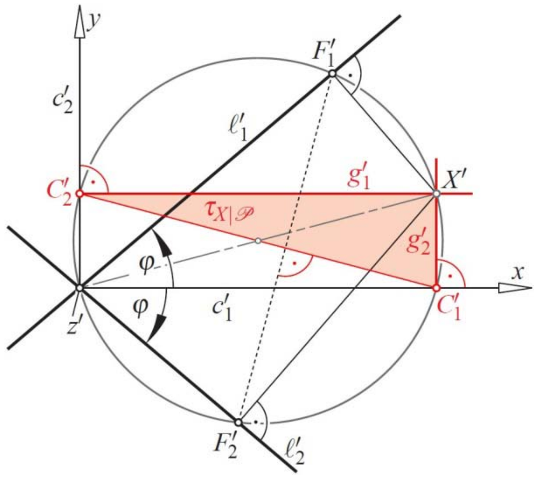

Figure 3. The tangent plane $\tau_{X|\mathcal{P}}$ at X to the bisecting paraboloid $\mathcal{P}$ is the orthogonal bisector plane $\sigma_{F_1F_2}$ of the respective pedal points F_1 and F_2 of X on the lines ℓ_1 and ℓ_2.

3. Let F_1 and F_2 be the pedal points of $X \in \mathcal{P}$ on the lines ℓ_1 and ℓ_2, respectively. Then, X is uniquely defined as the point of intersection between the orthogonal bisector plane $\sigma_{F_1F_2}$ of F_1 and F_2 and the planes orthogonal to ℓ_1 and ℓ_2 through the respective points F_1 and F_2. The generators g_1, g_2 of $\mathcal{P}$ through X pass through the pedal points C_1, C_2 of X on the vertex generators c_1 and c_2 of $\mathcal{P}$. The tangent plane $\tau_{X|\mathcal{P}}$ to $\mathcal{P}$ at X is spanned by g_1 and g_2.

Now, we project the scene orthogonally into the [*x*,*y*]-plane (Fig. 3): The top view of the *z*-axis is the common point of ℓ_1' and ℓ_2'. Since F_1 and F_2 are at equal distance to the [*x*,*y*]-plane, but on different sides, the bisecting plane $\sigma_{F_1F_2}$ intersects the [*x*,*y*]-plane along the orthogonal bisector line of the top views F_1' and F_2'. The Thales circle with diameter $X'z'$ passes through F_1' and F_2', and also through the pedal points C_i' of X' on c_i' for $i = 1$, 2. Since the arcs from C_1' to F_1' and to F_2' are of equal lengths, point C_1 lies on the trace of $\sigma_{F_1F_2}$, which must be a diameter of the Thales circle. Hence, this diameter coincides with the trace $[C_1,C_2]$ of $\tau_{X|\mathcal{P}}$, which proves the coincidence of $\tau_{X|\mathcal{P}}$ and $\sigma_{F_1F_2}$.

4. If g is the axis of a rotation which sends ℓ_1 to ℓ_2, then each point $X \in g$ has equal distances to ℓ_1 and ℓ_2, which implies $g \subset \mathcal{P}$.

Conversely, let g be the generator of $\mathcal{P}$, which intersects c_1 orthogonally at any point M. The reflection in c_1 exchanges ℓ_1 and ℓ_2 while g is mapped onto itself. Consequently, there are equal distances $\overline{g\ell_1} = \overline{g\ell_2}$ and congruent angles $\measuredangle\, g\ell_1 = \measuredangle\, g\ell_2$. The reflection in c_1 exchanges also the pedal points N_1, N_2 of M on ℓ_1 and ℓ_2; the midpoint of N_1N_2 lies on c_1 (Fig. 4).

The generator g is orthogonal to c_1 and also to N_1N_2, since $g \subset \tau_{M|\mathcal{P}}$ lies in the orthogonal bisector plane $\sigma_{N_1N_2}$. Therefore, g is orthogonal to the plane connecting M, N_1, and N_2. Furthermore, the lines $[M,N_1]$ and $[M,N_2]$ are the common perpendiculars of g with ℓ_1 and ℓ_2, respectively.

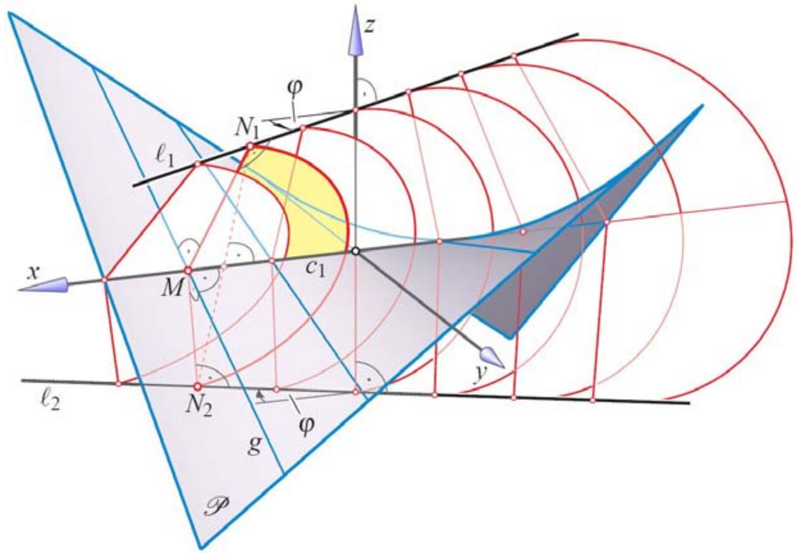

Figure 4. Gorge circles of hyperboloids of revolution through ℓ_1 and ℓ_2. The axes of the hyperboloids form a regulus of the bisecting orthogonal hyperbolic paraboloid $\mathscr{P}$ (Theorem 1,4. or Krames, 1983).

There is a rotation about g which sends N_1 to N_2. This rotation takes ℓ_1 into a line $\tilde{\ell}$ through N_2, which is orthogonal to MN_2 and includes with g an angle congruent to $\measuredangle\, g\ell_2$. We obtain $\tilde{\ell} = \ell_2$, since otherwise $\tilde{\ell}$ would be symmetric to ℓ_2 w.r.t. the meridian plane gN_2 and therefore, as a member of the complementary regulus, intersect ℓ_1.

Under a continuous rotation about g, the line ℓ_1 forms one regulus of a one-sheeted hyperboloid of revolution $\mathscr{H}$ (see Fig. 5). It is centered at M and its gorge circle passes through the pedal points N_1 and N_2 of M on the given lines ℓ_1, ℓ_2.

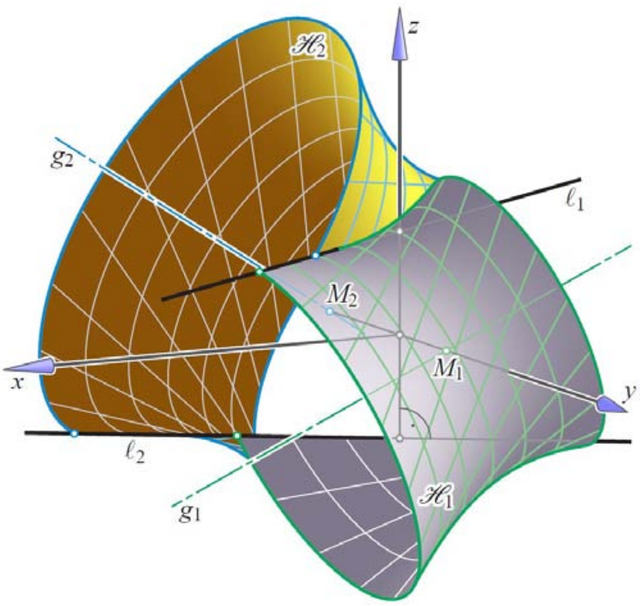

Figure 5. Two hyperboloids of revolution $\mathscr{H}_1$, $\mathscr{H}_2$ through two skew lines ℓ_1 and ℓ_2. The hyperboloids share the secondary semiaxis b and the distribution parameters $\pm b$ of their generators.

When M varies on c_1, we obtain a one-parameter family of one-sheeted hyperboloids of revolution through the skew generators ℓ_1 and ℓ_2 (Fig. 5). Due to a result of Wunderlich (1982) and Krames (1983), these two skew generators ℓ_1, ℓ_2 define already the secondary semiaxis b of these hyperboloids, namely $b = d \cot \varphi$, where $2d =$

$\overline{\ell_1\ell_2}$ and $2\varphi = \measuredangle\, \ell_1\ell_2$ (see also Odehnal et al, 2020, p. 37). Of course, the same holds for points $M \in c_2$. By the same token, $+b$ or $-b$ equals the distribution parameter of all generators of the hyperboloids. □

Remark 2: The complete intersection of any two hyperboloids of revolution $\mathcal{H}_1$, $\mathcal{H}_2$ through the two skew lines ℓ_1 and ℓ_2 (according to Theorem 1, 4., see Fig. 5) consists of two more lines which need not be real. They can be found as common transversals of ℓ_1, ℓ_2, and two other generators of the hyperboloids, one of each, and both skew to ℓ_1 and ℓ_2.

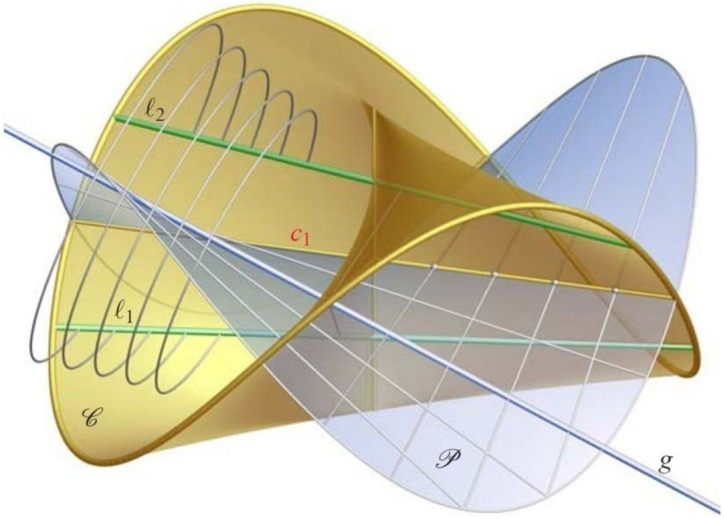

Figure 6. All pairs of skew lines (ℓ_1, ℓ_2) which share the bisecting orthogonal hyperbolic paraboloid $\mathcal{P}$ are located on a Plücker conoid $\mathcal{C}$. Generators g of $\mathcal{P}$ are axes of rotations with $\ell_1 \mapsto \ell_2$ (courtesy: G. Glaeser).

Let us focus on the paraboloid $\mathcal{P}$ with the equation (5) and ask the following: Where are all pairs (ℓ_1, ℓ_2) of lines for which $\mathcal{P}$ is the bisector? The answer, as given in the theorem below, was disclosed in Husty and Sachs (1994), but already reported at the turn to the 20th century in Schilling (1911), p. 54.

Theorem 2: *All pairs of skew lines (ℓ_1, ℓ_2) which share the bisecting orthogonal hyperbolic paraboloid $\mathcal{P}$ are located on a Plücker conoid (cylindroid) $\mathcal{C}$ in symmetric position w.r.t. the vertex generators c_1 and c_2 of $\mathcal{P}$.*

Proof. Let the lines ℓ_1 and ℓ_2 be given in the same way as in Theorem 1. Then, the bisector $\mathcal{P}$ remains the same if the quotient $(\sin 2\varphi)/d$ does not change. Obviously, all points of ℓ_1 and ℓ_2 satisfy

$$\mathcal{C}:\quad (x^2 + y^2)\, z - 2c\, xy = 0 \quad\text{where}\quad c := \frac{d}{\sin 2\varphi}. \tag{8}$$

This equation defines a Plücker conoid $\mathcal{C}$, as introduced in (2) (see Fig. 6). The surface $\mathcal{C}$ has the x- and y-axis as central generators c_1 and c_2 and the z-axis as double line. All pairs (ℓ_1, ℓ_2) are symmetric w.r.t. c_1 and c_2 and polar w.r.t. $\mathcal{P}$. □

Schilling's famous collection of mathematical models contains as model XXIII, no. 10, the pair of surfaces $\mathcal{C}$ and $\mathcal{P}$, each represented by strings with endpoints on a closed boundary curve of degree four (see Fig. 7[2] and compare with Fig. 8). The two boundary curves are even congruent, as we confirm below in Theorem 3.

[2]The displayed model belongs to the collection of the Institute of Discrete Mathematics and Geometry, Vienna University of Technology, \url{https://www.geometrie.tuwien.ac.at/modelle/models_show.php?mode=2&n=100&id=0}, retrieved March 2020.

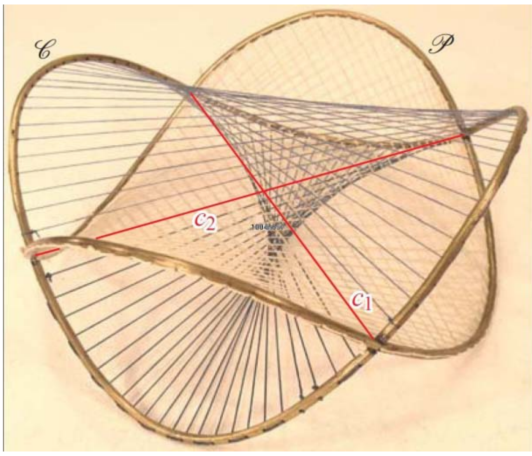

Figure 7. String model of a Plücker conoid $\mathcal{C}$ together with the surface formed by its normals along the central generators c_1 and c_2, an orthogonal hyperbolic paraboloid $\mathscr{P}$. This is model XXIII, no. 10, out of Schilling's famous collection of mathematical models. In addition, the lines c_1 and c_2, which are also vertex generators of $\mathscr{P}$, are marked in red color.

By the same token, all generators of the orthogonal hyperbolic paraboloid $\mathscr{P}$ are surface normals of the Plücker conoid along any central generator. This follows from (4): For $\varphi = 0$, the surface normal at the point $(r, 0, 0)$, $r \in \mathbb{R}$, has the direction of $(0, -2c, r)$. For $\varphi = \pi/2$, the normal at $(0, r, 0)$ has the direction of $(-2c, 0, r)$. Now we can confirm that the points $(r, -2ct, rt)$ and $(-2ct, r, rt)$ for all $(r,t) \in \mathbb{R}^2$ satisfy the paraboloid's equation

$$\mathscr{P}: \quad xy + 2cz = 0 \tag{9}$$

according to (5) in the case

$$c = \frac{d}{\sin 2\varphi} \ . \tag{10}$$

The same follows from Theorem 1,4. as the limit $\ell_1 \to \ell_2$, i.e., $d \to 0$: All generators of $\mathscr{P}$ are axes of one-sheeted hyperboloids of revolution which contact the conoid $\mathcal{C}$ along one of the central generators.

We summarize some properties of the pair of surfaces $\mathcal{C}$ and $\mathscr{P}$ (see Figs. 7 and 8), which share the distribution parameter $\delta = 2c$ at c_1 and c_2:

Theorem 3. *Let $\mathscr{P}$ be the orthogonal hyperbolic paraboloid* (9) *and $\mathcal{C}$ be the Plücker conoid satisfying* (8).

1. *The generators of $\mathscr{P}$ are the surface normals of $\mathcal{C}$ along its central generators c_1 and c_2.*
2. *Each generator g of $\mathscr{P}$ is the axis of concentric one-sheeted hyperboloids of revolution which intersect $\mathcal{C}$ along two skew generators ℓ_1, ℓ_2 being symmetric w.r.t. c_1 and c_2. The gorge circles lie in the tangent plane to $\mathcal{C}$ at the point M where $g \subset \mathscr{P}$ intersects the vertex generator of the complementary regulus.*
3. *The right cylinder $x^2 + y^2 = 4c^2$ with radius $2c$ equal to the width of $\mathcal{C}$ intersects $\mathscr{P}$ and $\mathcal{C}$ along two quartics which are symmetric w.r.t. the* $[x,y]$*-plane (Fig. 7).*
4. *The polarity in the paraboloid $\mathscr{P}$ maps the Plücker conoid $\mathcal{C}$ onto itself. Outside the torsal generators, there is a symmetric one-to-one correspondence between points Q_1, Q_2 on $\mathcal{C}$ such that Q_2 is the pole w.r.t. $\mathscr{P}$ of the tangent plane to $\mathcal{C}$ at Q_1, and vice versa.*

Proof. 2. We vary d and φ such that $c = d / \sin 2\varphi$ remains constant. The hyperboloids with the same axis g through $M \in c_1$ share the plane $[M, N_1, N_2]$ of the gorge circle, where the points N_1, N_2 are the pedal points of M on the corresponding pair of lines ℓ_1, ℓ_2. This plane orthogonal to g is tangent to $\mathcal{C}$ at M. The pedal points N_1 and N_2 belong to the pedal curve of M on $\mathcal{C}$, which is an ellipse with the minor axis OM along c_1 (note Fig. 1, right).

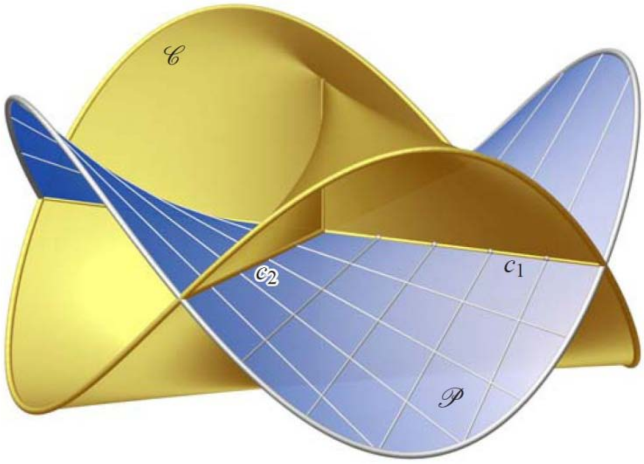

Figure 8. The surface normals of the Plücker conoid $\mathcal{C}$ along the two central generators c_1 and c_2 form the two reguli of an orthogonal hyperbolic paraboloid $\mathscr{P}$ (courtesy: G. Glaeser).

3. We plug $x = R\cos\varphi$ and $y = R\sin\varphi$ into the equation (9) of $\mathcal{C}$ and obtain $R^2 z - 2cR^2 \sin\varphi\cos\varphi = 0$. The same substitution in the equation (9) of $\mathscr{P}$ results in $R^2 \sin\varphi\cos\varphi + 2cz = 0$. The choice $R = 2c$ gives rise to two symmetric curves $z = \pm c\sin 2\varphi$ (Figs. 7 and 8).

4. We use the parametrization $\mathbf{x}(r, \varphi)$ from (3) and set $Q_i = (r_i, \varphi_i)$ for $i = 1,2$. Then, the tangent plane at Q_1 to $\mathcal{C}$ satisfies (4),

$$\tau_{Q_1|\mathscr{C}}:\ 2c\cos 2\varphi_1 (x\sin\varphi_1 - y\cos\varphi_1) + r_1 z = r_1 c\sin 2\varphi_1.$$

The polar plane of Q_2 w.r.t. $\mathscr{P}$ in (9) is given by

$$r_2 (x\sin\varphi_2 + y\cos\varphi_2) + 2cz = -2c\sin 2\varphi_2.$$

We obtain an identity of the two planes when we set

$$\varphi_2 = -\varphi_1 \ \text{ and } \ r_1 r_2 = -4c^2\cos 2\varphi. \qquad \square \qquad (11)$$

The correspondence of item 4 reveals: If points Q_1 is at the distance $r_1 = 2c$ to the double line, i.e., on the quartic c_{cyl} as mentioned in item 3, then the corresponding point Q_2 has a tangent plane which is inclined under 45°, since $r_2 = \delta = 2c\cos 2\varphi_2$. The polarity in $\mathscr{P}$ maps the ellipse $e_{Q_1} \subset (\mathcal{C} \cap \tau_{Q_1|\mathscr{C}})$ onto the quadratic tangent cone of $\mathcal{C}$ with the apex Q_2. The tangent planes of this cone, i.e., the planes spanned by Q_2 and any generator of $\mathcal{C}$, intersect $\mathcal{C}$ in ellipses passing through Q_2. All points of the ellipse e_{Q_1} are conjugate w.r.t. $\mathscr{P}$ to the point Q_2.

Remark: If $g_1, \ldots, g_4$ are concyclic generators of $\mathcal{C}$ (cf. Remark 1), then the bisecting paraboloids for any two of these four belong to a pencil of quadrics. Their common curve is a quartic with a double point at the ideal point of the z-axis. The infinitely many spheres which contact $g_1, \ldots, g_4$ are centered on this quartic. The top view of this spine curve is an equilateral hyperbola. For proofs and further details see Stachel (1995).

3. PLÜCKER'S CONOID AS LOCUS OF INSTANT SCREW AXES FOR SKEW GEARS

In spatial kinematics, the Plücker conoid $\mathcal{C}$ is well-known as the locus of instant axes ℓ_{12} of the relative screw motion for two wheels which rotate with constant velocities ω_1 and ω_2 about fixed skew axes ℓ_1 and ℓ_2, respectively. The axes of symmetry of the two axes of rotation ℓ_1 and ℓ_2 coincide with the central generators c_1,

c_2 of $\mathcal{C}$. The axodes of the relative screw motion are hyperboloids of revolution $\mathcal{H}_1$, $\mathcal{H}_2$ with mutual contact along ℓ_{12} (Fig. 9).[3] They are solutions of the purely geometric problem: For given skew axes ℓ_1, ℓ_2, find pairs of hyperboloids of revolution which contact each other along a line.

A standard proof of this result uses dual vectors for the representation of oriented lines and screws (see, e.g., Figliolini et al, 2007). Here we present another proof:

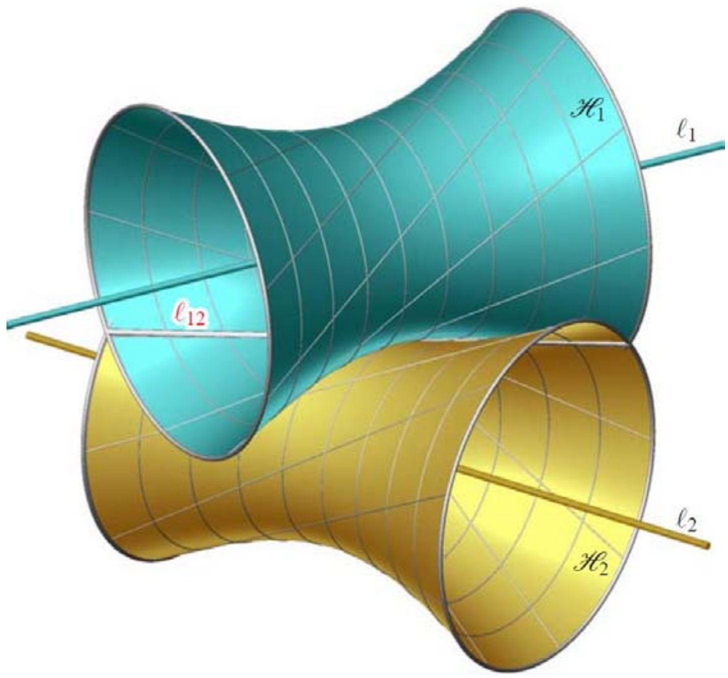

Figure 9. Two hyperboloids of revolution in contact along the line ℓ_{12} (courtesy: G. Glaeser).

The common surface normals of the two hyperboloids $\mathcal{H}_1$ and $\mathcal{H}_2$ along the line of contact ℓ_{12} form one regulus of an orthogonal hyperbolic paraboloid $\mathcal{P}$ which passes through the axes ℓ_1 and ℓ_2. The line ℓ_{12} is the vertex generator of the complementary regulus on $\mathcal{P}$. The other vertex generator of $\mathcal{P}$ intersects all three lines ℓ_{12}, ℓ_1, and ℓ_2 orthogonally. Therefore, it is the common perpendicular of ℓ_1 and ℓ_2. These conditions will prove to be sufficient for identifying the locus of the lines ℓ_{12} as a Plücker conoid.

We use the coordinate frame of Section 2 and define ℓ_1 and ℓ_2 by $z = \pm d$ and $x \sin\varphi = \pm\, y \cos\varphi$. Then the z-axis is the common perpendicular, and we can assume that ℓ_{12} is given by

$$z = a \quad \text{and} \quad x \sin\alpha = y \cos\alpha$$

(see Fig. 10). Now we intersect the orthogonal plane to ℓ_{12} through any point $X = (r\cos\alpha, r\sin\alpha, a) \in \ell_{12}$ with ℓ_1 and ℓ_2, and we obtain

[3] The various relations between the two fixed axes of rotations ℓ_1, ℓ_2, the relative axis ℓ_{12}, the angular velocities ω_1, ω_2, and the pitch of the relative screw motion can be visualized in the so-called *Ball-Disteli diagram*, which arises from $\mathcal{C}$ by a particular projection (see Figliolini et al., 2007, Fig. 7). It is noteworthy that we still obtain a Plücker conoid as the locus of relative screw axes when the two wheels perform helical motions with fixed pitches about fixed axes (Figliolini et al., 2007, Fig. 10). This is also a consequence of the following classical result Plücker's in connection with linear line complexes: The axes of all linear line complexes which are contained in a pencil belong to a Plücker conoid (Müller and Krames, 1931, p. 214).

$$X_1 = \left(\frac{r\cos\varphi}{\cos(\alpha-\varphi)},\ \frac{r\sin\varphi}{\cos(\alpha-\varphi)},\ d\right) \in \ell_1 \quad\text{and}\quad X_2 = \left(\frac{r\cos\varphi}{\cos(\alpha+\varphi)},\ \frac{-r\sin\varphi}{\cos(\alpha+\varphi)},\ -d\right) \in \ell_2 .$$

In the top view, the three points X, X_1, and X_2 appear already aligned. Therefore, they are collinear in space if and only if the segments X_1X and XX_2 have the same slope. This means,

$$\frac{a-d}{\tan(\alpha-\varphi)} = \frac{a+d}{\tan(\alpha+\varphi)},$$

hence

$$a\,\frac{\sin 2\varphi}{\cos(\alpha-\varphi)\cos(\alpha+\varphi)} = d\,\frac{\sin 2\alpha}{\cos(\alpha-\varphi)\cos(\alpha+\varphi)}.$$

After exclusion of the cases where $\cos(\alpha-\varphi)\cos(\alpha+\varphi) = 0$, i.e., $\alpha = \varphi \pm \pi/2$, we conclude

$$a = \frac{d}{\sin 2\varphi}\sin 2\alpha$$

as the relation between the altitude a and the polar angle α of the wanted line ℓ_{12} of contact. This is the equation (1) of a Plücker conoid in cylinder coordinates. In the excluded cases, the line ℓ_{12} intersects one of the given axes and is orthogonal to the other. Then, one hyperboloid degenerates into a cone and the other into a plane.

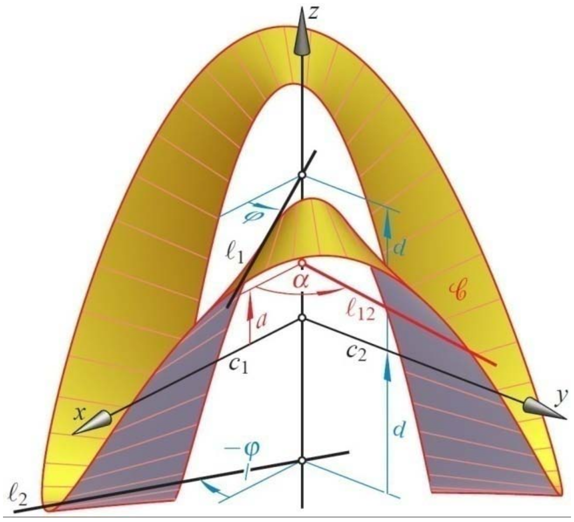

Figure 10. The axes ℓ_1, ℓ_2, the line of contact ℓ_{12}, and a portion of the Plücker conoid $\mathcal{C}$.

Theorem 4. *If the given skew lines ℓ_1 and ℓ_2 are axes of hyperboloids of revolution which contact each other along any line ℓ_{12}, then the lines ℓ_{12} are located on a Plücker cononoid $\mathcal{C}$ with the axes of symmetry of ℓ_1 and ℓ_2 as central generators. Conversely, on $\mathcal{C}$ each generator which is skew to ℓ_1 and ℓ_2 serves as a line of contact between such hyperboloids.*

Corollary 5. *Let g be any generator of the Plücker conoid $\mathcal{C}$ and n be an orthogonal transversal of g. If all points of intersection between n and $\mathcal{C}$ are real, then n meets two generators ℓ_1, ℓ_2 of $\mathcal{C}$ which are symmetric w.r.t. the central generators. In particular, at each point X of any central generator $c \subset \mathcal{C}$ the orthogonal transversals to other generators g of $\mathcal{C}$ are tangents of $\mathcal{C}$.*

Proof. According to the proof of Theorem 4, we can state: If an orthogonal transversal n of g meets any generator $\ell_1 \subset \mathcal{C}$, then it meets also the symmetric line ℓ_2.

However, we can also use the top view in Fig. 1, right, and argue as follows: The lines g and n span the tangent plane at any point $X \in g$. Each line $n \perp g$ sufficiently close to the double line intersects e_X at two points symmetric w.r.t. the minor axis of e_X. This shows that Theorem 4 can be concluded directly from the planar pedal curves e_X on the Plücker conoid. □

Remark. The complete intersection of the two contacting hyperboloids $\mathcal{H}_1$ and $\mathcal{H}_2$ in Fig. 9 consists of the line of contact ℓ_{12} with multiplicity two and two complex conjugate generators of the complementary regulus (cf. Phillips, 2003, pp. 119–122, and compare with Remark 2).

4. CONCLUSIONS

As explained above, there are various relations between Plücker conoids $\mathcal{C}$, one-sheeted hyperboloids of revolution $\mathcal{H}$, and orthogonal hyperbolic paraboloids $\mathcal{P}$. However, they show up in different, almost contrary ways:

In Section 2, the axes of the involved hyperboloids of revolution $\mathcal{H}$ are generators of $\mathcal{P}$, and the hyperboloids pass through pairs of lines (ℓ_1, ℓ_2) on $\mathcal{C}$ symmetrically placed w.r.t. the central generators c_1, c_2 (note Figs. 4 and 5). The orthogonal hyperbolic paraboloid $\mathcal{P}$ is orthogonal to $\mathcal{C}$ along the central generators (Fig. 8).

In Section 3, the axes ℓ_1, ℓ_2 of the hyperboloids $\mathcal{H}_1$ and $\mathcal{H}_2$ are two symmetrically placed generators of $\mathcal{C}$, and the hyperboloids contact each other along another generator ℓ_{12} of $\mathcal{C}$ (Fig. 9).

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