



## Properties of the Robin's Inequality

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August 27, 2020

# Properties of the Robin's Inequality

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*Abstract.* In mathematics, the Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part  $\frac{1}{2}$ . Many consider it to be the most important unsolved problem in pure mathematics. The Robin's inequality consists in  $\sigma(n) < e^\gamma \times n \times \ln \ln n$  where  $\sigma(n)$  is the divisor function and  $\gamma \approx 0.57721$  is the Euler-Mascheroni constant. The Robin's inequality is true for every natural number  $n > 5040$  if and only if the Riemann hypothesis is true. We prove the Robin's inequality is true for every natural number  $n > 5040$  when  $15 \nmid n$ , where  $15 \nmid n$  means that  $n$  is not divisible by 15. More specifically: every counterexample should be divisible by  $2^{20} \times 3^{13} \times 5^8 \times k_1$  or either  $2^{20} \times 3^{13} \times k_2$  or  $2^{20} \times 5^8 \times k_3$ , where  $2 \nmid k_1$ ,  $3 \nmid k_1$ ,  $5 \nmid k_1$ ,  $2 \nmid k_2$ ,  $3 \nmid k_2$ ,  $2 \nmid k_3$  and  $5 \nmid k_3$ .

## 1 Introduction

In mathematics, the Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part  $\frac{1}{2}$ . Many consider it to be the most important unsolved problem in pure mathematics [1]. It is of great interest in number theory because it implies results about the distribution of prime numbers [1]. It was proposed by Bernhard Riemann (1859), after whom it is named [1]. It is one of the seven Millennium Prize Problems selected by the Clay Mathematics Institute to carry a US 1,000,000 prize for the first correct solution [1]. The divisor function  $\sigma(n)$  for a natural number  $n$  is defined as the sum of the powers of the divisors of  $n$ ,

$$\sigma(n) = \sum_{k|n} k$$

where  $k | n$  means that the natural number  $k$  divides  $n$  [5]. In 1915, Ramanujan proved that under the assumption of the Riemann hypothesis, the inequality,

$$\sigma(n) < e^\gamma \times n \times \ln \ln n$$

holds for all sufficiently large  $n$ , where  $\gamma \approx 0.57721$  is the Euler-Mascheroni constant [3]. The largest known value that violates the inequality is  $n = 5040$ . In 1984, Guy Robin proved that the inequality is true for all  $n > 5040$  if and only if the Riemann hypothesis is true [3]. Using this inequality, we show an interesting result.

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2010 Mathematics Subject Classification: Primary 11M26.

Keywords: number theory, inequality, divisor, prime, counterexample.

## 2 Results

**Theorem 2.1** Given a natural number  $n = p_1^{a_1} \times p_2^{a_2} \times \dots \times p_m^{a_m}$  such that  $p_1, p_2, \dots, p_m$  are prime numbers, then we obtain the following inequality

$$\frac{\sigma(n)}{n} < \prod_{i=1}^m \frac{p_i}{p_i - 1}.$$

**Proof** For a natural number  $n = p_1^{a_1} \times p_2^{a_2} \times \dots \times p_m^{a_m}$  such that  $p_1, p_2, \dots, p_m$  are prime numbers, then we obtain the following formula

$$(2.1) \quad \sigma(n) = \prod_{i=1}^m \frac{p_i^{a_i+1} - 1}{p_i - 1}$$

from the Ramanujan's notebooks [2]. In this way, we have that

$$(2.2) \quad \frac{\sigma(n)}{n} = \prod_{i=1}^m \frac{p_i^{a_i+1} - 1}{p_i^{a_i} \times (p_i - 1)}.$$

However, for any prime power  $p_i^{a_i}$ , we have that

$$\frac{p_i^{a_i+1} - 1}{p_i^{a_i} \times (p_i - 1)} < \frac{p_i^{a_i+1}}{p_i^{a_i} \times (p_i - 1)} = \frac{p_i}{p_i - 1}.$$

Consequently, we obtain that

$$\frac{\sigma(n)}{n} < \prod_{i=1}^m \frac{p_i}{p_i - 1}.$$

■

**Theorem 2.2** Given some prime numbers  $p_1, p_2, \dots, p_m$ , then we obtain the following inequality,

$$\prod_{i=1}^m \frac{p_i}{p_i - 1} < \frac{\pi^2}{6} \times \prod_{i=1}^m \frac{p_i + 1}{p_i}.$$

**Proof** Given a prime number  $p_i$ , we obtain that

$$\frac{p_i}{p_i - 1} = \frac{p_i^2}{p_i^2 - p_i}$$

and that would be equivalent to

$$\frac{p_i^2}{p_i^2 - p_i} = \frac{p_i^2}{p_i^2 - 1 - (p_i - 1)}$$

and that is the same as

$$\frac{p_i^2}{p_i^2 - 1 - (p_i - 1)} = \frac{p_i^2}{(p_i - 1) \times \left(\frac{p_i^2 - 1}{p_i - 1} - 1\right)}$$

which is equal to

$$\frac{p_i^2}{(p_i - 1) \times \left(\frac{p_i^2 - 1}{(p_i - 1)} - 1\right)} = \frac{p_i^2}{(p_i - 1) \times \frac{p_i^2 - 1}{(p_i - 1)} \times \left(1 - \frac{(p_i - 1)}{p_i^2 - 1}\right)}$$

that is equivalent to

$$\frac{p_i^2}{(p_i - 1) \times \frac{p_i^2 - 1}{(p_i - 1)} \times \left(1 - \frac{(p_i - 1)}{p_i^2 - 1}\right)} = \frac{p_i^2}{p_i^2 - 1} \times \frac{1}{1 - \frac{(p_i - 1)}{p_i^2 - 1}}$$

which is the same as

$$\frac{p_i^2}{p_i^2 - 1} \times \frac{1}{1 - \frac{(p_i - 1)}{p_i^2 - 1}} = \frac{1}{1 - p_i^{-2}} \times \frac{1}{1 - \frac{1}{(p_i + 1)}}$$

and finally

$$\frac{1}{(1 - p_i^{-2})} \times \frac{1}{1 - \frac{1}{(p_i + 1)}} = \frac{1}{(1 - p_i^{-2})} \times \frac{p_i + 1}{p_i}.$$

In this way, we have that

$$\prod_{i=1}^m \frac{p_i}{p_i - 1} = \prod_{i=1}^m \frac{1}{1 - p_i^{-2}} \times \prod_{i=1}^m \frac{p_i + 1}{p_i}.$$

However, we know that

$$\prod_{i=1}^m \frac{1}{1 - p_i^{-2}} < \prod_{j=1}^{\infty} \frac{1}{1 - p_j^{-2}}$$

where  $p_j$  is the  $j^{\text{th}}$  prime number and we have that

$$\prod_{j=1}^{\infty} \frac{1}{1 - p_j^{-2}} = \frac{\pi^2}{6}$$

as a consequence of the result in the Basel problem [5]. Consequently, we obtain that

$$\prod_{i=1}^m \frac{p_i}{p_i - 1} < \frac{\pi^2}{6} \times \prod_{i=1}^m \frac{p_i + 1}{p_i}.$$

■

**Definition 2.3** We recall that an integer  $n$  is said to be squarefree if for every prime divisor  $p$  of  $n$  we have  $p^2 \nmid n$ , where  $p^2 \nmid n$  means that  $p^2$  does not divide  $n$  [3].

**Theorem 2.4** Given a squarefree number  $n = q_1 \times \dots \times q_m$  such that  $q_1, q_2, \dots, q_m$  are odd prime numbers,  $3 \nmid n$  and  $5 \nmid n$ , then we obtain the following inequality

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \sigma(n) \leq e^\gamma \times n \times \ln \ln(2^{19} \times n).$$

**Proof** This proof is very similar with the demonstration in Theorem 1.1 from the article reference [3]. By induction with respect to  $\omega(n)$ , that is the number of distinct prime factors of  $n$  [3]. Put  $\omega(n) = m$  [3]. We need to prove the assertion for those integers with  $m = 1$ . From the equation (2.1), we obtain that

$$(2.3) \quad \sigma(n) = (q_1 + 1) \times (q_2 + 1) \times \dots \times (q_m + 1)$$

when  $n = q_1 \times q_2 \times \dots \times q_m$ . In this way, for any prime number  $p_i \geq 7$ , then we need to prove

$$(2.4) \quad \frac{\pi^2}{6} \times \frac{3}{2} \times \left(1 + \frac{1}{p_i}\right) \leq e^\gamma \times \ln \ln(2^{19} \times p_i).$$

For  $p_i = 7$ , we have that

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \left(1 + \frac{1}{7}\right) \leq e^\gamma \times \ln \ln(2^{19} \times 7)$$

is actually true. For another prime number  $p_i > 7$ , we have that

$$\left(1 + \frac{1}{p_i}\right) < \left(1 + \frac{1}{7}\right)$$

and

$$e^\gamma \times \ln \ln(2^{19} \times 7) < e^\gamma \times \ln \ln(2^{19} \times p_i)$$

which clearly implies that the inequality (2.4) is true for every prime number  $p_i \geq 7$ . Now, suppose it is true for  $m - 1$ , with  $m \geq 1$  and let us consider the assertion for those squarefree  $n$  with  $\omega(n) = m$  [3]. So let  $n = q_1 \times \dots \times q_m$  be a squarefree number and assume that  $q_1 < \dots < q_m$  for  $q_m \geq 7$ .

**Case 1** :  $q_m \geq \ln(2^{19} \times q_1 \times \dots \times q_{m-1} \times q_m) = \ln(2^{19} \times n)$ .

By the induction hypothesis we have

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (q_1 + 1) \times \dots \times (q_{m-1} + 1) \leq e^\gamma \times q_1 \times \dots \times q_{m-1} \times \ln \ln(2^{19} \times q_1 \times \dots \times q_{m-1})$$

and hence

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (q_1 + 1) \times \dots \times (q_{m-1} + 1) \times (q_m + 1) \leq$$

$$e^\gamma \times q_1 \times \dots \times q_{m-1} \times (q_m + 1) \times \ln \ln(2^{19} \times q_1 \times \dots \times q_{m-1})$$

when we multiply the both sides of the inequality by  $(q_m + 1)$ . We want to show that

$$e^\gamma \times q_1 \times \dots \times q_{m-1} \times (q_m + 1) \times \ln \ln(2^{19} \times q_1 \times \dots \times q_{m-1}) \leq$$

$$e^\gamma \times q_1 \times \dots \times q_{m-1} \times q_m \times \ln \ln(2^{19} \times q_1 \times \dots \times q_{m-1} \times q_m) = e^\gamma \times n \times \ln \ln(2^{19} \times n).$$

Indeed the previous inequality is equivalent with

$$q_m \times \ln \ln(2^{19} \times q_1 \times \dots \times q_{m-1} \times q_m) \geq (q_m + 1) \times \ln \ln(2^{19} \times q_1 \times \dots \times q_{m-1})$$

or alternatively

$$\frac{q_m \times (\ln \ln(2^{19} \times q_1 \times \dots \times q_{m-1} \times q_m) - \ln \ln(2^{19} \times q_1 \times \dots \times q_{m-1}))}{\ln q_m} \geq$$

$$\frac{\ln \ln(2^{19} \times q_1 \times \dots \times q_{m-1})}{\ln q_m}.$$

From the reference [3], we have that if  $0 < a < b$ , then

$$(2.5) \quad \frac{\ln b - \ln a}{b - a} = \frac{1}{(b - a)} \int_a^b \frac{dt}{t} > \frac{1}{b}.$$

We can apply the inequality (2.5) to the previous one just using  $b = \ln(2^{19} \times q_1 \times \dots \times q_{m-1} \times q_m)$  and  $a = \ln(2^{19} \times q_1 \times \dots \times q_{m-1})$ . Certainly, we have that

$$\begin{aligned} & \ln(2^{19} \times q_1 \times \dots \times q_{m-1} \times q_m) - \ln(2^{19} \times q_1 \times \dots \times q_{m-1}) = \\ & \ln \frac{2^{19} \times q_1 \times \dots \times q_{m-1} \times q_m}{2^{19} \times q_1 \times \dots \times q_{m-1}} = \ln q_m. \end{aligned}$$

In this way, we obtain that

$$\begin{aligned} & \frac{q_m \times (\ln \ln(2^{19} \times q_1 \times \dots \times q_{m-1} \times q_m) - \ln \ln(2^{19} \times q_1 \times \dots \times q_{m-1}))}{\ln q_m} > \\ & \frac{q_m}{\ln(2^{19} \times q_1 \times \dots \times q_m)}. \end{aligned}$$

Using this result we infer that the original inequality is certainly satisfied if the next inequality is satisfied

$$\frac{q_m}{\ln(2^{19} \times q_1 \times \dots \times q_m)} \geq \frac{\ln \ln(2^{19} \times q_1 \times \dots \times q_{m-1})}{\ln q_m}$$

which is trivially true for  $q_m \geq \ln(2^{19} \times q_1 \times \dots \times q_{m-1} \times q_m)$  [3].

**Case 2** :  $q_m < \ln(2^{19} \times q_1 \times \dots \times q_{m-1} \times q_m) = \ln(2^{19} \times n)$ .

We need to prove

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \frac{\sigma(n)}{n} \leq e^\gamma \times \ln \ln(2^{19} \times n).$$

We know that  $\frac{3}{2} < 1.6 = \frac{4 \times 6}{3 \times 5}$ . Nevertheless, we could have that

$$\frac{3}{2} \times \frac{\sigma(n)}{n} \times \frac{\pi^2}{6} < \frac{4 \times 6 \times \sigma(n)}{3 \times 5 \times n} \times \frac{\pi^2}{6} = \frac{\sigma(3 \times 5 \times n)}{3 \times 5 \times n} \times \frac{\pi^2}{6} \leq e^\gamma \times \ln \ln(2^{19} \times n)$$

where this is possible because of  $3 \nmid n$  and  $5 \nmid n$ . If we apply the logarithm to the both sides of the inequality, then we obtain that

$$\begin{aligned} & \ln\left(\frac{\pi^2}{6}\right) + (\ln(3+1) - \ln 3) + (\ln(5+1) - \ln 5) + \sum_{j=3}^m (\ln(q_j + 1) - \ln q_j) \leq \\ & \gamma + \ln \ln \ln(2^{19} \times n). \end{aligned}$$

From the reference [3], we note that

$$\ln(p_1 + 1) - \ln p_1 = \int_{p_1}^{p_1+1} \frac{dt}{t} < \frac{1}{p_1}.$$

In addition, note also that  $\ln(\frac{\pi^2}{6}) < \frac{1}{2}$ . It is enough to prove that

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{q_1} + \dots + \frac{1}{q_m} \leq \sum_{p \leq q_m} \frac{1}{p} + \leq \gamma + \ln \ln \ln(2^{19} \times n)$$

where  $p \leq q_m$  means all the prime lesser than or equal to  $q_m$ . However, we know that

$$\gamma + \ln \ln q_m < \gamma + \ln \ln \ln(2^{19} \times n)$$

since  $q_m < \ln(2^{19} \times n)$  and therefore, we would only need to prove that

$$\sum_{p \leq q_m} \frac{1}{p} \leq \gamma + \ln \ln q_m$$

which is true according to the Lemma 2.1 from the article reference [3]. In this way, we finally show the Theorem is indeed satisfied. ■

**Theorem 2.5** *Given a natural number  $n = 2^{a_1} \times 3^{a_2} \times 5^{a_3} > 5040$  such that  $a_1, a_2, a_3 \geq 0$  are integers, then the Robin's inequality is true for  $n$ .*

**Proof** Given a natural number  $n = p_1^{a_1} \times p_2^{a_2} \times \dots \times p_m^{a_m} > 5040$  such that  $p_1, p_2, \dots, p_m$  are prime numbers, we need to prove that

$$\frac{\sigma(n)}{n} < e^\gamma \times \ln \ln n$$

that would be the same as

$$(2.6) \quad \prod_{i=1}^m \frac{p_i}{p_i - 1} < e^\gamma \times \ln \ln n$$

according to Theorem 2.1. Given a natural number  $n = 2^{a_1} \times 3^{a_2} \times 5^{a_3} > 5040$  such that  $a_1, a_2, a_3 \geq 0$  are integers, we have that

$$\prod_{i=1}^m \frac{p_i}{p_i - 1} \leq \frac{2 \times 3 \times 5}{1 \times 2 \times 4} = 3.75 < e^\gamma \times \ln \ln(5040) \approx 3.81.$$

However, we know for  $n > 5040$ , we have that

$$e^\gamma \times \ln \ln(5040) < e^\gamma \times \ln \ln n$$

and thus, the proof is completed. ■

**Theorem 2.6** *The Robin's inequality is true for every natural number  $n > 5040$  when  $15 \nmid n$ . More specifically: every counterexample should be divisible by  $2^{20} \times 3^{13} \times 5^8 \times k_1$  or either  $2^{20} \times 3^{13} \times k_2$  or  $2^{20} \times 5^8 \times k_3$ , where  $2 \nmid k_1$ ,  $3 \nmid k_1$ ,  $5 \nmid k_1$ ,  $2 \nmid k_2$ ,  $3 \nmid k_2$ ,  $2 \nmid k_3$  and  $5 \nmid k_3$ .*

**Proof** Given a natural number  $n = p_1^{a_1} \times p_2^{a_2} \times \dots \times p_m^{a_m} > 5040$  such that  $p_1, p_2, \dots, p_m$  are prime numbers, then we will check the Robin's inequality for  $n$ . We know this true when the greatest prime divisor of  $n$  is lesser than or equal to 5 according to Theorem 2.5. Another case is when  $3 \nmid n$  and  $5 \nmid n$ .

We need to prove the inequality (2.6) for that case. In addition, the inequality (2.6) would be true when

$$\frac{\pi^2}{6} \times \prod_{i=1}^m \frac{p_i + 1}{p_i} < e^\gamma \times \ln \ln n$$

according to Theorem 2.2. Using the properties of the equation (2.2), we obtain that will be equivalent to

$$\frac{\pi^2}{6} \times \frac{\sigma(n')}{n'} < e^\gamma \times \ln \ln n$$

where  $n' = q_1 \times \dots \times q_m$  is the squarefree representation of  $n$ . However, the Robin's inequality has been proved for all integers  $n$  not divisible by 2 (which are bigger than 10) [3]. Hence, we need to prove when  $2 \mid n'$ . In addition, we know the Robin's inequality is true for every  $n > 5040$  such that  $2^k \mid n$  for  $1 \leq k \leq 19$  [4]. Consequently, we only need to prove that for all  $n > 5040$  such that  $2^{20} \mid n$  and thus, we have that

$$e^\gamma \times n' \times \ln \ln(2^{19} \times \frac{n'}{2}) < e^\gamma \times n' \times \ln \ln n$$

because of  $2^{19} \times \frac{n'}{2} < n$  when  $2^{20} \mid n$  and  $2 \mid n'$ . In this way, we only need to prove that

$$\frac{\pi^2}{6} \times \sigma(n') \leq e^\gamma \times n' \times \ln \ln(2^{19} \times \frac{n'}{2}).$$

According to the equation (2.3) and  $2 \mid n'$ , we have that

$$\frac{\pi^2}{6} \times 3 \times \sigma(\frac{n'}{2}) \leq e^\gamma \times 2 \times \frac{n'}{2} \times \ln \ln(2^{19} \times \frac{n'}{2})$$

which is the same as

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \sigma(\frac{n'}{2}) \leq e^\gamma \times \frac{n'}{2} \times \ln \ln(2^{19} \times \frac{n'}{2})$$

which is true according to the Theorem 2.4. In addition, we know the Robin's inequality is true for every  $n > 5040$  such that  $3^i \mid n$  and  $5^j \mid n$  for  $1 \leq i \leq 12$  and  $1 \leq j \leq 7$  [4]. To sum up, we have finally proved this result. ■

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