The Hahn-Banach Theorem in Convex Optimization

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THE HAHN-BANACH THEOREM IN CONVEX OPTIMIZATION

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ABSTRACT

The main objective of this article is to present the separation theorems, important consequences of the Hahn-Theorem theorem. These results are the fundamentals of the convex optimization mathematical approach. Along this work begin to consider vector spaces, in general, then normed spaces and posteriorly Hilbert spaces. It ends with the presentation of applications of these results in convex programming and in minimax theorem, two important tools in operations research, management and economics, for instance.

Keywords: Hahn-Banach theorem, separation theorems, convex programming, minimax theorem.

1.INTRODUCTION

After a general overview on convex sets and convex functionals, the Hahn-Banach theorem is presented, with great generality, together with an important separation theorem.

Then those results are particularized for normed spaces and then concretized for a subclass of these spaces: the Hilbert spaces.

The fruitfulness of the results presented is emphasized in the last sections where it is shown that they permit to obtain results very important in the applications:

-First, the Kuhn-Tucker theorem, the main result of the complex programming so important in operations research,

-Then the minimax theorem, an important result in game theory, which consideration in management and economic models is becoming greater and greater.

2. CONVEX SETS AND FIELDS

Be L a real vector space.

Definition 2.1

A set $K \subset L$ is convex if and only if

$$\underset{x,y\in K}{\forall}\ \underset{\theta\in[0,1]}{\forall}\ \theta x+(1-\theta)y\in K \qquad (2.1).$$

Definition 2.2
The nucleus of a set$E \subset L$, designated$J(E)$, is the set of points $x \in E$ such that, given any $y \in L$, it is possible to determine $\varepsilon = \varepsilon(y) > 0$ such that $x + ty \in E$ since$|t| < \varepsilon$.

Definition 2.3
A convex set with non-empty nucleus is a convex field.

Theorem 2.1
The nucleus $J(K)$ of any convex set *K* is also a convex set.

Dem.: Suppose that $x, y \in J(K)$. Be $z = \theta x + (1-\theta)y, 0 \le \theta \le 1$. So, given any $a \in L$, it is possible to determine $\varepsilon_1 > 0, \varepsilon_2 > 0$ such that, for $|t_1| < \varepsilon_1, |t_2| < \varepsilon_2$, the points $x + t_1 a$ and $y + t_2 a$ belong both to *K*. So, the point $\theta(x + ta) + (1-\theta)(y + ta) = z + ta$ belongs to *K* for $|t| < \varepsilon = min\{\varepsilon_1, \varepsilon_2\}$, that is $z \in J(K)$.□

Theorem 2.2
The intersection of any family of convex sets is a convex set.

Dem.: Be $K = \underset{\alpha}{\cap} K_\alpha$, being each K_α a convex set. Consider any two points *x* and *y* from *K*. So $\theta x + (1-\theta)y, 0 \le \theta \le 1$, belongs to every $K_\alpha$and, in consequence, to *K*. So *K* is a convex set.□

Observation:
-The intersection of convex fields, being a convex set, it is not necessarily a convex field.

Definition 2.4
Be *A* a part, anyone, of a vector space *L*. Among the convex sets that contain *A* there is a minimal set: the intersection of the whole convex sets that contain *A* -there is at least one convex set that contains *A*: the space *L*. This minimal set is the convex hull of *A*.

3. HOMOGENEOUS CONVEX FUNCTIONALS

Definition 3.1
A functional *p*, defined in *L* is convex if and only if

$$\underset{x,y \in L}{\forall} \ \underset{\theta \in [0,1]}{\forall} p(\theta x + (1-\theta)y) \le \theta p(x) + (1-\theta)p(y) \qquad (3.1).$$

Definition 3.2
A functional *p* is positively homogeneous if and only if

$$\underset{x \in L}{\forall} \ \underset{\alpha > 0}{\forall} p(\alpha x) = \alpha p(x) \qquad (3.2).$$

Proposition 3.1
For any convex positively homogeneous functional it is always fulfilled:

$$i) p(x + y) \le p(x) + p(y) \quad (3.3),$$

ii) $p(0) = 0$ (3.4),

iii) $p(x) + p(-x) \geq 0, \forall_{x \in L}$ (3.5),

iv) $p(\alpha x) \geq \alpha p(x), \forall_{\alpha \in \mathbb{R}}$ (3.6).

Dem: *i)* In fact, $p(x+y) = 2p\left(\frac{x+y}{2}\right) \leq 2\left(p\left(\frac{x}{2}\right) + p\left(\frac{y}{2}\right)\right) = p(x) + p(y)$.

ii) $p(0) = p(\alpha 0) = \alpha p(0), \forall_{\alpha > 0}$. So $p(0) = 0$.

iii) $0 = p(0) = p\big(x + (-x)\big) \leq p(x) + p(-x), \forall_{x \in L}$.

iv) The result is evident for $\alpha \geq 0$. With $\alpha < 0$, $0 \leq p(\alpha x) + p(-\alpha x) = p(\alpha x) + p(|\alpha| x) = p(\alpha x) + |\alpha| p(x)$. So, $p(\alpha x) \geq -|\alpha| p(x)$, that is $p(\alpha x) \geq \alpha p(x)$. □

4. MINKOWSKY FUNCTIONALS

Definition 4.1
Be L any vector space and A a convex body in L which kernel contains 0. It is called Minkowsky functional of the convex body A, and called $p_A(x)$, the functional

$$p_A(x) = inf\left\{r: \frac{x}{r} \in A\right\} \qquad (4.1).$$

Theorem 4.1
A Minkowsky functional is convex positively homogeneous and assumes only positive values. Reciprocally, if *p(x)* is positively homogeneous functional, assuming only positive values, and K a positive number, so the set

$$A = \{x: p(x) \leq K\} \qquad (4.2)$$

is a convex body with kernel $\{x: p(x) < K\}$, which contains the point 0. If in (4.2) it is made $K = 1$, the initial functional p(x) will be the Minkowsky functional of A.

Dem: Given any element $x \in L$, $\frac{x}{r}$ belongs to A if r is great enough. So, the number $p_A(x)$ defined by (4.1) is positive and finite.

But, given $t > 0$ and $y = tx$, $p_A(y) = inf\left\{r > 0: \frac{y}{r} \in A\right\} = inf\left\{r > 0: \frac{tx}{r} \in A\right\} = inf\left\{tr' > 0: \frac{x}{r'} \in A\right\} = tinf\left\{r' > 0: \frac{x}{r'} \in A\right\} = tp_A(x)$. So,

$$p_A(tx) = tp_A(x), \forall_{t > 0} \quad (4.3),$$

and consequently $p_A(x)$ is positively homogeneous.

Suppose now that $x_1, x_2 \in L$. Given any $\varepsilon \in L$, choose the numbers $r_i, i = 1,2$ in order that $p_A(x_i) < r_i < p_A(x_i) + \varepsilon$. So $\frac{x_i}{r_i} \in A$. Making $r = r_1 + r_2$, the point $\frac{x_1+x_2}{r} = \frac{r}{rr_1}x_1 + \frac{r_2}{rr_2}x_2$ will belong to the set of points $S = \left\{z: z = \theta\frac{x_1}{r_1} + (1-\theta)\frac{x_2}{r_2}, \ \forall \theta \in [0,1]\right\}$. As A is a convex set, $S \subset A$ and , in particular, $\frac{x_1+x_2}{r} \in A$. So, $p_A(x_1 + x_2) \le r = r_1 + r_2 < p_A(x_1) +, p_A(x_2) + 2\varepsilon$. As ε is arbitrary,

$$p_A(x_1 + x_2) \le p_A(x_1) + p_A(x_2).$$

So, $p_A(\theta x + (1-\theta)y) \le p_A(\theta x) + p_A((1-\theta)y) = \theta p_A(x) + (1-\theta) p_A(y), \forall x, y \in L, \forall \theta \in [0,1]$, since it was already shown that $p_A(x)$ is positively homogeneous.

Look now to the set defined by (4.2). If $x, y \in A$ and $\theta \in [0,1]$, so $p(\theta x + (1-\theta)y) \le \theta p(x) + (1-\theta)p(y) \le K$. In consequence, A is a convex set. Suppose now that $p(x) < K, t > 0$ and $y \in L$. Under these conditions, $p(x \pm ty) \le p(x) + tp(\pm y)$. If $p(-y) = p(y) = 0$, so $x \pm ty \in A$ for any t. If at least one of the numbers (positive) $p(y), p(-y)$ is not nul, so $x \pm ty \in A$ for

$$t < \frac{K - p(x)}{max\{p(y), p(-y)\}}.$$

From the definitions it results that p is the Minkowsky functional of the set $\{x: p(x) \le 1\}$. □

Observation:
-Taking in account the Theorem 4.1, the Minkowsky functional allows to establish a correspondence between the positively homogeneous convex functionals, assuming only positive values, and the convex bodies to which kernels the origin belongs.

5. THE HAHN-BANACH-THEOREM

Definition 5.1
Consider a vector space L and its subspace L_0. Suppose that in L_0 is defined a linear functional f_0. A linear functional f defined in the whole space L is an extension of the functional f_0 if and only if

$$f(x) = f_0(x), \forall x \in L_0.$$

The Hahn-Banach theorem is essential in the in the resolution of the problem of finding an extension of a linear functional.

Theorem 5.1 (Hahn-Banach)
Be p a positively homogeneous convex functional defined in a real vector space L and L_0 an L subspace. If f_0 is a linear functional defined in L_0, fulfilling the condition

$$f_0(x) \leq p(x), \underset{x \in L_0}{\forall} \quad (5.1),$$

so, there is an extension f of f_0 defined in L, linear, and such that f(x)$\leq p(x), \underset{x \in L}{\forall}$.

Dem: Begin showing that if $L_0 \neq L$, there is an extension of f_0, f', defined in a subspace L' such that $L \subset L'$, to fulfill the condition (5.1).

Be z any element of L not belonging to L_0; if $L'$is the subspace generated by L_0 and z, each element of L' is expressed in the form $tz+x$, being $x \in L_0$. If $f'$is an extension (linear) of the functional f_0 to L', it will happen that $f'(tz+x) = tf'(z) + f_0(x)$ or, making $f'(z) = c$,

$$f'(tz+x) = tc + f_0(x).$$

Now choose c, fulfilling the condition (5.1) in L' , that is: in order that the inequality $f_0(x) + tc \leq p(x+tz)$, for any $x \in L_0$ and any real number t, is accomplished.

For $t > 0$ this inequality is equivalent to the condition $f_0\left(\frac{x}{t}\right) + c \leq p\left(\frac{x}{t} + z\right)$ or

$$c \leq \; p\left(\frac{x}{t} + z\right) - f_0\left(\frac{x}{t}\right) \qquad (5.2).$$

For $t < 0$ it is equivalent to the condition $f_0\left(\frac{x}{t}\right) + c \geq -p\left(-\frac{x}{t} - z\right)$, or

$$c \geq -p\left(-\frac{x}{t} - z\right) - f_0\left(\frac{x}{t}\right) \qquad (5.3).$$

Now it will be proved that there is always a number c satisfying simultaneously the conditions (5.2) and (5.3).

Given any two elements$y'$and y''belonging to L_0,

$$-f_0(y'') + p(y''+z) \geq -f_0(y') - p(-y'-z) \qquad (5.4),$$

since $f_0(y'') - f_0(y') \leq p(y''-y') = p\big((y''+z)-(y'+z)\big) \leq p(y''+z) + p(-y'-z)$.

Be $c'' = \inf_{y''}\big(-f_0(y'') + p(y''+z)\big)$ and $c' = \sup_{y'}\big(-f_0(y') - p(-y'-z)\big)$. As

$y'$and y'' are arbitrary, it results from (5.4) that $c'' \geq c'$. Choosing c in order that $c'' \geq c \geq c'$, it is defined the functional f' on L' through the formula

$$f'(tz+x) = tc + f_0(x).$$

This functional satisfies the condition (5.1). So, any functional f_0 defined in a subspace $L_0 \subset L$ and subject in L_0 to the condition (5.1), may be extended to a subspace L'. The extension f' satisfies the condition

$$f'(x) \leq p(x), \qquad \underset{x \in L'}{\forall}.$$

If L has an algebraic numerable base $(x_1, x_2, \dots, x_n, \dots)$ the functional in L is built by finite induction, considering the increasing sequence of subspaces

$$L^{(1)} = (L_0, x_1), L^{(2)} = \left(L^{(1)}, x_2\right), \dots$$

designating $\left(L^{(k)}, x_{k+1}\right)$ the L subspace generated by $L^{(k)}$ and x_{k+1}. In the general case, that is, when L has not an algebraic numerable base, it is mandatory to use a transfinite induction process, for instance the Haudsdorf maximal chain theorem.

Call $\mathcal{F}$ the set of the whole pairs(L', f'), at which L' is a L subspace that contains L_0 and f' is an extension of f_0 to L' that fulfills (5.1). Order partially $\mathcal{F}$so that

$$(L', f') \leq (L'', f'') \text{ if and only if } L' \subset L'' \text{and } f''_{|L'} = f'.$$

By the Haudsdorf maximal chain theorem, there is a chain, that is: a subset of $\mathcal{F}$ totally ordered, maximal, that is: not strictly contained in another chain. Call it Ω. Be Φ the family of the whole L' such that $(L', f') \in \Omega$. Φ is totally ordered by the sets inclusion; so, the union T of the whole elements of Φ is a L subspace. If $x \in \mathrm{T}$ then $x \in L'$ for some $L' \in \Phi$; define $\tilde{f}(x) = f'(x)$, where $f'$is the extension of f_0 that is in the pair (L', f')- the definition of $\tilde{f}$ is obviously coherent. It is easy to check that $\mathrm{T} = L$ and that $f = f'$ satisfies the condition (5.1).□

Now the Hahn-Banach theorem complex case, corresponding to the contribution of Hahn to the theorem, will be presented. But first:

Definition 5.2
A linear functional p, assuming only positive values, defined in a complex vector space L, is homogeneous convex if and only if, for any $x, y \in L$ and any complex number λ,

$$p(x + y) \leq p(x) + p(y),$$
$$p(\lambda x) = |\lambda| p(x).$$

Theorem 5.2 (Hahn-Banach)
Be p an homogeneous convex functional defined in a vector space L and f_0 a linear functional, defined in a subspace $L_0 \subset L$, fulfilling the condition

$$|f_0(x)| \leq p(x), x \in L_0.$$

Then, there is a linear functional f defined in L, satisfying the conditions

$|f(x)| \leq p(x), x \in L; f(x) = f_0(x), x \in L_0.$

Dem: Call L_R and L_{0R} the real vector spaces underlying, respectively, the spaces L and L_0 . As it is evident, p is a homogeneous convex functional in L_R and $f_{0R}(x) = Ref_0(x)$ a real linear functional in L_{0R} fulfilling the condition $|f_{0R}(x)| \leq p(x)$ and so,

$$f_{0R}(x) \leq p(x).$$

Then, from Theorem 5.1, there is a real linear functional f_R, defined in the whole L_R space, that satisfies the conditions

$$f_R(x) \leq p(x), x \in L_R; \; f_R(x) = f_{0R}(x), x \in L_{0R}.$$

But, $-f_R(x) = f_R(-x) \leq p(-x) = p(x),$ and

$$|f_R(x)| \leq p(x), x \in L_R \qquad (5.5).$$

Define in L the functional f making

$$f(x) = f_R(x) - if_R(ix).$$

It is immediate that f is a complex linear functional in L such that $f(x) = f_0(x), x \in L_0; Ref(x) = f_R(x), x \in L$. **It only misses to show that** $|f(x)| \leq p(x), \underset{x \in L}{\forall}$:

Proceed by absurd. Suppose that there is $x_0 \in L$ such that $|f(x_0)| > p(x_0)$. So, $f(x_0) = \rho e^{i\varphi}, \rho > 0$, and making $y_0 = e^{-i\varphi}x_0$, it would happen that $f_R(y_0) = Re\left[e^{-i\varphi}f(x_0)\right] = \rho > p(x_0) = p(y_0)$ that is contrary to (5.5).□

6. SEPARATION OF THE VECTOR SPACE CONVEX PARTS

The next theorem, very useful, consequence of the Hahn-Banach theorem, is about the separation of the vector space convex parts. Beginning with

Definition 6.1

Be *M* and *N* two subsets of a real vector space L. A linear functional *f* defined in L separates *M* and *N* if and only if there is a number *c* such that $f(x) \geq c,$ for $x \in M$ and $f(x) \leq c,$ for $x \in N$ that is, if $\inf_{x \in M} f(x) \geq \sup_{x \in N} f(x)$. A functional f separates strictly the sets M and N if and only if $\inf_{x \in M} f(x) > \sup_{x \in N} f(x)$.

Theorem 6.1 (Separation)

Suppose that *M* and *N* are two convex subsets of a vector space L such that the kernel of at least one of them, for instance the one of *M* , is non-empty and does not intersect the other set; So, there is a linear functional non-null on L that separates *M* and *N*.

Dem: Less than on translation, it is supposable that the point 0 belongs to the kernel of *M*, which is designated $\dot{M}$. So, given $y_0 \in N$, $-y_0$ belongs to the kernel of $M - N$ and 0 to the kernel of $M - N + y_0$. As $\dot{M} \cap N = \emptyset$, by hypothesis, 0 does not belong to the kernel of $M - N$ and y_0 does not belong to the one of $M - N + y_0$. Put $K = M - N + y_0$ and be p the Minkovsky functional of $\dot{K}$. So $p(y_0) \geq 1,$ since $y_0 \notin \dot{K}$. Define, now, the linear functional

$$f_0(\alpha y_0) = \alpha p(y_0).$$

Note that f_0 is defined in a space with dimension, constituted by elements αy_0, and it is such that

$$f_0(\alpha y_0) \leq p(\alpha y_0).$$

In fact, $p(\alpha y_0) = \alpha p(y_0)$, when $\alpha \geq 0$ and $f_0(\alpha y_0) = \alpha f_0(y_0) < 0 < p(\alpha y_0),$ when $\alpha > 0$. Under these conditions, after the Hahn-Banach theorem, it is possible to state the existence of linear functional f, defined in *L*, that extends f_0, and such that $f(y) \leq p(y), \forall_{y \in L}$.

Then it results

$$f(y) \leq 1, \forall_{y \in K} \text{ and } f(y_0) \geq 1.$$

In consequence:

-f separates the sets K and $\{y_0\}$, that is

- f separates the sets *M*-*N* and $\{y_0\}$, that is

-f separates the sets *M* and *N*.□

7. THE HAHN-BANACH THEOREM FOR NORMED SPACES

A vector space L with a norm is a normed space. The norm of an element $x \in L$ is usually denoted $\|x\|$. Every normed space is a metric space with the distance $d(x, y) = \|x - y\|$.

Definition 7.1
Consider a continuous linear functional *f* in a normed space *E*. It is called *f* norm, and designated $\|f\|$:

$$\|f\| = \sup_{\|x\| \leq 1} |f(x)|$$

that is: the supreme of the values assumed by $|f(x)|$ in the *E* unitary ball.

Observation:
-The class of the continuous linear functionals, with the norm above defined, is a normed vector space, called the *E* dual space, designated E'.

Theorem 5.1 is as follows, in normed spaces:

Theorem 7.1 (Hahn-Banach)
Name *L* a subspace of a real normed space *E* and f_0 a bounded linear functional in L. So, there is a linear functional defined in *E*, extension of f_0, such that

$$\|f_0\|_{L'} = \|f\|_{E'}.$$

Dem: It is enough to think in the functional $K\|x\|$ at which $K = \|f_0\|_{L'}$. As it is convex and positively homogeneous, it is possible to put $p(x) = K\|x\|$ and to apply Theorem 4.1.□

Observation:
-To see an interesting geometric interpretation of this theorem, consider the equation $\|f_0(x)\| = 1$. It defines, in L, an hiperplane at distance $\frac{1}{\|f_0\|}$ of 0. Considering the extension *f* of f_0 , with norm conservation, it is obtained an hiperplane in *E*, that contains the hiperplane considered behind in *L*, and that at the same distance from the origin.

The version for normed spaces of Theorem 5.2 is:

Theorem 7.2 (Hahn-Banach)
Be E a complex normed space and f_0 a bounded linear functional defined in a subspace $L \subset E$. So, there is a bounded linear functional *f*, defined in *E*, such that

$$f(x) = f_0(x), x \in L; \|f\|_{E'} = \|f_0\|_{L'}.$$

To end this section, two separation theorems, important consequences of the Hahn-Banach theorem, applied to the normed vector spaces, will be presented:

Theorem 7.3 (Separation)
Consider two convex sets *A* and *B* in a normed space *E*. If one of them, for instance *A*, has at least on interior point and $(intA) \cap B = \emptyset$, there is a continuous linear functional non-null that separates the sets *A* and *B*.Escreva uma equação aqui.

Theorem 7.4 (Separation)
Consider a closed convex set *A*, in a normed space *E*, and a point $x_0 \in E$, not belonging to *A*. So, there is a continuous linear functional, non-null, that separates strictly $\{x_0\}$ and *A*.

8. SEPARATION THEOREMS FOR HILBERT SPACES

Definition 8.1
A Hilbert space, designated *H* or *I*, is a complex vector space with inner product that, as a metric space, is complete.

Definition 8.2
An inner product, in a complex vector space *H* , is a sesquilinear hermitian functional, strictly positive on H.

Observation:
-Working with real vector spaces, "sesquilinear hermitian" must be replaced by "bilinear symmetric",

-The inner product of two vectors *x* and *y* of *H* , by this order is designated

-The norm of a vector x will be $\|x\| = \sqrt{[x, x]}$.

An important theorem, about the representation of continuous linear functionals by elements of the space is the Riesz representation theorem:

Theorem 8.1 (Riesz representation)
Every continuous linear functional $f(\cdot)$ may be represented in the form $f(x) = [x, \tilde{q}]$ where

$$\tilde{q} = \frac{\overline{f(q)}}{[q, q]} q.$$

From now on, only real Hilbert spaces will be considered.

Note that the separation theorems, seen in the former section, are valid in Hilbert spaces. But, due to the Riesz representation theorem, they may be stated in the following way:

Theorem 8.2 (Separation)
Consider two convex sets *A* and *B* in a Hilbert space *H*. If one of them, for instance *A*, has at least on interior point and $(intA) \cap B = \emptyset$, there is a non-null vector *v* such that

$$\sup_{x \in A} [v, x] \le \inf_{y \in B} [v, y].$$

Theorem 8.3 (Separation)
Consider a closed convex set *A*, in a Hilbert space *H*, and a point $x_0 \in H$, not belonging to *A*. So, there is a non-null vector *v*, such that

$$[v, x_0] < \inf_{x \in A} [v, x].$$

Another separation theorem:

Theorem 8.4 (Separation)
Two closed convex subsets *A* and *B*, of a Hilbert space, in a finite distance from each other, that is: such that:

$$\inf_{x \in A, y \in B} \|x - y\| = d > 0$$

may be strictly separated, that is:

$$\inf_{x \in A} [v, x] > \sup_{y \in B} [v, y].$$

It is also possible to demonstrate that:

Theorem 8.5 (Separation)
Being *H* a finite dimension Hilbert space, if *A* and *B* are disjoint and non-empty convex sets they always may be separated.

9. APPLICATION IN CONVEX PROGRAMMING

As application of separation theorems for convex sets, will be considered a class of convex programming problems, at which it is intended to minimize convex functionals subject to convex inequalities.

Begin by presenting a basic result that characterizes the minimum point of a convex functional subject to convex inequalities. Note that it is not necessary to impose any continuity conditions.

Theorem 9.1 (Kuhn-Tucker)

Be *f(x)*, $f_i(x), i = 1, \dots, n$, convex functionals defined in a convex subset *C* of a Hilbert space.

Consider the problem:

$$\min_{x \in C} f(x),$$
$$sub.: f_i(x) \leq 0, i = 1, \dots, n.$$

Be x_0 a point where the minimum, supposed finite, is reached. Also suppose that for each vector *u* in E_n, Euclidean space with dimension *n*, non-null and such that $u_k \geq 0$, there is a point *x* in *C* such that $\sum_1 u_k f_k(x) < 0$,designating u_k the components of *u*.

So,

i) There is a vector *v*, with non-negative components $\{v_k\}$, such that

$$\min_{x \in C} \left\{ f(x) + \sum_1^n v_k f_k(x) \right\} = f(x_0) + \sum_1^n v_k f_k(x_0) = f(x_0) \qquad (9.1),$$

ii) For every vector u in E_n with non-negative components, that is: belonging to the positive cone of E_n ,

$$f(x) + \sum_1^n v_k f_k(x)$$
$$\geq f(x_0) + \sum_1^n v_k f_k(x_0) \geq f(x_0) + \sum_1^n u_k f_k(x_0) \quad (9.2).$$

Corollary 9.1 (Lagrange duality)

In the conditions of Theorem 9.1

$$f(x_0) = \sup_{u \geq 0} \inf_{x \in C} f(x) + \sum_1^n u_k f_k(x).$$

Observation:
-This corollary has the utility of supplying a process to determine the problem optimal solution,

-If the whole v_k in expression (9.2) are positive, x_0 is a point that belongs to the border of the convex set defined by the inequalities,

-If the whole v_k are zero, the inequalities do not influence the problem, that is: the minimum is equal to the one of the free problem-without the inequality restrictions.

Considering non-finite inequalities:

Theorem 9.2 (Kuhn-Tucker in infinite dimension)
Be *C* a convex subset of a Hilbert space *H* and $f(x)$ a real convex functional defined in *C*. Be *I* a Hilbert space with a closed convex cone $\wp$, with non-empty interior, and $F(x)$ a convex transformation from *H* to *I* (convex in relation to the order introduced by cone $\wp$: if $x, y \in \wp, x \geq y \; if \; x - y \in \wp$). Be x_0 a $f(x)$ minimizing in *C* subjected to the inequality $F(x) \leq 0$.

Consider $\wp^* = \left\{x: [x, p] \geq 0, \begin{matrix} \forall \\ x \in \wp \end{matrix}\right\}$ – Dual cone.

Admit that given any $u \in \wp^*$ it is possible to determine *x* in *C* such that $[u, F(x)] < 0$. So, there is an element *v* in the dual cone $\wp^*$, such that for *x* in *C*

$$f(x) + [v, F(x)] \geq f(x_0) + [v, F(x_0)] \geq f(x_0) + [u, F(x_0)],$$

being *u* any element of $\wp^*$.

Corollary 9.2 (Lagrange duality in infinite dimension)
In the conditions of Theorem 9.2

$$f(x_0) = \sup_{v \in \wp^*} \inf_{x \in C} (f(x) + [v, F(x)]).$$

10. APPLICATION IN GAME THEORY

Now it will be seen how the result of convex sets strict separation permits to obtain a fundamental result of game theory: The minimax theorem.

Consider two players games with null sum:

-Be $\Phi(x, y)$ a real function of two variables $x, y \in H$.

-Be *A* and *B* convex sets in *H*.

-One of the players chooses strategies (points) in *A* in order to maximize $\Phi(x, y)$ (or minimize $-\Phi(x, y)$): it is the maximizing player.

-The other player chooses strategies (points) in *B* to minimize $\Phi(x, y)$ (or maximize $-\Phi(x, y)$): it is the minimizing player.

The function $\Phi(x, y)$ is the payoff function. $\Phi(x_0, y_0)$ represents, simultaneously, the win of the maximizing player and the loss of the minimizing player in a move at which they chose, respectively the strategies x_0 and y_0. So, the win of one of the players is equal to the loss of the other. That is why the game is a null sum game.

A game in these conditions has value *c* if

$$\sup_{x\in A} \inf_{y\in B} \Phi(x, y) = c = \inf_{y\in B} \sup_{x\in A} \Phi(x, y) \qquad (10.1).$$

If, for any (x_0, y_0), $\Phi(x_0, y_0) = c$, (x_0, y_0) is a pair of optimal strategies. There will be a saddle point if also

$$\Phi(x, y_0) \leq \Phi(x_0, y_0) \leq \Phi(x_0, y), x\in A, y \in B \qquad (10.2).$$

Theorem 10.1
Consider A and B convex sets closed in H, being A bounded. Be $\Phi(x, y)$a real functional defined for *x* in *A* and *y* in *B* fulfilling:

-$\Phi(x, (1-\theta)y_1 + \theta y_2) \leq (1-\theta)\Phi(x, y_1) + \theta\Phi(x, y_2)$ for *x* in *A* and y_1, y_2 in *B*, $0 \leq \theta \leq 1$ (that is: $\Phi(x, y)$ is convex in *y* for each *x*),

- $\Phi\big((1-\theta)x_1 + \theta x_2, y\big) \geq (1-\theta)\Phi\big(x_{1,}, y\big) + \theta\Phi(x_2, y)$ for *y* in *B* and x_1, x_2 in *A*, $0 \leq \theta \leq 1$ (that is: $\Phi(x, y)$ is concave in *x* for each *y*),

- $\Phi(x, y)$ is continuous in x for each y,

so (9.1) holds, that is: the game has a value.

The next corollary follows from the Theorem 10.1 hypothesis strengthen:

Corollary 10.1(Minimax)
Suppose that the functional $\Phi(x, y)$ of Theorem 10.1 is continuous in both variables, separately, and that *B* is also bounded. Then, there is an optimal pair of strategies, with the property of being a saddle point.

11. CONCLUSIONS

In the beginning of this paper, a general and relatively detailed overview on convex sets and convex functionals, with the study of the Minkowsky functional, determinant to the sequence of this work, was performed.

Then the Hahn-Banach theorem was presented with great generality, real and complex version, followed by an important separation theorem.

These results were particularized for normed spaces and then concretized for a subclass of these spaces: the Hilbert spaces. Better saying, they were reformulated for Hilbert spaces using the Riesz representation theorem.

The fruitfulness of the results presented is patent in the last sections where it is shown that they permit to obtain results important in the applications. Now the structures considered were the real Hilbert spaces.

The problems studied were convex optimization problems in which, it is well known, the separation theorems are a key tool.

First, the Kuhn-Tucker theorem, the main result of the complex programming so important in operations research.

Then the minimax theorem, an important result in game theory, which consideration in management and economic problems resolution is greater and greater.

Note that the problem studied in section 9 may be considered a game with one player only that chooses strategies (points) in a convex set in order to minimize a function(convex) subject to restrictions(convex). In this situation it makes no sense to consider the restriction of null sum: there is not a player losing to another one.

As for the problem studied in section 10, it is what may be called a hybrid problem. In fact, one tries to maximize a set of minimums or to minimize a set of maximums.

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