Gray Cycles of Maximum Length Related to K-Character Substitutions

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Abstract. Given a word binary relation τ we define a τ-Gray cycle over a finite language X to be a permutation $\left(w_{[i]}\right)_{0 \le i \le |X|-1}$ of X such that each word w_i is an image of the previous word w_{i-1} by τ. We introduce the complexity measure $\lambda(n)$, equal to the largest cardinality of a language X having words of length at most n, and such that some τ-Gray cycle over X exists. The present paper is concerned with the relation $\tau = \sigma_k$, the so-called k-character substitution, such that (u, v) belongs to σ_k if, and only if, the Hamming distance of u and v is k. We compute the bound $\lambda(n)$ for all cases of the alphabet cardinality and the argument n.

Keywords: Character · Complexity · Cycle · Gray · Substitution · Relation · Word

1 Introduction

In the framework of combinatorial algorithms, one of the most documented questions consists in the development of methods in order to generate, exactly once, all the objects in some specific class [15]. Many topics are concerned by such a problem: suffice it to mention sequence counting [1], signal encoding [16], and data compression [18].

The so-called *binary Gray codes* first appeared in [10]: given a binary alphabet A and some positive integer n, they referred to sequences with maximum length of pairwise different n-tuples of characters (that is, words in A^n), provided that any pair of consecutive items differ by exactly one character. Shortly after, a similar study was drawn in the framework of non-binary alphabets [8]. With regard to other famous combinatorial classes of objects, the term of *combinatorial Gray code*, for its part, appeared in [11]: actually, the difference between successive items, although being fixed, need not to be small [20]. Generating all permutations of a given n-element set constitutes a noticeable example [7]. Subsets of fixed size are also concerned [6], as well as cross-bifix free words [2], Debruijn sequences [9], set partitions [13], necklaces [19]: the list is far to be exhaustive. Combinatorial Gray sequences are often needed to be *cyclic* [3], in the sense that the initial term itself can be retrieved as successor of the last one. Such a condition justifies the terminology of *Gray cycle* [14, Sect. 7.2.1.1].

In view of some formal framework, we notice that each of the sequences we mentioned above involves some word binary relation $\tau \in A^* \times A^*$, where A stands

for a finite alphabet, and A^* for the free monoid it generates. For its part, the combinatorial class of objects can be modelled by some finite langage $X \subseteq A^*$. Given a sequence of words we denote in square brackets the corresponding indices: this will allow us to make a difference from w_i, the character in *position i* in a given word w. We define a *cyclic Gray sequence over X, with respect to τ* (for short: *τ-Gray cycle over X*) as every finite sequence of words $\left(w_{[i]}\right)_{i \in [0,|X|-1]}$ satisfying each of the three following conditions:

(G1) For every word $x \in X$, some $i \in [0, |X| - 1]$ exists such that we have $x = w_{[i]}$;

(G2) For every $i \in [1, |X| - 1]$, we have $w_{[i]} \in \tau\left(w_{[i-1]}\right)$; in addition, the condition $w_{[0]} \in \tau\left(w_{[|X|-1]}\right)$ holds;

(G3) For every pair $i, j \in [0, |X| - 1]$, $i \neq j$ implies $w_{[i]} \neq w_{[j]}$.

With this definition, the set X need not to be *uniform* that is, in the Gray cycle the terms may have a variable length. For instance, given the alphabet $A = \{0, 1\}$, take for τ the word binary relation Λ_1 which, with every word w associates all the strings located within a *Levenshtein distance* of 1 from w (see e.g. [17]); with such a relation, the sequence $(0, 00, 01, 11, 10, 1)$ is a Λ_1-Gray cycle over $X = A \cup A^2$. Actually, in addition to the topics we mentioned above, two other famous topics are involved by those Gray cycles. Firstly, with regard to graph theory, a τ-Gray cycle over X exists if, and only if, there is some hamiltonian circuit in the graph of the relation τ (see e.g. [20]). Secondly, given a binary word relation $\tau \subseteq A^* \times A^*$, and given $X \subseteq A^*$, if some τ-Gray cycle exists over X, then X is *τ-closed* [17] that is, the inclusion $\tau(X) \subseteq X$ holds, where $\tau(X)$ stands for the set of the images of the words in X under the relation τ. Such closed sets actually constitute a special subfamily in the famous *dependence systems* (see [5] or [12, Chap. 6]). Notice that, given a *τ-closed* set $X \subseteq A^*$, there do not necessarily exist non-empty τ-Gray cycles over X. A typical example is provided by τ being id_{A^*}, the identity over A^*, with respect to which every finite set $X \subseteq A^*$ is closed; however non-empty τ-Gray cycle can exist only over singletons.

In the present paper, given a positive integer n, and denoting by $A^{\leq n}$ the set of the words with length not greater than n, we consider the family of all sequences that can be a τ-Gray cycle over some subset X of $A^{\leq n}$. This is a natural question to study those sequences of maximum length which, of course, correspond to subsets X of maximum cardinality; such a length, which we denote by $\lambda_{A,\tau}(n)$, means introducing some complexity measure for the word binary relation τ. With regard to the preceding examples we have $\lambda_{\{0,1\},\Lambda_1}(2) = 6$; moreover, for every alphabet A and positive integer n, the identity $\lambda_{A,id_{A^*}}(n) = 1$ holds. We focus on the case where τ is σ_k, the so-called *k-character substitution*: with every word with length at least k, say w, this relation associates all the words w', with $|w'| = |w|$ and such that the character w'_i differs from w_i in exactly k values of $i \in [1, |w|]$. As commented in [12, 17], σ_k has noticeable inference in the famous framework of *error detection*. On the other hand, by definition, $w' \in \sigma_k(w)$ implies $|w'| = |w|$ therefore, if there is some σ_k-Gray cycle over X, then X is a uniform set. From this point of view, the classical Gray codes, which

allow to generate all n-tuples over A, correspond to σ_1-Gray cycles over A^n, furthermore we have $\lambda_{A,\sigma_1}(n) = |A|^n$. In addition, in the case where A is a binary alphabet, it can be easily proved that, for every $n \geq 3$, we have $\lambda_{A,\sigma_2}(n) = 2^{n-1}$ [14, Exercice 8, p. 77]. However, in the most general case, although an exhaustive description of σ_k-closed variable-length codes has been provided in [17], the question of computing some σ_k-Gray cycle of maximum length has remained open. In our paper we establish the following result:

Theorem *Let A be a finite alphabet, $k \geq 1$, and $n \geq k$. Then exactly one of the following conditions holds:*

(i) $|A| \geq 3$, $n \geq k$, and $\lambda_{A,\sigma_k}(n) = |A|^n$;

(ii) $|A| = 2$, $n = k$, and $\lambda_{A,\sigma_k}(n) = 2$;

(iii) $|A| = 2$, $n \geq k+1$, k is odd and $\lambda_{A,\sigma_k}(n) = |A|^n$;

(iv) $|A| = 2$, $n \geq k+1$, k is even, and $\lambda_{A,\sigma_k}(n) = |A|^{n-1}$.

In addition, in each case some σ_k-Gray cycle of maximum length can be explicitly computed.

We now shortly describe the contents of the paper. In Section 2, we recall the two famous examples of the *binary* (resp., $|A|$-*arity*) *reflected Gray code*. By applying some induction based methods, in Section 3 and Section 4, these sequences allow to compute special families of σ_k-Gray cycles with maximum length. In Section 5, in the case where A is a binary alphabet, and k an even positive integer, we also compute a family of Gray cycles with maximum length; in addition some further development is raised.

2   Preliminaries

Several definitions and notation have already been fixed. In the whole paper, A stands for some finite alphabet, with $|A| \geq 2$. Given a word $w \in A^*$, we denote by $|w|$ its length; in addition, for every $a \in A$, we denote by $|w|_a$ the number of occurrences of the character (or character) a in w.

The reflected binary Gray cycle
Let $A = \{0,1\}$, and $n \geq 1$. The most famous example of σ_1-Gray cycle over A^n is certainly the so-called *reflected binary Gray code* (see e.g. [10] or [14, p. 6]): in the present paper we denote it by $g^{n,1}$. It can be defined by the recurrent sequence initialized with $g_{[0]}^{n,1} = 0^n$, and satisfying the following property: for every $i \in [1, |A|^n - 1]$, a unique integer $j \in [1, n]$ exists such that, in both words $g_{[i]}^{n,1}$ and $g_{[i-1]}^{n,1}$ the corresponding characters in position j differ; in addition, the position j is chosen to be maximum in such a way that $g_{[i]}^{n,1} \notin \{g_{[0]}^{n,1}, \cdots, g_{[i-1]}^{n,1}\}$.

Example 1. In what follows we provide a column representation of $g^{2,1}$ and $g^{3,1}$:

$g^{2,1}$	$g^{3,1}$
00	000
01	001
11	011
10	010
	110
	111
	101
	100

By construction, for every $n \geq 1$, each of the following identities holds:

$$g^{n,1}_{[0]} = 0^n, \quad g^{n,1}_{[1]} = 0^{n-1}1, \quad g^{n,1}_{[2^n-2]} = 10^{n-2}1, \quad g^{n,1}_{[2^n-1]} = 10^{n-1} \tag{1}$$

The $|A|$-ary reflected Gray cycle

The preceding construction can be extended in order to obtain the so-called $|A|$-ary reflected Gray code [4, 8], a σ_1-Gray cycle over A^n, which we denote by $h^{n,1}$. Set $A = \{0, \cdots, p-1\}$ and denote by θ the cyclic permutation $(0, 1, \ldots p-1)$. The sequence $h^{n,1}$ is initialized with $h^{n,1}_{[0]} = 0^n$. In addition, for every $i \in [1, |A|^n - 1]$, a unique integer j exists such that c and d, the characters respectively in position j in $h^{n,1}_{[i-1]}$ and $h^{n,1}_{[i]}$, satisfy both the following conditions: (i) $d = \theta(c)$; (ii) j is the greatest integer in $[1, n]$ such that $h^{n,1}_{[i]} \notin \{h^{n,1}_{[0]}, \cdots, h^{n,1}_{[i-1]}\}$.

Example 2. For $A = \{0, 1, 2\}$ the sequence $h^{3,1}$ is the concatenation in this order of the three following subsequences:

$h^{3,1}$		
000	120	210
001	121	211
002	122	212
012	102	222
010	100	220
011	101	221
021	111	201
022	112	202
020	110	200

3 The case where we have $k \geq 1$ and $|A| \geq 3$

Let $n \geq k \geq 1$, $p \geq 3$, and $A = \{0, 1, \cdots, p-1\}$. We will indicate the construction of a peculiar σ_k-Gray cycle over A^n, namely $h^{n,k}$. This will be done by applying some induction over $k \geq 1$: in view of that we set $n_0 = n - k + 1$. The starting point corresponds to $h^{n_0,1}$, the p-arity reflected Gray code over A^{n_0} as reminded in Section 2. For the induction stage, starting with some σ_{k-1}-Gray cycle over

A^{n-1}, namely $h^{n-1,k-1}$, we compute the sequence $h^{n,k}$ as indicated in what follows: let $i \in [0, p^n - 1]$, and let $q \in [0, p-1]$, $r \in [0, p^{n-1} - 1]$ be the unique pair of non-negative integers such that $i = qp^{n-1} + r$. We set:

$$h^{n,k}_{[i]} = h^{n,k}_{[qp^{n-1}+r]} = \theta^{q+r}(0)h^{n-1,k-1}_{[r]} \tag{2}$$

As illustrated by Example 3, the resulting sequence $h^{n,k}$ is actually the concatenation in this order of p subsequences namely $C_0, \ldots, C_{p-1}$, with $C_q = \left(h^{n,k}_{[qp^{n-1}+r]}\right)_{0 \le r \le p^{n-1}-1}$, for each $q \in [0, p-1]$. Since θ is one-to-one, given a pair of different integers $q, q' \in [0, p-1]$, for every $r \in [0, p^{n-1} - 1]$, in each of the subsequences C_q, $C_{q'}$, the words $h^{n,k}_{[qp^{n-1}+r]}$ and $h^{n,k}_{[q'p^{n-1}+r]}$ only differ in their initial characters, which respectively are $\theta^{q+r}(0)$ and $\theta^{q'+r}(0)$. In addition, since $h^{n-1,k-1}$ is a σ_{k-1}-Gray cycle over A^{n-1}, we have $\left|h^{n,k}\right| = p\left|h^{n-1,k-1}\right| = p^n$.

Example 3. Let $A = \{0, 1, 2\}$, $n = 3$, $k = 2$, thus $p = 3$, $n_0 = 2$. By starting with the sequence $h^{n-1,k-1} = h^{2,1}$, $h^{n,k}$ is the concatenation of C_0, C_1, and C_2 :

$h^{n-1,k-1}$		$h^{n,k}$		
00		000	100	200
01		101	201	001
02		202	002	102
12		012	112	212
10		110	210	010
11		211	011	111
21		021	121	221
22		122	222	022
20		220	020	120

Proposition 1. $h^{n,k}$ *is a* σ_k-*Gray cycle over* A^n.

Proof. We argue by induction over $k \ge 1$. With regard to the base case, as indicated above $h^{n_0,1}$ is the $|A|$-ary reflected Gray sequence. In view of the induction stage, we assume that the finite sequence $h^{n-1,k-1}$ is a σ_{k-1}-Gray cycle over A^{n-1}, for some $k \ge 2$.

(i) We start by proving that $h^{n,k}$ satisfies Condition (G2). This will be done through the three following steps:

(i.i) Firstly, we prove that, for each $q \in [0, p-1]$, in the subsequence C_q two consecutive terms are necessarily in correspondence under σ_k. Given $r \in [0, p^{n-1}-1]$, by definition, we have $\theta^{r+q}(0) \in \sigma_1\left(\theta^{r+q-1}(0)\right)$. Since $h^{n-1,k-1}$ satisfies Condition (G2), we have $h^{n-1,k-1}_{[r]} \in \sigma_{k-1}\left(h^{n-1,k-1}_{[r-1]}\right)$. We obtain $\theta^{r+q}(0)h^{n-1,k-1}_{[r]} \in \sigma_k\left(\theta^{r+q-1}(0)h^{n-1,k-1}_{[r-1]}\right)$, thus according to (2): $h^{n,k}_{[qp^{n-1}+r]} \in \sigma_k\left(h^{n,k}_{[qp^{n-1}+r-1]}\right)$.

(i.ii) Secondly, we prove that, for each $q \in [1, p-1]$, the last term of C_{q-1} and the initial term of C_q are also connected by σ_k. Take $r = 0$ in Equation (2): it follows from $\theta^{p^{n-1}} = id_A$ that we have $h^{n,k}_{[qp^{n-1}]} = \theta^q(0)h^{n-1,k-1}_{[0]} =$

$\theta^{p^{n-1}+q}(0)h_{[0]}^{n-1,k-1}$. In (2) take $r = p^{n-1}-1$, moreover substitute $q-1 \in [0, p-2]$ to $q \in [1, p-1]$: we obtain $h_{[qp^{n-1}-1]}^{n,k} = \theta^{q+p^{n-1}-2}(0)h_{[p^{n-1}-1]}^{n-1,k-1}$. It follows from $p = |A| \geq 3$ that $\theta(0) \neq \theta^{-2}(0)$: since θ is one-to-one this implies $\theta^{q+p^{n-1}}(0) \neq \theta^{q+p^{n-1}-2}(0)$, thus $\theta^{q+p^{n-1}}(0) \in \sigma_1\left(\theta^{q+p^{n-1}-2}(0)\right)$. By induction we have $h_{[0]}^{n-1,k-1} = \sigma_{k-1}\left(h_{[p^{n-1}-1]}^{n-1,k-1}\right)$, thus $h_{[qp^{n-1}]}^{n,k} \in \sigma_k\left(\theta^{q+p^{n-1}-2}(0)h_{[qp^{n-1}-1]}^{n-1,k-1}\right)$ that is, $h_{[qp^{n-1}]}^{n,k} \in \sigma_k\left(h_{[qp^{n-1}-1]}^{n,k}\right)$.

(i.iii) At last, we prove that the first term of C_0 is an image under σ_k of the last term of C_{p-1}. In Equation (2), take $q = 0$ and $r = 0$: we obtain $h_{[0]}^{n,k} = 0h_{[0]}^{n-1,k-1}$. Similarly, by setting $q = p-1$ and $r = p^{n-1}-1$, we obtain $h_{[(p-1)p^{n-1}+p^{n-1}-1]}^{n,k} = \theta^{p^{n-1}+p-2}(0)h_{[p^{n-1}-1]}^{n-1,k-1}$, thus $h_{[p^n-1]}^{n,k} = \theta^{-2}(0)h_{[p^{n-1}-1]}^{n-1,k-1}$. Since $h^{n-1,k-1}$ is a σ_{k-1}-Gray cycle over A^{n-1}, we have $h_{[0]}^{n-1,k-1} \in \sigma_{k-1}\left(h_{[p^{n-1}-1]}^{n-1,k-1}\right)$. In addition, it follows from $p \geq 3$, that $\theta^{-2}(0) \neq 0$, thus $0 \in \sigma_1\left(\theta^{-2}(0)\right)$. We obtain $h_{[0]}^{n,k} \in \sigma_k\left(\theta^{-2}(0)h_{[p^{n-1}-1]}^{n-1,k-1}\right)$, thus $h_{[0]}^{n,k} \in \sigma_k\left(h_{[p^n-1]}^{n,k}\right)$ that is, the required property.

(ii) Now, we prove that, in the sequence $h^{n,k}$ all terms are pairwise different. Let $i, i' \in [0, p^n-1]$ such that $h_{[i]}^{n,k} = h_{[i']}^{n,k}$ and consider the unique 4-tuple of integers $q, q' \in [0, p-1]$, $r, r' \in [0, p^{n-1}-1]$ such that $i = qp^{n-1}+r$ and $i' = q'p^{n-1}+r'$. According to (2) we have $\theta^{q+r}(0)h_{[r]}^{n-1,k-1} = \theta^{q'+r'}(0)h_{[r']}^{n-1,k-1}$, thus $\theta^{q+r}(0) = \theta^{q'+r'}(0) \in A$ and $h_{[r]}^{n-1,k-1} = h_{[r']}^{n-1,k-1}$. Since $h^{n-1,k-1}$ satisfies (G3), the second equation implies $r = r'$, whence the first one implies $\theta^q(0) = \theta^{q'}(0)$, thus $q = q'$ mod p. Since we have $q, q' \in [0, p-1]$ we obtain $q = q'$, thus $i = i'$.

(iii) Finally, since $h^{n,k}$ satisfies (G3), we have $\left|\bigcup_{0 \leq i \leq p^n-1}\{h_{[i]}^{n,k}\}\right| = p^n$, hence $h^{n,k}$ satisfies Condition (G1). □

4   The case where A is a binary alphabet, with k odd

Let $A = \{0, 1\}$ and $n \geq k$. Classically, the cyclic permutation θ, which was introduced in Section 3, can be extended into a one-to-one monoid homomorphism onto A^*: in view of this, we set $\theta(\varepsilon) = \varepsilon$ and, for any non-empty n-tuple of characters $a_1, \cdots, a_n \in A$, $\theta(a_1 \cdots a_n) = \theta(a_1) \cdots \theta(a_n)$. Trivially, in the case where we have $n = k$, if a non-empty σ_k-Gray code exists over $X \subseteq A^n$, then we have $X = \{x, \theta(x)\}$, for some $x \in A^n$. In the sequel of the paper, we assume $n \geq k+1$. In what follows, we indicate the construction of a peculiar pair of σ_k-Gray cycles over A^n, namely $\gamma^{n,k}$ and $\rho^{n,k}$. This will be done by induction over k', the unique non-negative integer such that $k = 2k'+1$. Let $n_0 = n-2k' = n-k+1$.
– For the base case, $\gamma^{n_0,1}$ and $\rho^{n_0,1}$ are computed by applying some reversal (resp., shift) over the sequence $g^{n_0,1}$ from Section 2:

$$\gamma_{[0]}^{n_0,1} = g_{[0]}^{n_0,1} \text{ and } \gamma_{[i]}^{n_0,1} = g_{[2^{n_0}-i]}^{n_0,1} \quad (1 \leq i \leq 2^{n_0}-1); \tag{3}$$

$$\rho_{[0]}^{n_0,k} = g_{[2^{n_0}-1]}^{n_0,1} \text{ and } \rho_{[i]}^{n_0,k} = g_{[i-1]}^{n_0,1} \quad (1 \leq i \leq 2^{n_0}-1). \tag{4}$$

By construction, $\gamma^{n_0,1}$ and $\rho^{n_0,1}$ are σ_1-Gray Cycle over A^{n_0}. Moreover we have:

$$\gamma^{n_0,1}_{[0]} = g^{n_0,1}_{[0]} = 0^{n_0} \quad \text{and} \quad \rho^{n_0,1}_{[0]} = g^{n_0,1}_{[2^{n_0}-1]} = 10^{n_0-1}; \tag{5}$$

$$\gamma^{n_0,1}_{[2^{n_0}-1]} = g^{n_0,1}_{[1]} = 0^{n_0-1}1 \quad \text{and} \quad \rho^{n_0,1}_{[2^{n_0}-1]} = g^{n_0,1}_{[2^{n_0}-2]} = 10^{n-2}1. \tag{6}$$

Example 4. For $n_0 = 3$ we obtain the following sequences:

$g^{3,1}$	$\gamma^{3,1}$	$\rho^{3,1}$
000	000	100
001	100	000
011	101	001
010	111	011
110	110	010
111	010	110
101	011	111
100	001	101

– In view of the induction step, we assume that we have computed the σ_k-Gray cycles $\gamma^{n,k}$ and $\rho^{n,k}$. Notice that we have $n+2 = n_0 + 2(k'+1) = n_0 + (k+2) - 1$: below we explain the construction of the two corresponding 2^{n+2}-term sequences $\gamma^{n+2,k+2}$ and $\rho^{n+2,k+2}$. Let $i \in [0, 2^{n+2} - 1]$, and let $q \in [0,3]$, $r \in [0, 2^n - 1]$ be the unique pair of integers such that $i = q2^n + r$. Since we have $r \in [0, 2^n - 1]$, taking for q the value $q = 0$ (resp., 1, 2, 3), we state the corresponding equation (7a) (resp., (7b),(7c),(7d)):

$$\gamma^{n+2,k+2}_{[r]} = \theta^r(00)\gamma^{n,k}_{[r]}; \tag{7a}$$

$$\gamma^{n+2,k+2}_{[2^n+r]} = \theta^r(01)\rho^{n,k}_{[r]}; \tag{7b}$$

$$\gamma^{n+2,k+2}_{[2.2^n+r]} = \theta^r(11)\gamma^{n,k}_{[r]}; \tag{7c}$$

$$\gamma^{n+2,k+2}_{[3.2^n+r]} = \theta^r(10)\rho^{n,k}_{[r]}. \tag{7d}$$

Similarly the sequence $\rho^{n+2,k+2}$ is computed by substituting, in the preceding equations, the 4-tuple $(10, 11, 01, 00)$ to $(00, 01, 11, 10)$:

$$\rho^{n+2,k+2}_{[r]} = \theta^r(10)\gamma^{n,k}_{[r]}; \tag{8a}$$

$$\rho^{n+2,k+2}_{[2^n+r]} = \theta^r(11)\rho^{n,k}_{[r]}; \tag{8b}$$

$$\rho^{n+2,k+2}_{[2.2^n+r]} = \theta^r(01)\gamma^{n,k}_{[r]}; \tag{8c}$$

$$\rho^{n+2,k+2}_{[3.2^n+r]} = \theta^r(00)\rho^{n,k}_{[r]}. \tag{8d}$$

Example 5. (Example 4 continued) $\gamma^{5,3}$ is the concatenation, in this order, of the 4 following subsequences:

$\overbrace{\gamma^{3,1}}$	$\overbrace{\rho^{3,1}}$	$\overbrace{\gamma^{3,1}}$	$\overbrace{\rho^{3,1}}$
00 000	01 100	11 000	10 100
11 100	10 000	00 100	01 000
00 101	01 001	11 101	10 001
11 111	10 011	00 111	01 011
00 110	01 010	11 110	10 010
11 010	10 110	00 010	01 110
00 011	01 111	11 011	10 111
11 001	10 101	00 001	01 101

Lemma 1. *$\gamma^{n,k}$ and $\rho^{n,k}$ satisfy both the conditions* (G1) *and* (G3).

Proof. We argue by induction over $k' \geq 0$, with $k = 2k' + 1$. the base case corresponds to $k' = 0$ that is, $k = 1$ and $n = n_0$: as indicated above, $\gamma^{n_0,1}$ and $\rho^{n_0,1}$ are σ_1-Gray cycles over A^n. In view of the induction step we assume that, for some $k' \geq 0$, both the sequences $\gamma^{n,k}$ and $\rho^{n,k}$ are σ_k-Gray cycles over A^n.
(i) In order to prove that $\gamma^{n+2,k+2}$ satisfies Condition (G3), let $i, i' \in [0, 2^{n+2} - 1]$ such that $\gamma^{n+2,k+2}_{[i]} = \gamma^{n+2,k+2}_{[i']}$, and $q, q' \in [0,3]$, $r, r' \in [0, 2^n - 1]$ such that $i = q2^n + r$, $i' = q'2^n + r'$. According to the equations (7a)–(7d), words $x, x' \in A^2$, $w, w' \in A^n$ exist such that $\gamma^{n+2,k+2}_{[i]} = \theta^r(x)w$ and $\gamma^{n+2,k+2}_{[i']} = \theta^{r'}(x')w'$ that is, $\theta^r(x) = \theta^{r'}(x') \in A^2$ and $w = w'$. By the definition of θ, this implies either $x, x' \in \{00, 11\}$ or $x, x' \in \{01, 10\}$ that is, by construction, either $q, q' \in \{0, 2\}$, $x, x' \in \{00, 11\}$, $w = \gamma^{n,k}_{[r]} = \gamma^{n,k}_{[r']}$, or $q, q' \in \{1, 3\}$, $x, x' \in \{01, 10\}$, $w = \rho^{n,k}_{[r]} = \rho^{n,k}_{[r']}$. Since $\gamma^{n,k}$ and $\rho^{n,k}$ satisfies (G3), in any case we have $r = r'$. This implies $\theta^r(x) = \theta^r(x')$, thus $x = x'$. With regard to the equations (7a)–(7d), this corresponds to $q = q'$, thus $i = q2^n + r = q'2^n + r = i'$, therefore $\gamma^{n+2,k+2}$ satisfies Condition (G3).
(ii) By substituting $(10, 11, 01, 00)$ to $(00, 01, 11, 10)$, according to (8a)–(8d), similar arguments prove that $\rho^{n+2,k+2}_{[i]} = \rho^{n+2,k+2}_{[i']}$ implies $i = i'$, thus $\rho^{n+2,k+2}$ also satisfies (G3) (see Annex 1).
(iii) Since $\gamma^{n+2,k+2}$ satisfies (G3), we have $\bigcup_{0 \leq i \leq 2^{n+2}-1}\{\gamma^{n+2,k+2}_i\} = A^{n+2}$, hence it satisfies (G1). Similarly, since $\rho^{n+2,k+2}$ satisfies (G3) it satisfies (G1). $\square$

In order to prove that our sequences satisfy (G2), we prove the following property:

Lemma 2. *We have $\gamma^{n,k}_{[0]} \in \sigma_{k+1}\left(\rho^{n,k}_{[2^n-1]}\right)$ and $\rho^{n,k}_{[0]} \in \sigma_{k+1}\left(\gamma^{n,k}_{[2^n-1]}\right)$.*

Proof. We argue by induction over the integer $k' \geq 0$. The case $k' = 0$ corresponds to $k = 1$ and $n = n_0$: with such a condition, our property comes from the identities (5) and (6). For the induction step, we assume that, for some $k' \geq 0$, we have $\gamma^{n,k}_{[0]} \in \sigma_{k+1}\left(\rho^{n,k}_{[2^n-1]}\right)$ and $\rho^{n,k}_{[0]} \in \sigma_{k+1}\left(\gamma^{n,k}_{[2^n-1]}\right)$.

(i) In (7a), by taking $r = 0$ we obtain $\gamma_{[0]}^{n+2,k+2} = 00\gamma_{[0]}^{n,k}$, hence by induction:
$\gamma_{[0]}^{n+2,k+2} \in 00\sigma_{k+1}\left(\rho_{[2^n-1]}^{n,k}\right) \subseteq \sigma_{k+3}\left(11\rho_{[2^n-1]}^{n,k}\right)$. By setting $r = 2^n - 1$ in (8d), we obtain $\rho_{[2^{n+2}-1]}^{n+2,k+2} = 11\rho_{[2^n-1]}^{n,k}$, thus $\gamma_{[0]}^{n+2,k+2} \in \sigma_{k+3}\left(\rho_{[2^{n+2}-1]}^{n+2,k+2}\right)$.

(ii) Similarly, by setting $r = 0$ in (8a), and by induction we have: $\rho_{[0]}^{n+2,k+2} = 10\gamma_{[0]}^{n,k} \in \sigma_{k+3}\left(01\rho_{[2^n-1]}^{n,k}\right)$. By taking $r = 2^n - 1$ in (7d) we obtain $\gamma_{[2^{n+2}-1]}^{n+2,k+2} = 01\rho_{[2^n-1]}^{n,k}$, thus $\rho_{[0]}^{n+2,k+2} \in \sigma_{k+3}\left(\gamma_{[2^{n+2}-1]}^{n+2,k+2}\right)$. $\hfill\square$

Since the equations (7a)–(8d) look alike, one may be tempted to compress them thanks to some unique generic formula. Based on our tests, such a formula needs to introduce at least two additionnal technical parameters, which would make their handling tedious. In the proof of the following result, we have opted to report some case-by-case basis argumentation: this has the advantage of making use of arguments which, although being similar, actually are easy to read.

Proposition 2. *Both the sequences $\gamma^{n,k}$ and $\rho^{n,k}$ are σ_k-Gray cycles over A^n.*

Proof sketch Once more we argue by induction over $k' \geq 0$. Since $\gamma^{n_0,1}$ and $\rho^{n_0,1}$ are σ_1-Gray cycles over A^n, the property holds for $k' = 0$. In view of the induction stage, we assume that, for some $k' \geq 0$ both the sequences $\gamma^{n,k}$ and $\rho^{n,k}$ are σ_k-Gray cycles over A^n. According to Lemma 1, it remains to establish that $\gamma^{n+2,k+2}$ and $\rho^{n+2,k+2}$ satisfy Condition (G2) that is:

$$(\forall q \in \{0,1,2,3\})(\forall r \in [1, 2^n - 1]) \; \gamma_{[q2^n+r]}^{n+2,k+2} \in \sigma_{k+2}\left(\gamma_{[q2^n+r-1]}^{n+2,k+2}\right); \qquad (9)$$

$$(\forall q \in \{1,2,3\}) \; \gamma_{[q2^n]}^{n+2,k+2} \in \sigma_{k+2}\left(\gamma_{[q2^n-1]}^{n+2,k+2}\right); \qquad (10)$$

$$\gamma_{[0]}^{n+2,k+2} \in \sigma_{k+2}\left(\gamma_{[2^{n+2}-1]}^{n+2,k+2}\right). \qquad (11)$$

$$(\forall q \in \{0,1,2,3\})(\forall r \in [1, 2^n - 1]) \; \rho_{[q2^n+r]}^{n+2,k+2} \in \sigma_{k+2}\left(\rho_{[q2^n+r-1]}^{n+2,k+2}\right); \qquad (12)$$

$$(\forall q \in \{1,2,3\}) \; \rho_{[q2^n]}^{n+2,k+2} \in \sigma_{k+2}\left(\rho_{[q2^n-1]}^{n+2,k+2}\right); \qquad (13)$$

$$\rho_{[0]}^{n+2,k+2} \in \sigma_{k+2}\left(\rho_{[2^{n+2}-1]}^{n+2,k+2}\right). \qquad (14)$$

<u>*Condition* (9)</u>. (i) At first assume $q = 0$. According to (7a), and since $\gamma^{n,k}$ satisfies (G2), we have $\gamma_{[r]}^{n+2,k+2} = \theta^r(00)\gamma_{[r]}^{n,k} \in \theta^r(00)\sigma_k\left(\gamma_{[r-1]}^{n,k}\right)$, thus $\gamma_{[r]}^{n+2,k+2} \in \sigma_{k+2}\left(\theta^{r-1}(00)\gamma_{[r-1]}^{n,k}\right)$. In (7a), substitute $r-1$ to r (we have $0 \leq r-1 \leq 2^n - 2$): we obtain $\gamma_{[r-1]}^{n+2,k+2} = \theta^{r-1}(00)\gamma_{[r-1]}^{n,k}$, thus $\gamma_{[r]}^{n+2,k+2} \in \sigma_{k+2}\left(\gamma_{[r-1]}^{n+2,k+2}\right)$.

(ii) Now assume $q = 1$. According to (7b), and since $\rho^{n,k}$ satisfies (G2), we have $\gamma_{[2^n+r]}^{n+2,k+2} = \theta^r(01)\rho_{[r]}^{n,k} \in \sigma_{k+2}\left(\theta^{r-1}(01)\rho_{[r-1]}^{n,k}\right)$. In (7b), susbtitute $r-1$ to r: we obtain $\gamma_{[2^n+r-1]}^{n+2,k+2} = \theta^{r-1}(01)\rho_{[r-1]}^{n,k}$, thus $\gamma_{[2^n+r]}^{n+2,k+2} \in \sigma_{k+2}\left(\gamma_{[2^n+r-1]}^{n+2,k+2}\right)$.

(iii) For $q = 2$, the arguments are similar to those applied in (i) by substituting $\gamma_{[2 \cdot 2^n + r]}^{n+2, k+2}$ to $\gamma_{[r]}^{n+2, k+2}$, Equation (7c) to (7a), and 11 to 00.

(iv) Similarly, for $q = 3$ the proof is obtained by substituting in (ii) $\gamma_{[3 \cdot 2^n + r]}^{n+2, k+2}$ to $\gamma_{[2^n + r]}^{n+2, k+2}$, (7d) to (7b), and 10 to 01.

$\underline{\textit{Condition} \ (10).}$ (i) Assume $q = 1$ and take $r = 0$ in (7b). According to Lemma 2, we obtain $\gamma_{[2^n]}^{n+2, k+2} = 01 \rho_{[0]}^{n,k} \in 01 \sigma_{k+1} \left(\gamma_{[2^n - 1]}^{n,k} \right) \subseteq \sigma_{k+2} \left(11 \gamma_{[2^n - 1]}^{n,k} \right)$. Take $r = 2^n - 1$ in (7a): we obtain $\gamma_{[2^n - 1]}^{n+2, k+2} = 11 \gamma_{[2^n - 1]}^{n,k}$, thus $\gamma_{[2^n]}^{n+2, k+2} \in \sigma_{k+2} \left(\gamma_{[2^n - 1]}^{n+2, k+2} \right)$.

(ii) Now, assume $q = 2$, and set $r = 0$ in Equation (7c). According to Lemma 2 we have $\gamma_{[2 \cdot 2^n]}^{n+2, k+2} = 11 \gamma_{[0]}^{n,k} \in 11 \sigma_{k+1} \left(\rho_{[2^n - 1]}^{n,k} \right) \subseteq \sigma_{k+2} \left(10 \rho_{[2^n - 1]}^{n,k} \right)$. By taking $r = 2^n - 1$ in (7b), we obtain $\gamma_{[2 \cdot 2^n - 1)]}^{n+2, k+2} = 10 \rho_{[2^n - 1]}^{n,k}$, thus $\gamma_{[2 \cdot 2^n]}^{n+2, k+2} \in \sigma_{k+2} \left(\gamma_{[2 \cdot 2^n - 1]}^{n+2, k+2} \right)$.

(iii) For $q = 3$, substitute in (i) $\gamma_{[3 \cdot 2^n]}^{n+2, k+2}$ to $\gamma_{[2^n]}^{n+2, k+2}$, Equation (7d) to Equation (7b), (7c) to (7a), 10 to 01 and 00 to 11: similar arguments prove that $\gamma_{[3 \cdot 2^n]}^{n+2, k+2} \in \sigma_{k+2} \left(\gamma_{[3 \cdot 2^n - 1]}^{n+2, k+2} \right)$.

$\underline{\textit{Condition} \ (11).}$ Take $r = 0$ in (7a). According to Lemma 2, we have $\gamma_{[0]}^{n+2, k+2} = 00 \gamma_{[0]}^{n,k} \in 00 \sigma_{k+1} \left(\rho_{[2^n - 1]}^{n,k} \right) \subseteq \sigma_{k+2} \left(01 \rho_{[2^n - 1]}^{n,k} \right)$. By taking $r = 2^n - 1$ in (7d) we obtain $\gamma_{[2^{n+2} - 1]}^{n,k} = 01 \rho_{[2^n - 1]}^{n,k}$, thus $\gamma_{[0]}^{n+2, k+2} \in \sigma_{k+2} \left(\gamma_{[2^{n+2} - 1]}^{n,k} \right)$.

According to the structures of the equations (8a)–(8d), for proving the conditions (12)–(14), the method consists in substituting the word $\rho_{[r]}^{n+2, k+2}$ to $\gamma_{[r]}^{n+2, k+2}$, the 4-uple $(10, 11, 01, 00)$ to $(00, 01, 11, 10)$, and Equation (8a) $\big($resp., (8b), (8c), (8d)$\big)$ to Equation (7a) $\big($resp., (7b), (7c), (7d)$\big)$.   □

5   The case where we have $|A| = 2$ and k even

Beforehand, we recall some classical algebraic interpretation of the substitution σ_k in the framework of the binary alphabet $A = \{0, 1\}$. Denote by $\oplus$ the addition in the group $\mathbb{Z}/2\mathbb{Z}$ with identity 0. Given a positive integer n, and $w, w' \in A^n$, define $w \oplus w'$ as the unique word of A^n such that, for each $i \in [1, n]$: $(w \oplus w')_i = w_i \oplus w'_i$. With this notation the sets A^n and $(\mathbb{Z}/2\mathbb{Z})^n$ are in one-to-one correspondence. Moreover we have $w' \in \sigma_k(w)$ if, and only if, some word $u \in A^n$ exists such that $|u|_1 = k$ and $w = w' \oplus u$, therefore if k is even we have $|w|_1 = |w'|_1 \bmod 2$. Consequently, given a σ_k-Gray cycle $\left(\alpha_{[i]} \right)_{0 \le i \le m}$, for each $i \in [0, m]$ we have $\left| \alpha_{[i]} \right|_1 = \left| \alpha_{[0]} \right|_1 \bmod 2$. As a corollary, setting $\mathrm{Even}_1^n = \{ w \in A^* : |w|_1 = 0 \bmod 2 \}$ and $\mathrm{Odd}_1^n = \{ w \in A^* : |w|_1 = 1 \bmod 2 \}$:

Lemma 3. *With the condition of Section 5, given a σ_k-Gray cycle α over X, either we have $X \subseteq \mathrm{Even}_1^n$, or we have $X \subseteq \mathrm{Odd}_1^n$.*

Since $k-1$ is an odd integer, according to Proposition 2, the sequence $\gamma^{n-1,k-1}$ is a σ_{k-1}-Gray cycle over A^{n-1}. We set:

$$(\forall i \in [0, 2^{n-1}-1]) \quad \gamma_{[i]}^{n,k} = \theta^i(0)\gamma_{[i]}^{n-1,k-1} \quad \text{and} \quad \underline{\gamma}_{[i]}^{n,k} = \theta^i(1)\gamma_{[i]}^{n-1,k-1} \quad (15)$$

For instance, we have $\gamma_{[0]}^{6,4} = 000000$, $\underline{\gamma}_{[0]}^{6,4} = 100000$, $\gamma_{[1]}^{6,4} = 111100$.

Proposition 3. $\gamma^{n,k}$ (resp., $\underline{\gamma}^{n,k}$) is a σ_k-Gray cycle over Even_1^n (resp., Odd_1^n).

Proof. (i) According to Equation (15), since $\gamma^{n-1,k-1}$ satisfies (G3), both the sequences $\gamma^{n,k}$ and $\underline{\gamma}^{n,k}$ also satisfy (G3).

(ii) By Lemma 3, we have $\bigcup_{0 \le i \le 2^n - 1} \{\gamma^{n,k}\} \subseteq \mathrm{Even}_1^n$ and $\bigcup_{0 \le i \le 2^n - 1} \{\underline{\gamma}^{n,k}\} \subseteq \mathrm{Odd}_1^n$. In addition, according to (15), we have $|\gamma^{n,k}| = |\underline{\gamma}|^{n,k} = |\gamma^{n-1,k-1}| = 2^{n-1}$. This implies $\bigcup_{0 \le i \le 2^n - 1} \{\gamma^{n,k}\} = \mathrm{Even}_1^n$ and $\bigcup_{0 \le i \le 2^n - 1} \{\underline{\gamma}^{n,k}\} = \mathrm{Odd}_1^n$ that is, $\gamma^{n,k}$ and $\underline{\gamma}^{n,k}$ satisfy (G1).

(iii) Let $i \in [1, 2^{n-1}-1]$. Since $\gamma^{n-1,k-1}$ satisfies (G2), we have $\gamma_{[i]}^{n-1,k-1} \in \sigma_{k-1}\left(\gamma_{[i-1]}^{n-1,k-1}\right)$. According to (15), the initial characters of $\gamma_{[i]}^{n,k}$ and $\gamma_{[i-1]}^{n,k}$ (resp., $\underline{\gamma}_{[i]}^{n,k}$ and $\underline{\gamma}_{[i-1]}^{n,k}$) are different, hence we have $\gamma_{[i]}^{n,k} \in \sigma_k\left(\gamma_{[i-1]}^{n,k}\right)$ and $\underline{\gamma}_{[i]}^{n,k} \in \sigma_k\left(\underline{\gamma}_{[i-1]}^{n,k}\right)$. In addition, once more according to (15) it follows from $\gamma_{[0]}^{n-1,k-1} \in \sigma_{k-1}\left(\gamma_{[2^{n-1}-1]}^{n-1,k-1}\right)$ that $\gamma_{[0]}^{n,k} = 0\gamma_{[0]}^{n-1,k-1} \in \sigma_k\left(1\gamma_{[2^{n-1}-1]}^{n-1,k-1}\right) \subseteq \sigma_k\left(\gamma_{[2^{n-1}-1]}^{n,k}\right)$, hence $\gamma^{n,k}$ satisfies (G2). Similarly, $\underline{\gamma}_{[0]}^{n-1,k-1} \in \sigma_{k-1}\left(\underline{\gamma}_{[2^{n-1}-1]}^{n-1,k-1}\right)$ implies $\underline{\gamma}_{[0]}^{n,k} \in \sigma_k\left(\underline{\gamma}_{[2^{n-1}-1]}^{n,k}\right)$, hence $\underline{\gamma}^{n,k}$ satisfies (G2). $\qquad\square$

The following statement provides the description of the complexity $\lambda_{A,\tau}$:

Theorem 1. *Given a finite alphabet A and $n \ge k \ge 1$, exactly one of the four following properties holds:*
   (i) *$|A| \ge 3$, $n \ge k$, and $\lambda_{A,\sigma_k}(n) = |A|^n$;*
   (ii) *$|A| = 2$, $n = k$, and $\lambda_{A,\sigma_k}(n) = 2$;*
   (iii) *$|A| = 2$, $n \ge k+1$, k is odd and $\lambda_{A,\sigma_k}(n) = 2^n$;*
   (iv) *$|A| = 2$, $n \ge k+1$, k is even, and $\lambda_{A,\sigma_k}(n) = 2^{n-1}$.*
In addition, in each case some σ_k-Gray cycle of maximum length can be explicitly computed.

Proof. Recall that if some σ_k-Gray cycle exists over $X \subseteq A^{\le n}$, necessarily X is a uniform set that is, $X \subseteq A^m$ holds for some $m \ge n$; hence, in any case we have $\lambda_{A,\sigma_k}(n) \le |A|^n$. According to Proposition 1, if we have $|A| \ge 3$ and $n \ge k$, a σ_k-Gray cycle exists over A^n, hence Property (i) holds. Similarly, (iii) comes from Proposition 2. As indicated in the preamble of Section 3, Property (ii) trivially holds. Finally, according to Lemma 3, given a binary alphabet A, if k is even we have $\lambda_{A,\sigma_k}(n) \le 2^{n-1}$, hence (iv) comes from Proposition 3. $\qquad\square$

Further development Since our Gray cycles were constructed by applying recursive processes, it is legitimate to ask whether some method could exist for computing $\gamma_{[i]}^{n,k}$ by directly starting with $\gamma_{[i-1]}^{n,k}$, as in the case of the classical reflected Gray cycles. In view of some of our more recent studies, we strongly believe that such algorithms can actually be devised: we hope to develop this point in a further paper.

On the other hand, it could be of interest to study the behaviour of $\Lambda_{A,\tau}$ in the framework of other word binary relations τ, even in restraining to special families of sets $X \subseteq A^*$, such as variable-length codes.

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