Subprevarieties of Algebraic Systems Versus Extensions of Logics: Application to Some Many-Valued Logics

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SUBPREVARIETIES OF ALGEBRAIC SYSTEMS VERSUS EXTENSIONS OF LOGICS: APPLICATION TO SOME MANY-VALUED LOGICS

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Abstract. Here, we study applications of the factual interpretability of [equivalence between] the equality-free infinitary universal Horn theory (in particular, the sentential logic) of a class of algebraic systems (in particular, logical matrices) [with equality uniformly definable by a set of atomic equality-free formulas] in [and] the prevariety generated by the class, in which case the lattice of extensions of the former is a Galois retract of [dual to] that of all subprevarieties of the prevariety, the retraction [duality] retaining relative equality-free infinitary universal Horn axiomatizations. As representative instances, we explore:

(1) the classical (viz., Boolean) expansion of Belnap's four-valued logic that is not equivalent to any class of *pure* algebras but is equivalent to the quasivariety of *filtered De Morgan Boolean algebras* that are matrices with underlying algebra being a *De Morgan Boolean algebra*, truth predicate being a filter of it and equality being definable by a strong equivalence connective, proving that prevarieties of such structures form an eight-element non-chain distributive lattice, and so do extensions of the expansion involved;
(2) Kleene's three-valued logic that is neither interpretable in pure algebras nor equivalent to a prevariety of algebraic systems, but is interpretable into the quasivariety of *resolutional filtered Kleene lattices* that are matrices with underlying algebra being a *Kleene lattice* and truth predicate being a filter of it, satisfying the *Resolution* rule, proving that proper extensions of the logic form a four-element diamond.

1. Introduction

Appearance of any logical system/calculus inevitably raises a number of metalogical issues such as its semantics, derivable/admissible rules as well as both its [axiomatic] extensions and their semantics, (relative) axiomatizations etc. On the other hand, the principal meaning of universal logical investigations consists in developing generic tools of exploring such issues as for particular logics. In this connection, the work [11] has suggested a general algebraic approach going back to [9] as for axiomatic extensions, providing reduction of the problem involved to that of finding subquasivarieties of the equivalent quasivariety of algebras (if any, at all) of a given logical system. This advanced paradigm has been successfully applied to many particular logical systems in both [11] and [9] themselves as well as in further works [12], [13], [15], [16] and [19].

However, there are certain interesting logical systems having no equivalent quasivariety of *pure* algebras. This concerns both sequential (viz., Gentzen-style) and

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sentential (viz., Hilbert-style) calculi studied in [11] and [12]. Nevertheless, these instances do possess equivalent quasivarieties but of rather algebraic systems (in the sense of [6]) than pure algebras. This paper is, first of all, devoted to the primary task of factual extending [11] to such general quasivarieties in MAL'CEV's sense. And what is more, like in [14], to cover not necessarily finitary logics (mainly, potential infinitary extensions of normally finitary initial logics), we deal with the framework of *prevarieties* (in the terminology of [20]), that is, infinitary universal Horn model classes (more specifically, within infinitary logics of the form $L_{\infty\kappa}$, where κ is a regular infinite cadinal, while [20] implicitly deals with $L_{\infty\infty}$, in which case universal Horn theories are, generally speaking, proper classes). In this connection, we extend [14] beyond *pure* algebras. Finally, we exemplify our generic elaboration by applying it to, perhaps, most representative instances of Hilbert-style calculi — both Kleene's three-valued logic [4] and the logic introduced in [12] by supplementing Belnap's four-valued logic [2] with classical (viz., Boolean) negation.

The rest of the paper is as follows. The exposition of the material of the paper is entirely self-contained (of course, modulo very basic issues concerning Set Theory, Lattice Theory, Universal Algebra and Model Theory, not specified here explicitly, to be found, e.g., in [1], [3], [6] and [7]). Section 2 is a concise summary of basic issues underlying the paper, most of which have actually become a part of logical and algebraic folklore. Section 3 is a self-contained summary of Chapter 1 of [11] extended within infinitary framework and supplemented with the issues of semantics and prevarieties of algebraic systems, not involved therein *explicitly*. Then, Section 4 is entirely devoted to the issue of equational systems going back to [18].[1] In its turn, Section 5 provides, in particular, an extension of [14] beyond *pure* algebras, Finally, Section 6 (more specifically, Subsection 6.1/6.2) is a quite illustrative application of the generic tools elaborated in Section 4/5 to the classical expansion of Belnap's four-valued logic/Kleene's three-valued logic, respectively. Finally, Section 7 is an outline of further related work.

2. BASIC ISSUES

Notations like img, dom, ker, hom and Con and related notions are supposed to be clear.

2.1. **Set-theoretical background.** We follow the standard set-theoretical convention, according to which natural numbers (including 0) are treated as finite ordinals (viz., sets of lesser natural numbers), the ordinal of all them being denoted by ω. The proper class of all ordinals is denoted by ∞. Likewise, functions are viewed as binary relations. In addition, singletons are often identified with their unique elements, unless any confusion is possible.

Given a class K, the class of all [finite] subsets of K is denoted by $\wp_{[\omega]}(K)$.

Let S be a set. Given any equivalence relation θ on S, as usual, by ν_θ we denote the function with domain S defined by $\nu_\theta(a) \triangleq [a]_\theta \triangleq \theta[\{a\}]$, for all $a \in S$, in which case $\ker \nu_\theta = \theta$, whereas we set $(T/\theta) \triangleq \nu_\theta[T]$, for every $T \subseteq S$. Next, S-tuples (viz., functions with domain S) are normally written in either sequence $\bar{t}$ or vector $\vec{t}$ forms, its s-th component, where $s \in S$, being written as t_s in that case. Given two more sets A and B, any relation $R \subseteq (A \times B)$ (in particular, a mapping $R : A \to B$) determines the equally-denoted relation $R \subseteq (A^S \times B^S)$ (resp. mapping $R : A^S \to B^S$) point-wise, that is, $R \triangleq \{\langle \bar{a}, \bar{b}\rangle \in (A^S \times B^S) \mid \forall s \in S : a_s \; R \; b_s\}$.

[1]Though some general results presented in that section, concerning finitary logics, have more immediate arguments with using certain universal results of [11] concerning extensions of equivalent generalized logics, for the sake of self-containity, we have preferred to refrain from *explicit* involving the conceptions of *translation* and *equivalence* introduced therein in this connection.

Likewise, given, in addition, any $f : S \to A$ and any $g : S \to B$, we have $(f \times g) : S \to (A \times B), s \mapsto \langle f(s), g(s)\rangle$. Further, set $\Delta_S \triangleq \{\langle a, a\rangle | a \in S\}$, such functions being said to be *diagonal/identity*, and $S^+ \triangleq \bigcup_{i\in(\omega\setminus 1)} S^i$, elements of which being identified with ordinary finite non-empty tuples of elements of S. Then, any binary operation $\diamond$ on S determines the equally-denoted mapping $\diamond : S^+ \to S$ as follows: by induction on the length $l = \mathrm{dom}\,\bar{a} \in (\omega \setminus 1)$ of any $\bar{a} \in S^+$, put:

$$\diamond\bar{a} \triangleq \begin{cases} a_0 & \text{if } l = 1, \\ (\diamond(\bar{a}{\restriction}(l-1))) \diamond a_{l-1} & \text{otherwise.} \end{cases}$$

Given any $f : S \to S$, put $f^0 \triangleq \Delta_S$ and $f^1 \triangleq f$. Finally, given any $n \in (\omega \setminus 1)$ and any $\vec{T} \in \wp(S)^n$, we have $\chi^{\vec{T}} : (\bigcup(\mathrm{img}\,\vec{T})) \to n, a \mapsto \min\{i \in n \mid a \in T_i\}$. Then, for any $T \subseteq S$, $\chi^T_S \triangleq \chi^{\langle S,T\rangle} : S \to 2$ is the usual *characteristic function of T in S*.

Let A be a set. A $U \subseteq \wp(A)$ is said to be *upward-directed*, provided, for every $S \in \wp_\omega(U)$, there is some $T \in U$ such that $(\bigcup S) \subseteq T$. A *closure operator over A* is any unary operation C on $\wp(A)$ such that, for all $D \subseteq B \subseteq A$, it holds that $(C(C(B)) \cup B \cup C(D)) \subseteq C(B)$. This is said to be *inductive/finitary*, provided, for every upward-directed $U \subseteq \wp(A)$, it holds that $O(\bigcup U) \subseteq \bigcup O[U]$.

A *Galois retraction between* posets $\langle P, \leqq\rangle$ and $\langle Q, \lesssim\rangle$ is any couple $\langle f, g\rangle$ of anti-monotonic mappings $f : P \to Q$ and $g : Q \to P$ such that $(g \circ f) = \Delta_P$ and $(f \circ g) \subseteq \lesssim$. (Galois retractions are exactly Galois connections with injective/surjective left/right component; cf. [14] and [19].)

2.2. **Algebraic background.** Unless otherwise specified, abstract algebras are denoted by capital Fraktur letters (possibly, with indices), their carriers (viz., underlying sets) being denoted by corresponding Italic letters (with same indices, if any). Likewise, unless otherwise specified, we deal with a fixed but arbitrary algebraic (viz., functional) signature F constituted by function (viz., operation) symbols of arity in a fixed regular infinite cardinal κ, treated as *(propositional) connectives.* Then, algebraic systems (viz., first-order model structures) are denoted by Calligraphic letters (possibly, with indices), their *underlying algebras* (viz., F-reducts) being denoted by corresponding Fraktur letters (with same indices, if any).

Given any $S \in \wp(\infty)$ put $V_S \triangleq \{x_\beta | \beta \in S\}$ and $(\exists_S) \triangleq (\exists V_S)$. The absolutely-free F-algebra $\mathfrak{Tm}_F$ freely-generated by the set V_κ, elements of which being viewed as *(propositional) variables*, is referred to as the *term F-algebra*, elements of its carrier Tm_F (viz., F-terms) being treated as *(propositional) F-formulas*, its endomorphisms being referred to as *F-substitutions.*

Model L-structures (viz., algebraic systems of the signature L; cf. [6]) with underlying algebra $\mathfrak{Tm}_F$ are said to be *Lindenbaum.*

3. Generalized logics versus algebraic systems

Fix any first-order signature $L = \langle F, R\rangle$, where R is a relational signature, i.e., a set of predicate (viz., relation) symbols of arity in κ, disjoint with F, to be identified with F alone, whenever $R = \varnothing$. Strict atomic equality-free formulas of L with variables in V_κ are called *L-formulas*, the set of all them being denoted by Fm_L. Given any $\Sigma \subseteq (\mathrm{Fm}_L \cup \mathrm{Tm}_F)$, the set of all variables actually occurring in an element of Σ is denoted by $\mathrm{Var}(\Sigma)$. Then, subsets/elements of $\wp_{[\omega]}(\mathrm{Fm}_L) \times \mathrm{Fm}_L$ are referred to as *[finitary] L-calculi/-rules.* As usual, any L-rule $\langle\Gamma, \Phi\rangle$ is normally written as $\Gamma \to \Phi$ and is identified with the infinitary Horn formula $(\bigwedge \Gamma) \to \Phi$,[2] Φ/any element of Γ being referred to as the/a *conclusion/premise of* the rule. As

[2]In this way, [providing $\kappa = \omega$, finitary] L-calculi are nothing but strict equality-free [first-order] universal Horn theories of $L_{\infty\kappa[\omega\omega]}$.

usual, L-rules without premises are called *L-axioms* and are identified with their conclusions.

A *(generalized) L-logic* is any closure operator C over Fm_L that is *structural* in the sense that $\sigma[C(X)] \subseteq C(\sigma[X])$, for all $X \subseteq \mathrm{Fm}_L$ and all $\sigma \in \hom(\mathfrak{Tm}_F, \mathfrak{Tm}_F)$. It is said to be *[in]consistent*, if $C(\varnothing) \neq [=] \mathrm{Fm}_L$. This is said to *satisfy* an L-rule $\Gamma \to \Phi$, provided $\Phi \in C(\Gamma)$, L-axioms satisfied in C being called *theorems of C*. Next, C is uniquely determined by, and so naturally identified with the L-calculus of all L-rules satisfied in C (in this way, logics become particular cases of calculi). Further, a *(proper) extension of* an L-logic C is any L-logic $C' \supseteq (\supsetneq) C$, in which case C is said to be a *(proper) sublogic of C'*. Then, C' is said to be *axiomatized by* an L-calculus $\mathcal{C}$ *relatively to* C, whenever C' is the least extension of C satisfying each rule of $\mathcal{C}$. Finally, an extension of an L-logic is said to be *axiomatic*, whenever it is relatively axiomatized by a set of L-axioms.

Given any class M of model L-structures, we have the L-logic Cn_{M}, constituted by all L-rules true in M and said to be *defined by* M or called the one *of* M.

Let $\mathcal{A}$ be an L-structure.

Elements of $\mathrm{Con}(\mathcal{A}) \triangleq \{\theta \in \mathrm{Con}(\mathfrak{A}) | \forall r \in R : \theta[r^{\mathcal{A}}] \subseteq r^{\mathcal{A}}\} \ni \Delta_A$ are called *congruences of $\mathcal{A}$*.[3] Then, $\mathcal{A}$ is said to be *simple*, whenever $\mathrm{Con}(\mathcal{A}) = \{\Delta_A\}$. Given any $\theta \in \mathrm{Con}(\mathfrak{A})$, we have the *quotient* Σ-structure $\mathcal{A}/\theta$ with underlying algebra $\mathfrak{A}/\theta$ and relations $r^{\mathcal{A}/\theta} \triangleq (r^{\mathcal{A}}/\theta)$, where $r \in R$.

Next, $\mathcal{A}$ is said to be *inconsistent*, if, for every $r \in R$ of arity $\alpha \in \kappa$, it holds that $r^{\mathcal{A}} = A^\alpha$, and *consistent*, otherwise.

Finally, $\mathcal{A}$ is said to be *finitely-generated/finite/one-element*, whenever $\mathfrak{A}$ is so.

Let $\mathcal{A}$ and $\mathcal{B}$ be two L-structures. A *(strict) [surjective] homomorphism from $\mathcal{A}$ [on]to $\mathcal{B}$* is any $h \in \hom(\mathfrak{A}, \mathfrak{B})$ such that $r^{\mathcal{A}} \subseteq (=) h^{-1}[r^{\mathcal{B}}]$, for every $r \in R$[, while $h[A] = B$], the set of all them being denoted by $\hom^{[\mathrm{S}]}_{(\mathrm{S})}(\mathcal{A}, \mathcal{B})$, in which case (we set $(\mathfrak{A} \upharpoonright h^{-1}[\mathcal{B}]) \triangleq \mathcal{A}$, while

$$(\ker h) \in \mathrm{Con}(\mathfrak{A} \upharpoonright h^{-1}[\mathcal{B}]), \tag{3.1}$$

whereas) we have $\hom(\mathfrak{Tm}_F, \mathfrak{B}) \supseteq [=] \{h \circ g | g \in \hom(\mathfrak{Tm}_F, \mathfrak{A})\}$, and so:

$$(\hom^{[\mathrm{S}]}_{\mathrm{S}}(\mathcal{A}, \mathcal{B}) \neq \varnothing) \Rightarrow (\mathrm{Cn}_{\mathcal{B}} \subseteq [=] \mathrm{Cn}_{\mathcal{A}}), \tag{3.2}$$

$$(\hom^{\mathrm{S}}(\mathcal{A}, \mathcal{B}) \neq \varnothing) \Rightarrow (\mathrm{Cn}_{\mathcal{B}}(\varnothing) \subseteq \mathrm{Cn}_{\mathcal{A}}(\varnothing)). \tag{3.3}$$

Then, $\mathcal{A}$ is said to be a *substructure of $\mathcal{B}$*, whenever $\Delta_A \in \hom_{\mathrm{S}}(\mathcal{A}, \mathcal{B})$, in which case we set $(\mathcal{B} \upharpoonright A) \triangleq \mathcal{A}$. Likewise, $\mathcal{A} \preceq \mathcal{B}$ means that $\Delta_A \in \hom(\mathcal{A}, \mathcal{B})$. (In this way, $\preceq$ is a partial ordering on the class of all L-structures.) Injective/bijective strict homomorphisms from $\mathcal{A}$ to $\mathcal{B}$ are referred to as *embeddings/isomorphisms of/from $\mathcal{A}$ into/onto $\mathcal{B}$*, in case of existence of which $\mathcal{A}$ is said to be *embeddable/isomorphic into/to $\mathcal{B}$*. (Note that, for any $\theta \in \mathrm{Con}(\mathfrak{A})$ [resp., $\theta \in \mathrm{Con}(\mathcal{A})$], $\nu_\theta \in \hom^{\mathrm{S}}_{[\mathrm{S}]}(\mathcal{A}, \mathcal{A}/\theta)$.)

Let I be a set and $\overline{\mathcal{A}}$ an I-tuple of L-structures. In case $\{\mathfrak{A}_i \mid i \in I\} \subseteq \{\mathfrak{A}\}$, where $\mathfrak{A}$ is an F-algebra, by $\mathfrak{A} \upharpoonright (\bigcap_{i \in I} \mathcal{A}_i)$ we denote the *intersection of $\overline{\mathcal{A}}$ over* $\mathfrak{A}$, being the L-structure with underlying algebra $\mathfrak{A}$ and relations given by $r^{\mathfrak{A} \upharpoonright (\bigcap_{i \in I} \mathcal{A}_i)} \triangleq (A^\alpha \cap (\bigcap_{i \in I} r^{\mathcal{A}_i}))$, for every $r \in R$ of arity $\alpha \in \kappa$. In general, we then have the *direct product* $(\prod_{i \in I} \mathcal{A}_i) \triangleq ((\prod_{i \in I} \mathfrak{A}_i) \upharpoonright (\bigcap_{i \in I} ((\prod_{i \in I} \mathfrak{A}_i) \upharpoonright \pi_i^{-1}[\mathcal{A}_i])))$ *of* $\overline{\mathcal{A}}$, any substructure $\mathcal{B}$ of it being called a *subdirect product of* $\overline{\mathcal{A}}$, whenever, for each $i \in I$, $\pi_i[B] = A_i$. (In case $I = 2$, as usual, $\mathcal{A}_0 \times \mathcal{A}_1$ stands for the direct product involved.)

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[3] According to Example 6 of [18], in case of infinitary signatures (as opposed to that of finitary ones), structures need not have the greatest congruence that excludes adaptation of "finitary" methods to the general infinitary case.

Given a class K of L-structures, the class of all L-structures being isomorphic/[non-one-element] substructures to/of members of K is denoted by $(\mathbf{I}/\mathbf{S}_{[>1]})\mathsf{K}$, respectively. Likewise, the class of all (sub)direct products of [finite] tuples constituted by {consistent} members of K is denoted by $\mathbf{P}^{(\mathrm{SD})\{*\}}_{[\omega]}\mathsf{K}$, respectively.

A [Lindenbaum] L-structure $\mathcal{A}$ is said to be a *[Lindenbaum] model of* an L-calculus $\mathcal{C}$, whenever every L-rule of $\mathcal{C}$ is satisfied in $\mathcal{A}$, that is, in $\mathrm{Cn}_{\mathcal{A}}$, the class of all them being denoted by $\mathrm{Mod}(\mathcal{C})$ (note that it is closed under intersections, and so under $\mathbf{I}$, $\mathbf{S}$ and $\mathbf{P}$, in view of (3.2)) [resp., $\mathrm{Lin}(\mathcal{C})$].

Given any $\Gamma \subseteq \mathrm{Fm}_L$, we have the Lindenbaum L-structure $\Gamma\!\uparrow$ with relations $r^{\Gamma\uparrow} \triangleq \{\bar{\varphi} \in \mathrm{Tm}^{\alpha}_{F} \mid r(\bar{\varphi}) \in \Gamma\}$, for every $r \in R$ of arity $\alpha \in \kappa$. Conversely, given any Lindenbaum L-structure $\mathcal{A}$, we have $(\mathcal{A}\!\downarrow) \triangleq \{r(\bar{\varphi}) \mid r \in R, \bar{\varphi} \in r^{\mathcal{A}}\} \subseteq \mathrm{Fm}_L$. This provides an isomorphism between the poset $\wp(\mathrm{Fm}_L)$ ordered by inclusion and the poset of all Lindenbaum L-structures ordered by $\preceq$.

Given any L-logic C, taking its structurality and the diagonal F-substitution into account, it is routine checking that:

$$\mathrm{Lin}(C) = \{\Gamma\!\uparrow \mid \Gamma \in (\mathrm{img}\, C)\}. \tag{3.4}$$

Theorem 3.1 (Completeness theorem)**.** *Any L-logic C is defined by* $\mathrm{Lin}(C)$*, and so by* $\mathrm{Mod}(C)$*.*

Proof. Consider any L-rule $\Gamma \to \Phi$ not satisfied in C, in which case $\Phi \notin C(\Gamma) \supseteq \Gamma$, and so, taking (3.4) into account, the rule is not true in $(C(\Gamma)\!\uparrow) \in \mathrm{Lin}(C) \subseteq \mathrm{Mod}(C)$ under the diagonal F-substitution, as required. □

3.1. **Sentential logics and logical matrices.** Let $R \triangleq \{D\}$, where D is unary (truth predicate). Then, any L-formula $D(\varphi)$ is identified with the F-term φ, unless any confusion is possible. In general, "(sentential) F-" means "L-". Then, sentential F-structures are traditionally called *(logical) F-matrices* (cf. [5]), any F-matrix $\mathcal{A}$ being identified with the couple $\langle \mathfrak{A}, D^{\mathcal{A}}\rangle$ with natural identifying elements of A^1 with those of A. This is said to be *n-valued/truth[-non]-empty*, where $n \in \omega$, provided $|A| = n/D^{\mathcal{A}} = [\neq]\varnothing$, respectively. In that case, an F-logic is said to be *[minimally] n-valued*, whenever it is defined by a single n-valued F-matrix [but is defined by a single m-valued F-matrix, for no $m \in n$]. Furthermore, an F-logic C is said to be *non-pseudo-axiomatic* (cf. [11] for the case $\kappa = \omega$), provided $\bigcap_{\beta\in\kappa} C(x_\beta) \subseteq C(\varnothing)$ (the converse inclusion always holds by the monotonicity of C).

Remark 3.2. Given an F-logic C, we have the F-logic $C_{+/-0}$, defined by $C_{+/-0}(X) \triangleq C(X)$, for all non-empty $X \subseteq \mathrm{Fm}_L$, and $C_{+/-0}(\varnothing) \triangleq (\varnothing/(\bigcap_{\beta\in\kappa} C(x_\beta)))$, being the greatest/least theorem-less/non-pseudo-axiomatic sublogic/extension of C, called the *theorem-less/non-pseudo-axiomatic version of C*. Then, the mappings

$$\begin{array}{rcl} C & \mapsto & C_{+0}, \\ C & \mapsto & C_{-0}, \end{array}$$

are inverse to one another isomorphisms between the poset of all non-pseudo-axiomatic L-logics ordered by $\subseteq$ and that of all theorem-less ones. □

Remark 3.3. Since any rule with[out] premises is [not] satisfied in any truth-empty matrix, given any class M of F-matrices and any non-empty class S of truth-empty F-matrices, the logic of $\mathsf{S} \cup \mathsf{M}$ is the theorem-less version of the logic of M. □

Proposition 3.4. *The logic of any class M of truth-non-empty Σ-matrices is non-pseudo-axiomatic.*

Proof. Consider any $\phi \in (\bigcap_{\beta\in\kappa} \mathrm{Cn}_{\mathsf{M}}(x_\beta))$, any $\mathcal{A} \in \mathsf{M}$ and any $h \in \hom(\mathfrak{Tm}_F, \mathfrak{A})$. Then, in view of the infiniteness and the regularity of κ, by induction on construction of any $\psi \in \mathrm{Tm}_F$, it is routine checking that $|\operatorname{Var}(\psi)| < \kappa = |V_\kappa|$. In particular, $V \triangleq \operatorname{Var}(\phi) \neq V_\kappa$. Therefore, there is some $\beta \in \kappa$ such that $x_\beta \notin V$. Take any $a \in D^{\mathcal{A}} \neq \varnothing$. Let $g \in \hom(\mathfrak{Tm}_F, \mathfrak{A})$ extend $(h \restriction V) \cup ((V_\kappa \setminus V) \times \{a\})$. Then, we have $g(x_\beta) = a \in D^{\mathcal{A}}$, and so, as $\phi \in \mathrm{Cn}_{\mathsf{M}}(x_\beta)$, we get $h(\phi) = g(\phi) \in D^{\mathcal{A}}$, as required. □

In case F contains a unary connective $\sim$ (weak negation), an F-matrix/-logic is said to be *paraconsistent*, provided it does not satisfy the *Ex Contradictione Quodlibet* rule:

$$\{D(x_0), D(\sim x_0)\} \to D(x_1). \tag{3.5}$$

Then, in case F contains also a binary connective $\vee$ (disjunction), an F-matrix/-logic is said to be *paracomplete*, provided it does not satisfy the *Excluded Middle law* axiom:

$$D(x_0 \vee \sim x_0), \tag{3.6}$$

3.2. **Prevarieties of algebraic systems.** Let $\approx$ be a binary *equality* symbol, without loss of generality, not belonging to $F \cup R$. Set $(R + \approx) \triangleq (R \cup \{\approx\})$ and $(L{+}\approx) \triangleq \langle F, R + \approx\rangle$. Then, $(L{+}\approx)$-rules are referred to as *L-pre-identities*, finitary $(L{+}\approx)$-axioms[rules] being, as usual (cf., e.g., [6]), referred to as *L-[quasi-]identities*. Next, any $(L + \approx)$-structure $\mathcal{A}$ is identified with the couple $\langle \mathcal{A}\restriction L, \approx^{\mathcal{A}}\rangle$. Further, given a class K of L-structures, put $(\mathsf{K} + \approx) \triangleq \{\langle \mathcal{A}, \Delta_A\rangle \mid \mathcal{A} \in \mathsf{K}\}$ and $\mathrm{Cn}^{\approx}_{\mathsf{K}} \triangleq \mathrm{Cn}_{\mathsf{K}+\approx}$, said to be *equationally defined by* K. Then, an L-pre-identity is said to be *(equationally) true/valid/satisfied in* K, whenever it is true in $\mathsf{K} + \approx$. Respectively, given a set of [(finitary) $(L + \approx)$-rules]-axioms $\mathcal{C}$, a(n) *(equational) model of* $\mathcal{C}$ is any L-structure equationally satisfying every pre-identity in $\mathcal{C}$, the class of all them being denoted by $\mathrm{Mod}_{\approx}(\mathcal{C})$ and called the *L-[pre(quasi)]variety axiomatized by* $\mathcal{C}$ (since $\mathbf{I}$, $\mathbf{S}$ and $\mathbf{P}$ retain the diagonality of binary relations, prevarieties are closed under $\mathbf{I}$, $\mathbf{S}$ and $\mathbf{P}$; cf. [6]). Likewise, the *relative sub[pre(quasi)]variety* $\mathrm{Mod}_{\approx}(\mathcal{C}) \cap \mathsf{K}$ *of* K is said to be *axiomatized by* $\mathcal{C}$ *relatively to* K, the reservation "relative" being omitted, whenever K is a [pre(quasi)]variety. Finally, the least L-[pre(quasi)]variety including K, being clearly axiomatized by the set of all L-[pre(quasi)-]identities true in K, is said to be *generated by* K and denoted by $[\mathbf{P}(\mathbf{Q})]\mathbf{V}(\mathsf{K})$.

By $\mathcal{E}_L$ we denote the $(L + \approx)$-calculus constituted by the following L-pre-identities:

$$x_0 \approx x_0, \tag{3.7}$$

$$(x_0 \approx x_1) \to (x_1 \approx x_0), \tag{3.8}$$

$$\{x_0 \approx x_1, x_1 \approx x_2\} \to (x_0 \approx x_2), \tag{3.9}$$

$$\{x_i \approx x_{\alpha+i} \mid i \in \alpha\} \to (f(x_k)_{k\in\alpha} \approx f(x_{\alpha+k})_{k\in\alpha}), \tag{3.10}$$

$$(\{x_j \approx x_{\beta+j} \mid j \in \beta\} \cup \{r(x_l)_{l\in\beta}\}) \to r(x_{\beta+l})_{l\in\beta}, \tag{3.11}$$

for all $f \in F$ of arity $\alpha \in \kappa$ and all $r \in R$ of arity $\beta \in \kappa$.

4. Equational systems

An *equational L-system* (*for* a class M of L-structures) is any $\varepsilon \in \wp(\mathrm{Fm}_L)$ such that $\operatorname{Var}(\varepsilon) \subseteq V_2$ (and the L-pre-identities:

$$\varepsilon \quad \to \quad (x_0 \approx x_1), \tag{4.1}$$

$$\varepsilon[x_1/x_0] \tag{4.2}$$

are true in M, in which case it[, being finite,] is so for $\mathbf{P}[\mathbf{Q}]\mathbf{V}(\mathsf{M})$). In this way, we have the mapping $\tau_\varepsilon : \mathrm{Fm}_{L+\approx} \to \wp(\mathrm{Fm}_L)$, given by:

$$\tau_\varepsilon(\Phi) \triangleq \begin{cases} \varepsilon[x_0/\phi, x_1/\psi] & \text{if } \Phi = (\phi \approx \psi), \phi, \psi \in \mathrm{Tm}_F, \\ \{\Phi\} & \text{otherwise.} \end{cases}$$

Then, for any L-pre-identity $\Psi = (\Gamma \to \Phi)$, we have the L-calculus $\tau_\varepsilon(\Psi) \triangleq \{(\bigcup \tau_\varepsilon[\Gamma]) \to \Upsilon \mid \Upsilon \in \tau_\varepsilon(\Phi)\}$.

Lemma 4.1. *Let M be a class of L-structures, ε an equational L-system for it, $\mathcal{A} \in \mathsf{M}$, $\Phi \in \mathrm{Fm}_{L+\approx}$ and $h \in \hom(\mathfrak{Tm}_F, \mathfrak{A})$. Then, $(\mathcal{A}{+}\approx) \models \Phi[h] \Leftrightarrow \mathcal{A} \models (\bigwedge \tau_\varepsilon(\Phi))[h]$, in which case Cn_{M} and $\mathrm{Cn}^{\approx}_{\mathsf{M}} = \mathrm{Cn}^{\approx}_{\mathbf{PV}(\mathsf{M})}$ are equivalent (in the sense of [11][4]) with the identity unary operation on Fm_L and τ_ε. In particular, an L-pre-identity Ψ is true in M iff each rule in $\tau_\varepsilon(\Psi)$ is so.*

Proof. By the validity of (4.1) and (4.2) in $\mathcal{A}$. □

Lemma 4.2. *Let ε be an equational L-system and $\mathcal{A}$ an L-structure. Suppose $\theta^{\mathcal{A}}_\varepsilon \triangleq \{\langle a, b\rangle \mid \mathcal{A} \models (\bigwedge \varepsilon)[x_0/a, x_1/b]\} \in \mathrm{Con}(\mathcal{A})$. Then, ε is an equational L-system for $\mathcal{A}/\theta^{\mathcal{A}}_\varepsilon$.*

Proof. Since $\theta \triangleq \theta^{\mathcal{A}}_\varepsilon \in \mathrm{Con}(\mathcal{A})$, we have $\nu_\theta \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{A}, \mathcal{A}/\theta)$. Hence, for all $a, b \in A$, we get $((\mathcal{A}/\theta) \models (\bigwedge \varepsilon)[x_0/\nu_\theta(a), x_1/\nu_\theta(b)) \Leftrightarrow (\mathcal{A} \models (\bigwedge \varepsilon)[x_0/a, x_1/b) \Leftrightarrow (a \; \theta \; b) \Leftrightarrow (\nu_\theta(a) = \nu_\theta(b))$, that is, ε is an equational L-system for $\mathcal{A}/\theta$, as required. □

Theorem 4.3. *Let ε be an equational L-system and C an L-logic. Then, the following are equivalent:*

(i) $(\bigcup \tau_\varepsilon[\mathcal{E}_L]) \subseteq C$;
(ii) *for every $\mathcal{A} \in \mathrm{Mod}(C)$, $\theta^{\mathcal{A}}_\varepsilon \in \mathrm{Con}(\mathcal{A})$;*
(iii) *there is some class M of L-structures such that C is defined by M and ε is an equational L-system for M.*

Proof. First, (i)⇒(ii) is immediate. Next, (ii)⇒(iii) is by (3.2), Theorem 3.1 and Lemma 4.2, when taking $\mathsf{M} \triangleq \{\mathcal{A}/\theta^{\mathcal{A}}_\varepsilon \mid \mathcal{A} \in \mathrm{Mod}(C)\}$. Finally, (iii)⇒(i) is by Lemma 4.1 and the fact that all pre-identities of $\mathcal{E}_L$ are true in every L-structure. □

Corollary 4.4. *Let $\mathsf{M} \cup \{\mathcal{A}\}$ be a class of L-structures and ε an equational L-system for M. Then, $\mathcal{A} \in \mathrm{Mod}(\mathrm{Cn}_{\mathsf{M}})$ iff there is some $\theta \in \mathrm{Con}(\mathcal{A})$ such that $(\mathcal{A}/\theta) \in \mathbf{PV}(\mathsf{M})$.*

Proof. First, assume there is some $\theta \in \mathrm{Con}(\mathcal{A})$ such that $(\mathcal{A}/\theta) \in \mathbf{PV}(\mathsf{M})$. Then, $(\mathcal{A}/\theta) \in \mathrm{Mod}_{\approx}(\mathrm{Cn}^{\approx}_{\mathsf{M}}) \subseteq \mathrm{Mod}(\mathrm{Cn}_{\mathsf{M}})$. Moreover, $\nu_\theta \in \hom_{\mathrm{S}}(\mathcal{A}, \mathcal{A}/\theta)$. Hence, by (3.2), we get $\mathcal{A} \in \mathrm{Mod}(\mathrm{Cn}_{\mathsf{M}})$. Conversely, assume $\mathcal{A} \in \mathrm{Mod}(\mathrm{Cn}_{\mathsf{M}})$. Then, by Theorem 4.3(iii)⇒(ii), $\theta \triangleq \theta^{\mathcal{A}}_\varepsilon \in \mathrm{Con}(\mathcal{A})$, in which case $\nu_\theta \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{A}, \mathcal{A}/\theta)$, and so, by (3.2), $(\mathcal{A}/\theta) \in \mathrm{Mod}(\mathrm{Cn}_{\mathsf{M}})$, while, by Lemma 4.2, ε is an equational L-system for $\mathcal{A}/\theta$. Let $\mathcal{P}$ be the set of all L-pre-identities true in M. Then, by Lemma 4.1, the L-rules of $\bigcup \tau_\varepsilon[\mathcal{P}]$ are all true in M, and so in $\mathcal{A}/\theta$. Therefore, by Lemma 4.1, we eventually conclude that the pre-identities of $\mathcal{P}$ are all true in $\mathcal{A}/\theta$, that is, $(\mathcal{A}/\theta) \in \mathbf{PV}(\mathsf{M})$, as required. □

Since L-rules are L-pre-identities, combining (3.2), Theorem 3.1, Lemma 4.1 and Corollary 4.4, we eventually get:

Theorem 4.5. *Let M be a class of L-structures and ε an equational L-system for it. Then, the following hold:*

[4] More precisely, in a sense immediately extending that adopted therein to infinitary logics and translations as well as to non-countable κ.

(i) *the mappings*

$$\begin{array}{rcl} C & \mapsto & (\mathbf{PV}(\mathsf{M}) \cap \mathrm{Mod}(C)), \\ \mathsf{S} & \mapsto & \mathrm{Cn}_{\mathsf{S}} \end{array}$$

are inverse to one another dual isomorphisms between the lattice of all extensions of Cn_{M} *and that of all subprevarieties of* $\mathbf{PV}(\mathsf{M})$*;*

(ii) *for any L-calculus $\mathcal{C}$, the extension of Cn_{M} relatively axiomatized by $\mathcal{C}$ corresponds to the subprevariety of $\mathbf{PV}(\mathsf{M})$ relatively axiomatized by $\mathcal{C}$, while, conversely, for any $(L+\approx)$-calculus $\mathcal{P}$, the subprevariety of $\mathbf{PV}(\mathsf{M})$ relatively axiomatized by $\mathcal{P}$ corresponds to the extension of Cn_{M} relatively axiomatized by $\bigcup \tau_\varepsilon[\mathcal{P}]$, and so axiomatic extensions of Cn_{M} correspond exactly to relative subvarieties of $\mathbf{PV}(\mathsf{M})$;*

(iii) *the subprevariety of $\mathbf{PV}(\mathsf{M})$ generated by any $\mathsf{K} \subseteq \mathbf{PV}(\mathsf{M})$ corresponds to the extension of Cn_{M} defined by K.*

Lemma 4.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be L-structures, $h \in \hom_{\mathrm{S}}(\mathcal{A}, \mathcal{B})$ and ε an equational L-system for $\mathcal{A}$. Then, h is injective.*

Proof. Then, by (3.3), the L-axioms (4.2) are true in $\mathcal{C} \triangleq (\mathcal{B}{\restriction}(\mathrm{img}\, h))$, for $h \in \hom^{\mathrm{S}}(\mathcal{A}, \mathcal{C})$. Consider any $a, b \in A$ such that $h(a) = h(b)$. Then, by (4.2), we have $\mathcal{C} \models (\bigwedge \varepsilon)[x_0/h(a), x_1/h(b)]$, in which case we get $\mathcal{A} \models (\bigwedge \varepsilon)[x_0/a, x_1/b]$, for $h \in \hom_{\mathrm{S}}(\mathcal{A}, \mathcal{C})$, and so, by (4.1), we eventually get $a = b$, as required. □

As an immediate consequence of Lemma 4.6, we first have:

Corollary 4.7. *Any L-structure with equational L-system is simple.*

Since prevarieties are closed under $\mathbf{I}$, while any L-structure $\mathcal{A}$ is isomorphic to $\mathcal{A}/\Delta_A$, by Corollaries 4.4 and 4.7, we eventually get:

Corollary 4.8. *Let M be a class of L-structures and ε an equational L-system for it. Then, $\mathbf{PV}(\mathsf{M})$ is the class of all simple models of Cn_{M}.*

5. Discrimination-refutation

Let $\mathsf{K} \cup \{\mathcal{A}\}$ be a class of L-structures. First, put

$$\hom^{[\mathrm{S}]}_{(\mathrm{S})}(\mathcal{A}, \mathsf{K}) \triangleq (\bigcup\{\hom^{[\mathrm{S}]}_{(\mathrm{S})}(\mathcal{A}, \mathcal{B}) \mid \mathcal{B} \in \mathsf{K}\}).$$

Next, $\mathcal{A}$ is said to be *discriminated/refuted by* K, whenever, for each $r \in R$ of arity $\alpha \in \kappa$ and every $\bar{a} \in (A^\alpha \setminus r^{\mathcal{A}})$, there are some $\mathcal{B} \in \mathsf{K}$ and some $h \in \hom(\mathcal{A}, \mathcal{B})$ such that $\bar{a} \notin h^{-1}[r^{\mathcal{B}}]$, the class of all L-structures refuted by K being denoted by $\Re(\mathsf{K})$. The meaning of this notion is explained by the following two key observations:

Lemma 5.1. *Suppose $\mathcal{A}$ is discriminated by K[, while either both $\mathcal{A}$ and R are finite or both $\mathcal{A}$ is finitely-generated and both K and all members of it are finite].Then, $\hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{A}, \mathbf{P}^{\mathrm{SD}*}_{[\omega]}\mathbf{IS}\mathsf{K}) \neq \varnothing$. In particular, $\mathcal{A} \in \mathrm{Mod}(\mathrm{Cn}_{\mathsf{K}})$.*

Proof. Let S be the set of all consistent L-structures with underlying algebra $\mathfrak{A}$. Then, $I \triangleq \{\langle \mathcal{C}, \theta\rangle \mid \mathcal{A} \preceq \mathcal{C} \in \mathsf{S}, \theta \in \mathrm{Con}(\mathcal{C}), (\mathcal{C}/\theta) \in \mathbf{IS}\mathsf{K}\}$ is a set. For every $i = \langle \mathcal{C}, \theta\rangle \in I$, put $\mathcal{C}_i \triangleq \mathcal{C} \in \mathsf{S}$, $\theta_i \triangleq \theta \in \mathrm{Con}(\mathcal{C}_i) \subseteq \mathrm{Con}(\mathfrak{A})$ and $\mathcal{B}_i \triangleq (\mathcal{C}_i/\theta_i) \in \mathbf{IS}\mathsf{K}$, in which case $\mathcal{A} \preceq \mathcal{C}_i = (\mathfrak{A} \uparrow \nu_{\theta_i}^{-1}[\mathcal{B}_i])$ and $\nu_{\theta_i} \in \hom^{\mathrm{S}}_{\{\mathrm{S}\}}(\mathcal{A}\{\mathcal{C}_i\}, \mathcal{B}_i)$, and so $\mathcal{B}_i$ is consistent, for $\mathcal{C}_i \in \mathcal{S}$ is so. [In case both $\mathcal{A}$ and R are finite, both S and $\mathrm{Con}(\mathfrak{A}) \subseteq \wp(A^2)$ are finite, and so is $I \subseteq (\mathsf{S} \times \mathrm{Con}(\mathfrak{A}))$. Likewise, assume both $\mathcal{A}$ is finitely-generated and both K and all members of it are finite, in which case both $\mathbf{S}\mathsf{K}$ and all members of it are finite, and so $\mathcal{H} \triangleq \{\langle g, \mathcal{D}\rangle \mid \mathcal{D} \in \mathbf{S}\mathsf{K}, g \in \hom(\mathcal{A}, \mathcal{D})\}$ is finite. Consider any $i \in I$. Then, by the Choice Axiom, there are some $\mathcal{D}_i \in \mathbf{S}\mathsf{K}$ and some isomorphism e_i from $\mathcal{B}_i$ onto $\mathcal{D}_i$, in which case $g_i \triangleq (e_i \circ \nu_{\theta_i}) \in \hom(\mathcal{A}, \mathcal{D}_i)$, and so $\langle g_i, \mathcal{D}_i\rangle \in \mathcal{H}$. And what is more, $(\ker g_i) = (\ker \nu_{\theta_i}) = \theta_i$, for e_i is injective, while $\mathcal{C}_i = (\mathfrak{A} \upharpoonright \nu_{\theta_i}^{-1}[\mathcal{B}_i]) = (\mathfrak{A} \upharpoonright \nu_{\theta_i}^{-1}[\mathfrak{B}_i \upharpoonright e_i^{-1}[\mathcal{D}_i]]) = (\mathfrak{A} \upharpoonright g_i^{-1}[\mathcal{D}_i])$. In this way, $\{\langle i, \langle g_i, \mathcal{D}_i\rangle\rangle \mid i \in I\} : I \to \mathcal{H}$ is injective, and so I is finite, for $\mathcal{H}$ is so. Thus, anyway, I is finite.] Then, $h : A \to (\prod_{i\in I} B_i), a \mapsto \langle [a]_{\theta_i}\rangle_{i\in I}$, in which case $(\pi_i \circ h) = \nu_{\theta_i}$, for each $i \in I$, is a homomorphism from $\mathfrak{A}$ to $\mathfrak{D} \triangleq (\prod_{i\in I} \mathfrak{B}_i)$, and so is a surjective one onto $\mathfrak{E} \triangleq (\mathfrak{D}{\upharpoonright}(\operatorname{img} h))$, in which case $\pi_i[E] = \pi_i[h[A]] = (A/\theta_i) = B_i$, for each $i \in I$. Moreover, $\mathcal{A} \preceq (\mathfrak{A} \upharpoonright \bigcap_{i\in I} \mathcal{C}_i)$. Conversely, consider any $r \in R$ of arity $\alpha \in \kappa$ and any $\bar{a} \in (A^\alpha \setminus r^{\mathcal{A}})$. Then, there are some $\mathcal{F} \in \mathsf{K}$ and some $g \in \hom(\mathcal{A}, \mathcal{F})$ such that $\bar{a} \notin g^{-1}[r^{\mathcal{F}}]$, in which case $\mathcal{A} \preceq \mathcal{G} \triangleq (\mathfrak{A} \upharpoonright g^{-1}[\mathcal{F}])$, and so $\bar{a} \notin r^{\mathcal{G}}$, in which case $\mathcal{G}$ is consistent, and so $\mathcal{G} \in \mathsf{S}$, while $\vartheta \triangleq (\ker g) \in \mathrm{Con}(\mathcal{G})$, in view of (3.1). Therefore, by the Homomorphism Theorem, $e \triangleq (g \circ \nu_\vartheta^{-1}) \in \hom_{\mathrm{S}}(\mathcal{G}/\theta, \mathcal{F})$ is injective, in which case $(\mathcal{G}/\vartheta) \in \mathbf{IS}\mathsf{K}$, and so $i \triangleq \langle \mathcal{G}, \vartheta\rangle \in I$, in which case $\mathcal{G} = \mathcal{C}_i$, and so $\bar{a} \notin r^{\mathcal{C}_i}$. Thus, $\mathcal{A} = (\mathfrak{A} \upharpoonright \bigcap_{i\in I} \mathcal{C}_i) = (\mathfrak{A} \upharpoonright \bigcap_{i\in I}(\mathfrak{A} \upharpoonright (\pi_i \circ h)^{-1}[\mathcal{B}_i])) = (\mathfrak{A} \upharpoonright h^{-1}[\mathfrak{D} \upharpoonright (\bigcap_{i\in I}(\mathfrak{D} \upharpoonright \pi_i^{-1}[\mathcal{B}_i]))] = (\mathfrak{A} \upharpoonright h^{-1}[\prod_{i\in I} \mathcal{B}_i]) = (\mathfrak{A} \upharpoonright h^{-1}[(\prod_{i\in I} \mathcal{B}_i){\upharpoonright}E])$, and so $h \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{A}, (\prod_{i\in I} \mathcal{B}_i){\upharpoonright}E) \subseteq \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{A}, \mathbf{P}^{\mathrm{SD}*}_{[\omega]}\mathbf{IS}\mathsf{K})$, as required. Finally, (3.2) completes the argument. □

Conversely, we have:

Lemma 5.2. *Let* K *be a class of* L*-structures and* C *the logic of it. Then, any* $\mathcal{A} \in \mathrm{Lin}(C)$ *is discriminated by* K.

Proof. Consider any $r \in R$ of arity $\alpha \in \kappa$ and any $\bar{a} \in (A^\alpha \setminus r^{\mathcal{A}})$, in which case $\Phi \triangleq r(\bar{a}) \in (\mathrm{Fm}_L \setminus (\mathcal{A}{\downarrow}))$, and so, by (3.4), $\Phi \in (\mathrm{Fm}_L \setminus C(\mathcal{A}{\downarrow}))$, that is, there are some $\mathcal{B} \in \mathsf{K}$ and some $h \in \hom(\mathcal{A}, \mathcal{B})$ such that $\mathcal{B} \models (\bigwedge(\mathcal{A}{\downarrow}))[h]$ but $\mathcal{B} \not\models \Phi[h]$, i.e., $h \in \hom(\mathcal{A}, \mathcal{B})$ but $\bar{a} \notin h^{-1}[r^{\mathcal{B}}]$. □

First of all, combining (3.2) and Theorem 3.4 with Lemmas 5.1 and 5.2, we get the following relative generalization of Theorem 4.5 extending [14] beyond *pure* algebras:

Theorem 5.3. *Let* P *be a class of* L*-structures and* $\mathsf{M} \subseteq \mathsf{P}$. *Suppose* $\mathsf{P} \subseteq \mathrm{Mod}(\mathrm{Cn}_{\mathsf{M}})$ *is closed under* $\mathbf{I}$, $\mathbf{S}$ *and* $\mathbf{P}$ *(in particular,* $\mathsf{P} \subseteq \Re(\mathsf{M})$ *is a prevariety). Then, the following hold:*

(i) *the mappings*

$$\begin{array}{rcl} C & \mapsto & (\mathsf{P} \cap \mathrm{Mod}(C)), \\ \mathsf{S} & \mapsto & \mathrm{Cn}_{\mathsf{S}} \end{array}$$

form a Galois retraction between the lattice of all extensions of Cn_{M} *and that of all subprevarieties of* P*;*

(ii) *for any* L*-calculus* $\mathcal{C}$*, the extension of* Cn_{M} *relatively axiomatized by* $\mathcal{C}$ *is mapped to the subprevariety of* P *relatively axiomatized by* $\mathcal{C}$*;*

(iii) *given any* $\mathsf{K} \subseteq \mathsf{P}$, $\Re(\mathsf{K}) \cap \mathsf{P}$*, being the subprevariety of* P *relatively axiomatized by an* L*-calculus* $\mathcal{C}$*, is mapped to the extension of* Cn_{M} *defined by* K *and relatively axiomatized by* $\mathcal{C}$.

Next, combining Lemmas 4.6 and 5.1, we immediately get:

Proposition 5.4. *Let* ε *be an equational* L*-system for* $\mathsf{K} \cup \{\mathcal{A}\}$. *Suppose* $\mathcal{A}$ *is discriminated by* K*[, while either both* $\mathcal{A}$ *and* R *are finite or both* $\mathcal{A}$ *is finitely-generated and both* K *and all members of it are finite]. Then,* $\mathcal{A} \in \mathbf{IP}^{\mathrm{SD}*}_{[\omega]}\mathbf{IS}\mathsf{K} \subseteq \mathbf{PV}(\mathsf{K})$.

Note that $x_0 \approx x_1$ is an equational $(L + \approx)$-system for $(\mathsf{K} \cup \{\mathcal{A}\}) + \approx$. Hence, as a particular case of Proposition 5.4 valuable within Universal Algebra, we have:

Corollary 5.5 (cf. Remark 1.2 of [19] for the purely-algebraic case)**.** *Suppose* $\{R = \varnothing,\}$ $\mathcal{A} + \approx$ *is discriminated by* $\mathsf{K} + \approx$*[, while either both* $\mathcal{A}$ *and* R *are finite or both* $\mathcal{A}$ *is finitely-generated and both* K *and all members of it are finite]. Then,* $\mathcal{A} \in \mathbf{IP}^{\mathrm{SD}}_{[\omega]}\mathbf{IS}_{\{>1\}}\mathsf{K} \subseteq \mathbf{PV}(\mathsf{K})$.

It is the purely-algebraic particular case (with $R = \varnothing$) that justifies the term "discriminated" chosen here.

In addition, we also have:

Proposition 5.6. *Let* $\mathfrak{I}$ *be a set of* $(L[+\approx])$*-axioms. Suppose* $\mathcal{A} \in \mathrm{Mod}_{[\approx]}(\mathfrak{I})$ *is discriminated by* K*. Then, it is so by* $\mathrm{Mod}_{[\approx]}(\mathfrak{I}) \cap \mathbf{S}\mathsf{K}$.

Proof. Consider any $r \in R$ of arity $\alpha \in \kappa$ and any $\bar{a} \in (A^\alpha \setminus r^{\mathcal{A}})$. Then, there are some $\mathcal{B} \in \mathsf{K}$ and some $h \in \hom(\mathcal{A}, \mathcal{B})$ such that $\bar{a} \notin h^{-1}[r^{\mathcal{B}}]$, in which case $\mathcal{C} \triangleq (\mathcal{B}\restriction(\mathrm{img}\, h)) \in \mathbf{S}\mathsf{K}$, $\bar{a} \notin h^{-1}[r^{\mathcal{C}}]$ and $h \in \hom(\mathcal{A}[+\approx], \mathcal{C}[+\approx])$ is surjective, and so $\mathcal{C} \in \mathrm{Mod}_{[\approx]}(\mathfrak{I})$, in view of (3.3), as required. □

6. Applications and examples

[Bounded] lattices are supposed to be of the functional signature $F_{+[,01]} \triangleq \{\wedge, \vee[, \bot, \top]\}$ with binary $\wedge$ (conjunction) and $\vee$ (disjunction) [as well as nullary $\bot$ and $\top$]. Let $F \supseteq F_+$ be a functional signature. Given any $\phi, \psi \in \mathrm{Tm}_F$, the formal expression $\phi \lessapprox \psi$ stands for the equation $(\phi \wedge \psi) \approx \phi$. Given any F-algebra $\mathfrak{A}$ such that $\mathfrak{A}\restriction F_+$ is a lattice, the partial ordering/[prime] filters|ideals of the latter is/are denoted by $\leqslant^{\mathfrak{A}}$/referred to as those of $\mathfrak{A}$.

Given any $n \in (\omega \setminus 1)$, by $\mathfrak{D}_{n[,01]}$ we denote the [bounded] distributive lattice over the chain n ordered by the natural partial ordering.

During this section, we entirely follow Subsection 3.1.

6.1. Filtered De Morgan Boolean algebras versus the classical expansion of Belnap's four-valued logic.

Boolean algebras are supposed to be of the classical signature $F_{-} \triangleq (F_+ \cup \{\neg\})$ with unary $\neg$ (classical negation), the secondary binary "material" implication $\supset$ and respective equivalence $\leftrightarrow$ connectives being, as usual, defined by:

$$(x_0 \supset x_0) \triangleq (\neg x_0 \vee x_1),$$
$$(x_0 \leftrightarrow x_1) \triangleq ((x_0 \supset x_1) \wedge (x_1 \supset x_0)).$$

The ordinary two-element Boolean algebra with carrier 2 is denoted by $\mathfrak{B}_2$. (More precisely, $(\mathfrak{B}_2\restriction F_{+,01}) \triangleq \mathfrak{D}^2_{2,01}$ and $\neg^{\mathfrak{B}_2} i \triangleq (1 - i)$, for all $i \in 2$.)

During this subsection, we deal with the functional signature $F_{\simeq} \triangleq (F_{-} \cup \{\sim\})$ with unary $\sim$ (weak negation) and the secondary binary "strong equivalence" connective $\equiv$ defined by $(x_0 \equiv x_1) \triangleq ((x_0 \leftrightarrow x_1) \wedge ({\sim}x_0 \leftrightarrow {\sim}x_1))$ and treated as an equational $\langle F_{\simeq}, \{D\}\rangle$-system.

According to [12] (cf. [15]), a *De Morgan Boolean algebra* is any $F_{\simeq}$-algebra $\mathfrak{A}$ such that $\mathfrak{A}\restriction F_{-}$ is a Boolean algebra, and the following *De Morgan* identities are true in $\mathfrak{A}$:

$$ {\sim}{\sim}x_0 \approx x_0, \tag{6.1}$$
$$ {\sim}(x_0 \vee x_1) \approx {\sim}x_0 \wedge {\sim}x_1, \tag{6.2}$$

the variety of all them being denoted by DMBA. De Morgan Boolean algebras also satisfy the following identities to be used tacitly throughout the rest of this subsection:

$$ {\sim}(x_0 \wedge x_1) \approx {\sim}x_0 \vee {\sim}x_1, \tag{6.3}$$
$$ {\sim}\bot \approx \top, \tag{6.4}$$

$$\sim\top \approx \bot, \tag{6.5}$$
$$\sim\neg x \approx \neg\sim x. \tag{6.6}$$

A *filtered De Morgan Boolean algebra* is any $F_{\simeq}$-matrix with underlying algebra being a De Morgan Boolean algebra and satisfying the following quasi-identities:

$$D(x_0 \equiv x_1) \rightarrow (x_0 \approx x_1), \tag{6.7}$$
$$\{D(x_0), D(x_1)\} \rightarrow D(x_0 \wedge x_1), \tag{6.8}$$
$$D(x_0 \wedge x_1) \rightarrow D(x_0), \tag{6.9}$$
$$D(\top), \tag{6.10}$$

the quasivariety of all them being denoted by FDMBA. The identity $(x_0 \leftrightarrow x_0) \approx \top$ is well known to be true in Boolean algebras, in which case the identity $(x_0 \equiv x_0) \approx \top$ is true in De Morgan Boolean algebras, and so, by (6.7) and (6.10), $x_0 \equiv x_1$ is an equational $(F_{\simeq} \cup \{D\})$-system for FDMBA (this fact is used tacitly throughout the rest of the paper).

By $\mathfrak{DMB}_4$ we denote the Boolean De Morgan algebra defined as follows: put $(\mathfrak{DMB}_4{\restriction}F_-) \triangleq \mathfrak{B}_2^2$ and $\sim^{\mathfrak{DMB}_4}\vec{a} \triangleq \langle 1-a_{1-i}\rangle_{i\in 2}$, for all $\vec{a} \in 2^2$. In this connection, we use the following standard abbreviations going back to [2]:

$$\begin{aligned}\mathsf{t} &\triangleq \langle 1,1\rangle,\\ \mathsf{f} &\triangleq \langle 0,0\rangle,\\ \mathsf{b} &\triangleq \langle 1,0\rangle,\\ \mathsf{n} &\triangleq \langle 0,1\rangle.\end{aligned}$$

Since $\{\mathsf{b},\mathsf{t}\}$ is a prime filter of $\mathfrak{B}_2^2$, we have $((a \in \{\mathsf{b},\mathsf{t}\}) \Leftrightarrow (b \in \{\mathsf{b},\mathsf{t}\})) \Leftrightarrow ((a \leftrightarrow^{\mathfrak{B}_2^2} b) \in \{\mathsf{b},\mathsf{t}\})$, for all $a, b \in 2^2$. And what is more, $a = b$ iff, for each $i \in 2$, $((\sim^{\mathfrak{DMB}_4})^i a \in \{\mathsf{b},\mathsf{t}\}) \Leftrightarrow ((\sim^{\mathfrak{DMB}_4})^i b \in \{\mathsf{b},\mathsf{t}\})$, for all $a, b \in 2^2$. Hence, $x_0 \equiv x_1$ is an equational L-system for $\mathcal{DMB}_4 \triangleq \langle \mathfrak{DMB}_4, \{\mathsf{b},\mathsf{t}\}\rangle$ defining the classical expansion CB_4 of Belnap's logic DB_4 [12], and so $\mathcal{DMB}_4 \in \mathsf{FDMBA}$. In particular, Theorem 4.5 is well-applicable to CB_4. In this way, in view of the following primary result, to find its extensions is to find subprevarieties of FDMBA:

Proposition 6.1. *Every filtered De Morgan Boolean algebra is discriminated by $\mathcal{DMB}_4$. In particular,* $\mathsf{FDMBA} = \mathbf{PV}(\mathcal{DMB}_4)$.

Proof. Consider any $\mathcal{A} \in \mathsf{FDMBA}$ and any $a \in (A \setminus D^{\mathcal{A}})$. Then, as $D^{\mathcal{A}}$ is a filter of the Boolean algebra $\mathfrak{A}{\restriction}F_-$, by the Prime Ideal Theorem, there is some prime filter X of $\mathfrak{A}{\restriction}F_-$ such that $D^{\mathcal{A}} \subseteq X \not\ni a$. Then, by the following claim, $h \triangleq h^{\mathfrak{A}}_X \in \hom_S(\langle \mathfrak{A}, X\rangle, \mathcal{DMB}_4)$, in which case $h \in \hom(\mathcal{A}, \mathcal{DMB}_4)$, while $a \notin h^{-1}[D^{\mathcal{DMB}_4}]$, and so $\mathcal{A}$ is discriminated by $\mathcal{DMB}_4$:

Claim 6.2. *Let $\mathfrak{A} \in \mathsf{DMBA}$, X a prime filter of $\mathfrak{A}{\restriction}F_-$ and $Y \triangleq (\sim^{\mathfrak{A}})^{-1}[A \setminus X]$. Then, $h^{\mathfrak{A}}_X \triangleq (\chi^X_A \times \chi^Y_A) \in \hom(\mathfrak{A}, \mathfrak{DMB}_4)$.*

Proof. First, by (6.2), (6.3), (6.4) and (6.5), Y is a prime filter of $\mathfrak{A}{\restriction}F_-$. Moreover, given a prime filter Z of a Boolean algebra $\mathfrak{B}$, $\chi^Z_B \in \hom(\mathfrak{B}, \mathfrak{B}_2)$. Therefore, $h \triangleq h^{\mathfrak{A}}_X \in \hom(\mathfrak{A}{\restriction}F_-, \mathfrak{B}_2^2)$. Finally, using (6.1), it is routine checking that $h(\sim^{\mathfrak{A}} b) = \sim^{\mathfrak{DMB}_4} h(b)$, for all $b \in A$, as required. □

In this way, Proposition 5.4 completes the argument. □

The relative subvariety IFDMBA of FDMBA, constituted by all its inconsistent members, is relatively axiomatized by the axiom $D(x_0)$ and corresponds to the inconsistent extension of CB_4. In view of (6.7) and (6.10), a filtered De Morgan Boolean algebra is inconsistent iff it is one-element.

The relative subvariety of FDMBA, constituted by all its non-paracomplete members, is denoted by $\mathsf{NPCFDMBA}$ and corresponds to the least non-paracomplete extension of CB_4.

Note that $\mathfrak{DMB}_2 \triangleq (\mathfrak{DMB}_4 \restriction \{\mathsf{f}, \mathsf{t}\})$ is the only proper subalgebra of $\mathfrak{DMB}_4$, and so $\mathcal{DMB}_2 \triangleq \langle \mathfrak{DMB}_2, \{\mathsf{t}\}\rangle = (\mathcal{DMB}_4 \restriction \{\mathsf{f}, \mathsf{t}\})$ is the only proper submatrix of $\mathcal{DMB}_4$. Moreover, both (3.6) and the identity:

$$\sim x_0 \approx \neg x_0 \tag{6.11}$$

are true in $\mathfrak{DMB}_2$, while neither (3.6) nor (6.11) is true in $\mathcal{DMB}_4$ under $[x_0/\mathsf{n}]$. In this way, as $\mathcal{DMB}_2$, being non-one-element, is consistent, by Propositions 5.4, 5.6 and 6.1, we immediately get:

Corollary 6.3. *Relative subvarieties of* FDMBA *form the three-element chain:*

$$[\mathbf{Q}]\mathbf{V}(\varnothing) = \mathsf{IFDMBA} \subsetneq \mathsf{NPCFDMBA} = \mathbf{PV}(\mathcal{DMB}_2) \subsetneq \mathsf{FDMBA},$$

$\mathsf{NPCFDMBA}$ *being relatively axiomatized by* (6.11).

Thus, the logic of $\mathcal{DMB}_2$, being the definitional expansion of the classical logic PC of $\langle \mathfrak{B}_2, \{1\}\rangle$ given by (6.11), is the only proper consistent axiomatic extension of CB_4. The rest of this section is devoted to finding all non-axiomatic ones.

A filtered De Morgan Boolean algebra is said to be *truth-singular* if it satisfies the following quasi-identity:

$$D(x_0) \to (x_0 \approx \top), \tag{6.12}$$

the quasivariety of all them being denoted by $\mathsf{TSFDMBA}$.

Remark 6.4. It is routine checking that the rule $D(x_0) \to D(\sim x_0 \vee (x_0 \equiv \top))$ is true in FDMBA. In this way, $\mathsf{TSFDMBA}$ is the subquasivariety of FDMBA relatively axiomatized by the *Modus Ponens* rule for the one more "material" implication $\sim x_0 \vee x_1$:

$$\{D(x_0), D(\sim x_0 \vee x_1)\} \to D(x_1), \tag{6.13}$$

and so the extension of CB_4 corresponding to $\mathsf{TSFDMBA}$ is relatively axiomatized by (6.13) as well. □

Recall that the quasi-identity $((x_0 \leftrightarrow x_1) \approx \top) \to (x_0 \approx x_1)$ is true in Boolean algebras, and so in DMBA. Therefore, for any $\mathsf{K} \subseteq \mathsf{DMBA}$, we have $(\mathsf{K} + \top) \triangleq \{\langle \mathfrak{A}, \{\top^{\mathfrak{A}}\}\rangle \mid \mathfrak{A} \in \mathsf{K}\} \subseteq \mathsf{TSFDMBA}$. In particular, by (6.10), we get:

$$\mathsf{TSFDMBA} = (\mathsf{DMBA} + \top). \tag{6.14}$$

First, we have:

Proposition 6.5. *Every truth-singular filtered De Morgan Boolean algebra is discriminated by* $\mathfrak{DMB}_4 + \top$. *In particular,* $\mathsf{TSFDMBA} = \mathbf{PV}(\mathfrak{DMB}_4 + \bot)$.

Proof. Consider any $\mathcal{A} \in \mathsf{TSFDMBA}$ and any $a \in (A \setminus D^{\mathcal{A}})$. Then, as $D^{\mathcal{A}}$ is a filter of the Boolean algebra $\mathfrak{A} \restriction F_-$, by the Prime Ideal Theorem, there is some prime filter X of $\mathfrak{A} \restriction F_-$ such that $D^{\mathcal{A}} \subseteq X \not\ni a$. Then, by Claim 6.2, $h \triangleq h^{\mathfrak{A}}_X \in \hom_{\mathrm{S}}(\langle \mathfrak{A}, X\rangle, \mathcal{DMB}_4)$, in which case, by (6.14), $h \in \hom(\mathcal{A}, \mathfrak{DMB}_4 + \top)$, while $a \notin h^{-1}[D^{\mathcal{DMB}_4}] \supseteq h^{-1}[D^{\mathfrak{DMB}_4+\top}]$, and so $\mathcal{A}$ is discriminated by $\mathfrak{DMB}_4 + \top$. In this way, Proposition 5.4 completes the argument. □

A filtered De Morgan Boolean algebra is said to be *non-idempotent*, provided it satisfies the following quasi-identity:

$$(\sim x_0 \approx x_0) \to (x_0 \approx x_1). \tag{6.15}$$

The quasivariety of all [truth-singular] non-idempotent filtered De Morgan Boolean algebras is denoted by $\mathsf{[TS]NIFDMBA}$.

Lemma 6.6. *Let* $\mathcal{A} \in \mathsf{NIFDMBA}$. *Suppose* $|A| > 1$. *Then,* $\hom(\mathcal{A}, \mathcal{DMB}_2) \neq \varnothing$.

Proof. Given any $n \in \omega$, by induction on any $i \in (n+1)$, define:

$$\varphi^n_i \triangleq \begin{cases} x_0 \vee {\sim} x_0 & \text{if } i = 0, \\ ((x_i \vee {\sim} x_i) \vee {\sim}\varphi^n_{i-1}) \wedge \varphi^n_{i-1} & \text{otherwise,} \end{cases}$$

in which case the identity ${\sim}\varphi^n_i \lessapprox \varphi^n_i$ is true in $\mathcal{DMB}_4$. Then, by induction on any $j \in ((n+1) \setminus i)$, it is routine checking that the quasi-identity:

$$({\sim} x_i \approx x_i) \to ({\sim}\phi^n_j \approx \phi^n_j) \tag{6.16}$$

is true in $\mathcal{DMB}_4$. Further, given any $k \in \omega$ and any $l \in (\omega \setminus 1)$, it is routine checking that the first-order clause

$$\begin{gathered}(\textstyle\bigwedge(\{{\sim} x_{i+1} \lessapprox x_{i+1} \mid i \in k\} \cup \{D(x_0)\} \cup \\ \{x_{j+1} \lessapprox {\sim} x_{j+1} \mid j \in ((k+l) \setminus k)\} \cup \\ \{(\wedge\langle\langle x_{i+1}\rangle_{i\in k}, x_0\rangle) \lessapprox (\vee\langle x_{j+1}\rangle_{j \in ((k+l)\setminus k)})\})) \to \\ (\textstyle\bigvee\{{\sim} x_{m+1} \approx x_{m+1} \mid m \in (k+l)\})\end{gathered}$$

is true in $\mathcal{DMB}_4$. Therefore, taking (6.16) with $j = n = (k+l)$ into account, we see that the quasi-identity

$$\begin{gathered}(\{{\sim} x_{i+1} \lessapprox x_{i+1} \mid i \in k\} \cup \{D(x_0)\} \cup \\ \{x_{j+1} \lessapprox {\sim} x_{j+1} \mid j \in ((k+l) \setminus k)\} \\ \cup \{(\wedge\langle\langle x_{i+1}\rangle_{i\in k}, x_0\rangle) \lessapprox (\vee\langle x_{j+1}\rangle_{j \in ((k+l)\setminus k)})\}) \to ({\sim}\varphi^{k+l}_{k+l} \approx \varphi^{k+l}_{k+l})\end{gathered} \tag{6.17}$$

is true in $\mathcal{DMB}_4$, and so in $\mathcal{A}$, in view of Proposition 6.1. In this way, as $\mathcal{A}$ is both non-idempotent and non-one-element, combining (6.15) and (6.17), we conclude that the filter $\mathcal{F} \triangleq \{a \in A \mid (\wedge^{\mathfrak{A}}\langle\langle b_i \vee^{\mathfrak{A}} {\sim}^{\mathfrak{A}} b_i\rangle_{i\in k}, c\rangle) \leqslant^{\mathfrak{A}} a, \bar{b} \in A^k, k \in \omega, c \in D^{\mathcal{A}}\} \supseteq D^{\mathcal{A}}$ of $\mathfrak{A}{\restriction}F_-$ is disjoint with its ideal $\mathcal{I} \triangleq \{a \in A \mid a \leqslant^{\mathfrak{A}} (\vee^{\mathfrak{A}}\langle d_j \wedge^{\mathfrak{A}} {\sim}^{\mathfrak{A}} d_j\rangle_{j\in l}), \bar{d} \in A^l, l \in (\omega \setminus 1)\}$, in which case, by the Prime Ideal Theorem, there is a prime filter $\mathcal{G} \supseteq \mathcal{F}$ of $\mathfrak{A}{\restriction}F_-$ disjoint with $\mathcal{I}$, and so $(a \in \mathcal{G}) \Leftrightarrow ({\sim}^{\mathfrak{A}} a \notin \mathcal{G})$, for all $a \in A$. Then, by Claim 6.2, $h^{\mathfrak{A}}_{\mathcal{G}} \in \hom(\mathcal{A}, \mathcal{DMB}_2)$, as required. □

Corollary 6.7. *Any* $\mathcal{A} \in [\mathsf{TS}]\mathsf{NIFDMBA}$ *is discriminated by* $\mathcal{DMB}_4[(\mathfrak{DMB}_4 + \top)] \times \mathcal{DMB}_2$.

Proof. Consider any $a \notin D^{\mathcal{A}}$, in which case $\mathcal{A}$ is consistent, and so not one-element. Then, first, by Proposition 6.1[6.5], there is some $f \in \hom(\mathcal{A}, \mathcal{DMB}_4[\mathfrak{DMB}_4 + \top])$ such that $f(a) \notin D^{\mathcal{DMB}_4[\mathfrak{DMB}_4+\top]}$. Moreover, by Lemma 6.6, there is some $g \in \hom(\mathcal{A}, \mathcal{DMB}_2) \neq \varnothing$. In this way, $h \triangleq (f \times g) \in \hom(\mathcal{A}, \mathcal{DMB}_4[(\mathcal{DMB}_4 + \top)] \times \mathcal{DMB}_2)$, while we have $\pi_0(h(a)) = f(a) \notin D^{\mathcal{DMB}_4[\mathfrak{DMB}_4+\top]}$, and so we get $h(a) \notin D^{\mathcal{DMB}_4[(\mathfrak{DMB}_4+\top)] \times \mathcal{DMB}_2}$, as required. □

Clearly, [for any $\mathcal{A} \in \mathsf{FDMBA}$,] $([\mathcal{A}\times]\mathcal{DMB}_2) \in \mathsf{NIFDMBA}$, for $\mathfrak{DMB}_2 \not\models ((\exists_1)({\sim} x_0 \approx x_0))$[, and so:

$$(\mathfrak{A} \times \mathfrak{DMB}_2) \not\models ((\exists_1)({\sim} x_0 \approx x_0)), \tag{6.18}$$

for $\pi_1 \in \hom(\mathfrak{A} \times \mathfrak{DMB}_2, \mathfrak{DMB}_2)$ is surjective]. In this way, [as $\mathcal{DMB}_2 = (\mathfrak{DMB}_2 + \top)$], by [(6.14),] Proposition 5.4, Corollaries 6.3 and 6.7, we eventually get:

Proposition 6.8. $\mathsf{NPCFDMBA} \subseteq [\mathsf{TS}]\mathsf{NIFDMBA} = \mathbf{PV}((\mathcal{DMB}_4[(\mathfrak{DMB}_4 + \top)]) \times \mathcal{DMB}_2)$.

Proposition 6.9. $\mathsf{NIFDMBA} \cup \mathsf{TSFDMBA} = \mathbf{PV}(\{\mathcal{DMB}_4 \times \mathcal{DMB}_2, \mathfrak{DMB}_4 + \top\})$ *is the subquasivariety of* FDMBA *relatively axiomatized by the quasi-identity:*

$$\{\sim x_0 \approx x_0, D(x_1)\} \to (x_1 \approx \top). \tag{6.19}$$

Proof. First, consider any filtered De Morgan Boolean algebra $\mathcal{A}$ satisfying (6.19). Assume $\mathcal{A} \notin \mathsf{NIFDMBA}$. Then, there is some $a \in A$ such that $\sim^{\mathfrak{A}} a = a$. Consider any $b \in D^{\mathcal{A}}$. Then, by (6.19)$[x_0/a, x_1/b]$, we get $b = \top^{\mathfrak{A}}$, so $\mathcal{A}$ is truth-singular. Thus, $\mathcal{A} \in (\mathsf{NIFDMBA} \cup \mathsf{TSFDMBA})$. Next, by Propositions 6.5 and 6.8, $(\mathsf{NIFDMBA} \cup \mathsf{TSFDMBA}) \subseteq \mathbf{PV}(\{\mathcal{DMB}_4 \times \mathcal{DMB}_2, \mathfrak{DMB}_4 + \top\})$, Finally, (6.19) is true in both $\mathcal{DMB}_4 \times \mathcal{DMB}_2$, in view of (6.18), and $\mathfrak{DMB}_4 + \top$, in view of (6.14). □

The quasivariety of all non-paraconsistent filtered De Morgan Boolean algebras is denoted by $\mathsf{NPFDMBA}$. It corresponds to the least non-paraconsistent extension of CB_4.

Lemma 6.10. *Let $\mathcal{A}$ be a consistent non-paraconsistent filtered De Morgan Boolean algebra. Then,* $\hom(\mathcal{A}, \mathfrak{DMB}_4 + \top) \neq \varnothing$.

Proof. First, $\mathcal{F} \triangleq D^{\mathcal{A}}$ is a filter of $\mathfrak{A}{\restriction}F_-$. Then, in view of (6.1), (6.2) and (6.3), $\mathcal{I} \triangleq \sim^{\mathfrak{A}}[D^{\mathcal{A}}]$ is an ideal of $\mathfrak{A}{\restriction}F_-$. Moreover, since $\mathcal{A}$ is consistent but not paraconsistent, in view of (6.1), $(\mathcal{F} \cap \mathcal{I}) = \varnothing$. Hence, by the Prime Ideal Theorem, there is a prime filter $\mathcal{G} \supseteq \mathcal{F}$ of $\mathfrak{A}{\restriction}F_-$ disjoint with $\mathcal{I}$, in which case $\mathcal{F} \subseteq \mathcal{H} \triangleq (\sim^{\mathcal{A}})^{-1}[A \setminus \mathcal{G}]$, and so, by Claim 6.2, $h^{\mathfrak{A}}_{\mathcal{G}} \in \hom(\mathcal{A}, \mathfrak{DMB}_4 + \top)$, as required. □

Corollary 6.11. *Any $\mathcal{A} \in \mathsf{NPFDMBA}$ is discriminated by $\mathcal{DMB}_4 \times (\mathfrak{DMB}_4 + \top)$.*

Proof. Consider any $a \notin D^{\mathcal{A}}$, in which case $\mathcal{A}$ is consistent. Then, first, by Proposition 6.1, there is some $f \in \hom(\mathcal{A}, \mathcal{DMB}_4)$ such that $f(a) \notin D^{\mathcal{DMB}_4}$. Moreover, by Lemma 6.10, there is some $g \in \hom(\mathcal{A}, \mathfrak{DMB}_4 + \top) \neq \varnothing$. In this way, $h \triangleq (f \times g) \in \hom(\mathcal{A}, \mathcal{DMB}_4 \times (\mathfrak{DMB}_4 + \top))$, while $\pi_0(h(a)) = f(a) \notin D^{\mathcal{DMB}_4}$, and so $h(a) \notin D^{\mathcal{DMB}_4 \times (\mathfrak{DMB}_4 + \top)}$, as required. □

Clearly, $(\mathcal{DMB}_4 \times (\mathfrak{DMB}_4 + \top)) \in \mathsf{NPFDMBA}$, for $(\mathfrak{DMB}_4 + \top) \not\models ((\exists_1)(D(\sim x_0) \wedge D(x_0)))$, and so $(\mathcal{DMB}_4 \times (\mathfrak{DMB}_4 + \top)) \not\models ((\exists_1)(D(\sim x_0) \wedge D(x_0)))$, for $\pi_1 \in \hom(\mathcal{DMB}_4 \times (\mathfrak{DMB}_4 + \top), \mathfrak{DMB}_4 + \top)$ is surjective. In this way, by Proposition 5.4 and Corollary 6.11, we eventually get:

Proposition 6.12. $\mathsf{NPFDMBA} = \mathbf{PV}(\mathcal{DMB}_4 \times (\mathfrak{DMB}_4 + \top))$.

Theorem 6.13. *Prevarieties of filtered De Morgan Boolean algebras form the eight-element non-chain distributive lattice depicted at Figure 1.*

Proof. We use Corollary 6.3 tacitly.

First, note that (3.5) is not true in $\mathcal{DMB}_4$ under the assignment $[x_0/\mathsf{b}, x_1/\mathsf{f}]$, so $\mathsf{NPFDMBA} \subsetneq \mathsf{FDMBA}$. Next, any filtered De Morgan Boolean algebra satisfying the identity $\top \approx \bot$ is one-element, and so inconsistent, in view of (6.10), so, by (6.1) and (6.5), $\mathsf{TSFBDMA} \subseteq \mathsf{NPFBDMA}$. Moreover, by (6.1), (6.7), (6.8) and (6.9), the following quasi-identity is true in FDMBA:

$$\{D(x_0), D(\sim x_0)\} \to (x_0 \approx \sim x_0), \tag{6.20}$$

so $\mathsf{NIFDMBA} \subseteq \mathsf{NPFDMBA}$, for any one-element filtered De Morgan Boolean algebra is inconsistent, in view of (6.10). Further, (6.19) is not true in $\mathcal{DMB}_4 \times (\mathfrak{DMB}_4 + \top)$ under $[x_0/\langle \mathsf{b}, \mathsf{b}\rangle, x_1/\langle \mathsf{b}, \mathsf{t}\rangle]$. Thus, by Propositions 6.9 and 6.12, $(\mathsf{NIFDMBA} \cup \mathsf{TSFDMBA}) \subsetneq \mathsf{NPFDMBA}$. Furthermore, $\mathcal{DMB}_4 \times \mathcal{DMB}_2$ is not truth-singular, while $\mathfrak{DMB}_4 + \top$ is not non-idempotent. Hence, by Propositions 6.5 and 6.8, $\mathsf{NIFDMBA} \nsubseteq \mathsf{TSFDMBA}$, while $\mathsf{TSFDMBA} \nsubseteq \mathsf{NIFDMBA}$. Finally, (3.6)

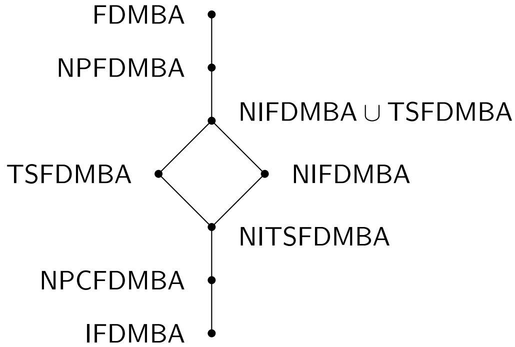

FIGURE 1. The lattice of prevarieties of filtered De Morgan Boolean algebras.

is not true in $\mathfrak{BDM}_4+\top$ under $[x_0/\mathsf{b}]$, and so, by (3.3), in $(\mathfrak{BDM}_4+\top)\times\mathcal{DMB}_2$, for π_0 is a surjective homomorphism from the latter onto the former, in which case, by Proposition 6.8, we get NPCFDMBA $\subsetneq$ TSNIFDMBA. In this way, the eight quasivarieties of filtered De Morgan Boolean algebras involved do form the lattice depicted at Figure 1. It only remains to argue that there is no more prevariety of filtered De Morgan Boolean algebras. For take any prevariety P of filtered De Morgan Boolean algebras. Consider the following eight exhaustive cases:

(1) P $\subseteq$ IFDMBA.
Then, P $=$ IFDMBA.
(2) P $\subseteq$ NPCFDMBA but P $\not\subseteq$ IFDMBA.
Take any consistent $\mathcal{A}\in$ P, in which case it is not one-element, and so, by (6.14) and Proposition 6.8, $\{\langle 0,\bot^{\mathfrak{A}}\rangle,\langle 1,\top^{\mathfrak{A}}\rangle\}$ is an embedding of $\mathcal{DMB}_2$ into $\mathcal{A}$. Then, P $=$ NPCFDMBA.
(3) P $\subseteq$ TSNIFDMBA but P $\not\subseteq$ NPCFDMBA.
We use (6.14) tacitly. Take any paracomplete $\mathcal{A}\in$ P, in which case there is some $a\in A$ such that $b\triangleq(a\vee^{\mathfrak{A}}\sim^{\mathfrak{A}}a)\neq\top^{\mathfrak{A}}$. If $c\triangleq(\sim^{\mathfrak{A}}b\vee^{\mathfrak{A}}\neg^{\mathfrak{A}}b)$ was equal to $\top^{\mathfrak{A}}$, then b would be equal to $\sim^{\mathfrak{A}}b$, so it would be equal to $\top^{\mathfrak{A}}$, in view of (6.15). Likewise, if $d\triangleq\neg^{\mathfrak{A}}\sim^{\mathfrak{A}}b$ was equal to $\top^{\mathfrak{A}}$, then $\sim^{\mathfrak{A}}b$ would be equal to $\bot^{\mathfrak{A}}$, so b would be equal to $\top^{\mathfrak{A}}$, in view of (6.1) and (6.4). In this way, the mapping $e:(2^2\times\Delta_2)\to A$, given by:
$$\begin{aligned}
e(\langle\mathsf{t},\mathsf{t}\rangle) &\triangleq \top^{\mathfrak{A}},\\
e(\langle\mathsf{b},\mathsf{t}\rangle) &\triangleq b,\\
e(\langle\mathsf{n},\mathsf{t}\rangle) &\triangleq d,\\
e(\langle\mathsf{f},\mathsf{t}\rangle) &\triangleq \sim^{\mathfrak{A}}c,\\
e(\langle\mathsf{f},\mathsf{f}\rangle) &\triangleq \bot^{\mathfrak{A}},\\
e(\langle\mathsf{b},\mathsf{f}\rangle) &\triangleq \sim^{\mathfrak{A}}b,\\
e(\langle\mathsf{n},\mathsf{f}\rangle) &\triangleq \sim^{\mathfrak{A}}d,\\
e(\langle\mathsf{t},\mathsf{f}\rangle) &\triangleq c,
\end{aligned}$$
is a strict homomorphism from $(\mathfrak{BDM}_4+\top)\times\mathcal{DMB}_2$ to $\mathcal{A}$, in which case it is injective, in view of Lemma 4.6, and so, by Proposition 6.8, P $=$ TSNIFDMBA.

(4) $\mathsf{P} \subseteq \mathsf{TSFDMBA}$ but $\mathsf{P} \nsubseteq \mathsf{NIFDMBA}$.
We use (6.14) tacitly. Then, $\mathcal{A}$ is not one-element, in which case $\bot^{\mathfrak{A}} \neq \top^{\mathfrak{A}}$, while there is some $a \in A$ such that $\sim^{\mathfrak{A}} a = a$, in which case, by (6.6), $\sim^{\mathfrak{A}} \neg^{\mathfrak{A}} a = \neg^{\mathfrak{A}} a$, while, by (6.4) and (6.5), $a \neq \top^{\mathfrak{A}} \neq \neg^{\mathfrak{A}} a$. In this way, the mapping $e : 2^2 \to A$, given by:

$$\begin{aligned} e\mathsf{b} &\triangleq a, \\ e\mathsf{n} &\triangleq \neg^{\mathfrak{A}} a, \\ e\mathsf{t} &\triangleq \top^{\mathfrak{A}}, \\ e\mathsf{f} &\triangleq \bot^{\mathfrak{A}}, \end{aligned}$$

is a strict homomorphism from $\mathfrak{DMB}_4 + \top$ to $\mathcal{A}$, in which case it is injective, in view of Lemma 4.6, and so, by Proposition 6.5, we eventually get $\mathsf{P} = \mathsf{TSFBDMBA}$.

(5) $\mathsf{P} \nsubseteq \mathsf{NPFDMBA}$.
Then, there is a paraconsistent filtered De Morgan Boolean algebra $\mathcal{A} \in \mathsf{P}$, in which case, by the following claim, $\mathcal{DMB}_4 \in \mathsf{P}$:

Claim 6.14. *Let $\mathcal{A}$ be a paraconsistent filtered De Morgan Boolean algebra. Then, $\mathcal{DMB}_4$ is embeddable into $\mathcal{A}$.*

Proof. In that case, $\mathcal{A}$ is consistent, and so non-one-element, while there is some $a \in A$ such that $\{a, \sim^{\mathfrak{A}} a\} \subseteq D^{\mathcal{A}}$, in which case, by (6.20), $a = \sim^{\mathfrak{A}} a$, and so, by (6.6), $\neg^{\mathfrak{A}} a = \sim^{\mathfrak{A}} \neg^{\mathfrak{A}} a$. Therefore, the mapping $e : 2^2 \to A$, given by:

$$\begin{aligned} e\mathsf{b} &\triangleq a, \\ e\mathsf{n} &\triangleq \neg^{\mathfrak{A}} a, \\ e\mathsf{t} &\triangleq \top^{\mathfrak{A}}, \\ e\mathsf{f} &\triangleq \bot^{\mathfrak{A}}, \end{aligned}$$

is a strict homomorphism from $\mathcal{DMB}_4$ to $\mathcal{A}$, in which case it is injective, in view of Lemma 4.6, as required. □

Thus, by Proposition 6.1, $\mathsf{P} = \mathsf{FDMBA}$.

(6) $\mathsf{P} \subseteq \mathsf{NPFDMBA}$ but $\mathsf{P} \nsubseteq (\mathsf{NIFDMBA} \cup \mathsf{TSFDMBA})$.
Then, by Proposition 6.9, there are some $\mathcal{A} \in \mathsf{P}$, $c \in A$ and $b \in D^{\mathcal{A}}$ such that $\sim^{\mathfrak{A}} c = c$, while $b \neq \top^{\mathfrak{A}}$, in which case $\mathcal{A}$ is not one-element. Put:

$$a \triangleq \begin{cases} c & \text{if } (c \vee^{\mathfrak{A}} b) \neq \top^{\mathfrak{A}}, \\ \neg^{\mathfrak{A}} c & \text{otherwise.} \end{cases}$$

Then, in view of (6.6), both $\sim^{\mathfrak{A}} a = a$ and $\sim^{\mathfrak{A}} \neg^{\mathfrak{A}} a = \neg^{\mathfrak{A}} a$, while $a \vee^{\mathfrak{A}} b \neq \top^{\mathfrak{A}}$, for $b \neq \top^{\mathfrak{A}}$, whereas $(a \vee^{\mathfrak{A}} b) \in D^{\mathcal{A}}$, in view of (6.9), for $b \in D^{\mathcal{A}}$. Moreover, since $\mathcal{A}$ is not one-element, and so consistent, by (3.5) and (6.8), we have $(\{a, \neg^{\mathfrak{A}} a, a \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} b\} \cap D^{\mathcal{A}}) = \varnothing$. Moreover, by (6.7), (6.8) and (6.10), the following quasi-identity:

$$\{D(\neg{\sim} x_0), D(x_0)\} \to (x_0 \approx \top) \tag{6.21}$$

is true in FDMBA, and so in $\mathcal{A}$. Therefore, $(\neg^{\mathfrak{A}} a \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} \sim^{\mathfrak{A}} b) \notin D^{\mathcal{A}}$, for $\top^{\mathfrak{A}} \neq (a \vee^{\mathfrak{A}} b) \in D^{\mathcal{A}}$. And what is more, the quasi-identity:

$$\{x_0 \approx {\sim} x_0, D(x_1), D(\neg x_0 \vee {\sim} x_1)\} \to (\bot \approx \top)$$

is true in $\mathcal{DMB}_4 \times (\mathfrak{DMB}_4 + \top)$, and so in $\mathcal{A}$, in view of Proposition 6.12. Therefore, $(\neg^{\mathfrak{A}} a \vee^{\mathfrak{A}} \sim^{\mathfrak{A}} b) \notin D^{\mathcal{A}}$. In this way, by (6.9) and (6.10), we have the strict homomorphism h from $\mathcal{DMB}_4 \times (\mathfrak{DMB}_4 + \top)$ to $\mathcal{A}$ defined by:

$$\begin{aligned}
h\langle \mathsf{f}, \mathsf{f}\rangle &\triangleq \bot^{\mathfrak{A}},\\
h\langle \mathsf{t}, \mathsf{t}\rangle &\triangleq \top^{\mathfrak{A}},\\
h\langle \mathsf{b}, \mathsf{b}\rangle &\triangleq a,\\
h\langle \mathsf{n}, \mathsf{n}\rangle &\triangleq \neg^{\mathfrak{A}} a,\\
h\langle \mathsf{b}, \mathsf{t}\rangle &\triangleq a \vee^{\mathfrak{A}} b,\\
h\langle \mathsf{b}, \mathsf{f}\rangle &\triangleq a \wedge^{\mathfrak{A}} \sim^{\mathfrak{A}} b,\\
h\langle \mathsf{n}, \mathsf{f}\rangle &\triangleq (\neg^{\mathfrak{A}} a \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}} b),\\
h\langle \mathsf{n}, \mathsf{t}\rangle &\triangleq (\neg^{\mathfrak{A}} a \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} \sim^{\mathfrak{A}} b),\\
h\langle \mathsf{f}, \mathsf{n}\rangle &\triangleq \neg^{\mathfrak{A}} a \wedge^{\mathfrak{A}} b,\\
h\langle \mathsf{t}, \mathsf{n}\rangle &\triangleq (\neg^{\mathfrak{A}} a \vee^{\mathfrak{A}} \sim^{\mathfrak{A}} b),\\
h\langle \mathsf{f}, \mathsf{b}\rangle &\triangleq (a \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}} \sim^{\mathfrak{A}} b),\\
h\langle \mathsf{t}, \mathsf{b}\rangle &\triangleq (a \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} b),\\
h\langle \mathsf{t}, \mathsf{f}\rangle &\triangleq ((a \wedge^{\mathfrak{A}} \sim^{\mathfrak{A}} b) \vee^{\mathfrak{A}} (\neg^{\mathfrak{A}} a \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}} b)),\\
h\langle \mathsf{f}, \mathsf{t}\rangle &\triangleq ((a \vee^{\mathfrak{A}} b) \wedge^{\mathfrak{A}} (\neg^{\mathfrak{A}} a \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} \sim^{\mathfrak{A}} b)),\\
h\langle \mathsf{b}, \mathsf{n}\rangle &\triangleq ((a \vee^{\mathfrak{A}} b) \wedge^{\mathfrak{A}} (\neg^{\mathfrak{A}} a \vee^{\mathfrak{A}} \sim^{\mathfrak{A}} b)),\\
h\langle \mathsf{n}, \mathsf{b}\rangle &\triangleq ((\neg^{\mathfrak{A}} a \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}} b) \vee^{\mathfrak{A}} (a \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}} \sim^{\mathfrak{A}} b)).
\end{aligned}$$

Then, by Lemma 4.6, h is injective. Hence, $(\mathcal{DMB}_4 \times (\mathfrak{DMB}_4 + \top)) \in \mathsf{P}$, so, by Proposition 6.12, $\mathsf{P} = \mathsf{NPFDMBA}$.

(7) $\mathsf{P} \subseteq \mathsf{NIFBDMA}$ but $\mathsf{P} \nsubseteq \mathsf{TSFBDMA}$.
Take any non-truth-singular $\mathcal{A} \in \mathsf{P}$. Then, there is some $a \in D^{\mathcal{A}}$ such that $a \neq \top^{\mathfrak{A}}$, in which case $\mathcal{A}$ is not one-element. Put $b \triangleq (a \vee^{\mathfrak{A}} \sim^{\mathfrak{A}} a)$, in which case $b \in D^{\mathcal{A}}$, in view of (6.9), while $\sim^{\mathfrak{A}} b \leqslant^{\mathfrak{A}} b$, whereas $b \neq \top^{\mathfrak{A}}$, for, otherwise, we would have $\sim^{\mathfrak{A}} a = \neg^{\mathfrak{A}} a$, in which case, by (6.21), we would get $a = \top^{\mathfrak{A}}$. Moreover, as $\mathcal{A}$ is both non-idempotent and non-one-element, $\sim^{\mathfrak{A}} b \neq b$. Therefore, by (6.20), $\sim^{\mathfrak{A}} b \notin D^{\mathcal{A}}$, in which case, by (6.8), $(\sim^{\mathfrak{A}} b \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} b) \notin D^{\mathcal{A}}$, while, by (6.21), $\neg^{\mathfrak{A}} \sim^{\mathfrak{A}} b \notin D^{\mathcal{A}}$. Moreover, since $\mathcal{A}$ is non-one-element, and so consistent, by (6.9), $\bot^{\mathfrak{A}} \notin D^{\mathcal{A}}$, so, by (6.8), $\neg^{\mathfrak{A}} b \notin D^{\mathcal{A}}$. In this way, by (6.9) and (6.10), we have the strict homomorphism h from $\mathcal{DMB}_4 \times \mathcal{DMB}_2$ to $\mathcal{A}$ defined by:

$$\begin{aligned}
h\langle \mathsf{f}, \mathsf{f}\rangle &\triangleq \bot^{\mathfrak{A}},\\
h\langle \mathsf{t}, \mathsf{t}\rangle &\triangleq \top^{\mathfrak{A}},\\
h\langle \mathsf{b}, \mathsf{t}\rangle &\triangleq b,\\
h\langle \mathsf{b}, \mathsf{f}\rangle &\triangleq \sim^{\mathfrak{A}} b,\\
h\langle \mathsf{n}, \mathsf{f}\rangle &\triangleq \neg^{\mathfrak{A}} b,\\
h\langle \mathsf{n}, \mathsf{t}\rangle &\triangleq \neg^{\mathfrak{A}} \sim^{\mathfrak{A}} b,\\
h\langle \mathsf{t}, \mathsf{f}\rangle &\triangleq (\sim^{\mathfrak{A}} b \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} b),\\
h\langle \mathsf{f}, \mathsf{t}\rangle &\triangleq (b \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}} \sim^{\mathfrak{A}} b).
\end{aligned}$$

Then, by Lemma 4.6, h is injective. Therefore, $(\mathcal{DMB}_4 \times \mathcal{DMB}_2) \in \mathsf{P}$, so, by Proposition 6.8, $\mathsf{P} = \mathsf{NIFDMBA}$.

(8) $\mathsf{P} \subseteq (\mathsf{TSFDMBA} \cup \mathsf{NIFDMBA})$ but $\mathsf{P} \nsubseteq \mathsf{NIFDMBA}$.
Then, $(\mathsf{P} \cap \mathsf{NIFDMBA}) \subseteq \mathsf{NIFDMBA}$ but $(\mathsf{P} \cap \mathsf{NIFDMBA}) \nsubseteq \mathsf{TSFDMBA}$, in which case, by Case (7), $(\mathsf{P} \cap \mathsf{NIFDMBA}) = \mathsf{NIFDMBA}$. Likewise, $(\mathsf{P} \cap \mathsf{TSFDMBA}) \subseteq \mathsf{TSFDMBA}$ but $(\mathsf{P} \cap \mathsf{TSFDMBA}) \nsubseteq \mathsf{NITSFDMBA}$, in

which case, by Case (4), $(\mathsf{P} \cap \mathsf{TSFDMBA}) = \mathsf{TSFBDMA}$. In this way, $\mathsf{P} = (\mathsf{TSFDMBA} \cup \mathsf{NIFDMBA})$.

This completes the argument. □

First of all, as, according to Theorem 6.13, $\mathsf{NPFDMBA}$ is the greatest proper subprevariety of FDMBA, we have:

Corollary 6.15. *CB_4 is a maximal paraconsistent logic in the sense that it is paraconsistent and has no proper paraconsistent extension.*

Such a maximal paraconsistency has been proved for certain *three*-valued paraconsistent logics in [10] and [16]. In this way, CB_4 becomes a first *non-artificial* instance of a *four*-valued maximal paraconsistent logic. On the other hand, the *logic of paradox LP* [8] , being initially defined by the three-valued matrix $\mathcal{K}_3 \triangleq \langle \mathfrak{K}_3, 3 \setminus 1\rangle$ (cf. [10]), where, for every $n \in (\omega \setminus 2)$, $\mathfrak{K}_n$ is the chain Kleene lattice over n, in view of (3.2), is equally defined by the n-valued matrix $\mathcal{K}_n \triangleq \langle \mathfrak{K}_n, n \setminus 1\rangle$, where $n \in (\omega \setminus 4)$, for $(\{\langle 0, 0\rangle, \langle n-1, 2\rangle\} \cup ((n \setminus \{0, n-1\}) \times \{1\})) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{K}_n, \mathcal{K}_3)$. Thus, for every $n \in (\omega \setminus 4)$ (in particular, $n = 4$), LP is a maximal paraconsistent n-valued logic as well but is not minimally so, as opposed to CB_4, in view of the following corollary:

Corollary 6.16. *Let $\mathcal{A} \in \mathrm{Mod}(CB_4)$ be paraconsistent. Then, $4 \leqslant |A|$. In particular, any class M of F-matrices defining CB_4 contains a member $\mathcal{B}$ such that $4 \leqslant |B|$, and so CB_4 is minimally 4-valued.*

Proof. Then, by (3.2), Corollary 4.4 and Proposition 6.1, there is some $\theta \in \mathrm{Con}(\mathcal{A})$ such that $(\mathcal{A}/\theta) \in \mathsf{FDMBA}$ is paraconsistent. Hence, by Claim 6.14, $\mathcal{DMB}_4$ is embeddable into $\mathcal{A}$. In this way, we eventually get $4 = |DMB_4| \leqslant |A/\theta| \leqslant |A|$, as required. □

To justify the non-*purely*-algebraic framework involved here, we finally prove:

Corollary 6.17. *There is no quasivariety Q of F-algebras such that CB_4 and $\mathrm{Cn}^{\approx}_{\mathsf{Q}}$ are equivalent in the sense of* [11].

Proof. Clearly, $\Delta_{2^2} \in \hom^{\mathrm{S}}(\mathfrak{DMB}_4 + \top, \mathcal{DMB}_4)$, Hence, in view of (3.3), (6.14), Proposition 6.5 and Theorem 6.13, the logic C of $\mathfrak{DMB}_4 + \top$ is a proper extension of CB_4 satisfying same axioms and being equivalent to the variety DMBA (more precisely, to $\mathrm{Cn}^{\approx}_{\mathsf{DMBA}}$) in the sense of [11]. In this way, Theorem 3.16 of [11] completes the argument. □

6.2. **Resolutional filtered Kleene lattices versus Kleene's three-valued logic.** During this subsection, we deal with the functional signature $F_{\sim} \triangleq (F_+ \cup \{\sim\})$ with unary $\sim$ (weak negation).

Recall that a *De Morgan lattice* (cf. [12], [13]) is any $F_{\sim}$-algebra, whose F_+-reduct is a distributive lattice and that satisfies the identities (6.1) and (6.2), and so (6.3). Then, a *Kleene lattice* (cf. [13], [14]) is any De Morgan lattice satisfying the identity:

$$(x_0 \wedge {\sim} x_0) \lessapprox (x_1 \vee {\sim} x_1). \tag{6.22}$$

Given any $n \in (\omega \setminus 1)$, by $\mathfrak{K}_n$ we denote the Kleene lattice such that $(\mathfrak{K}_n {\restriction} F_+) \triangleq \mathfrak{D}_n$ and ${\sim}^{\mathfrak{K}_n} i \triangleq (n - 1 - i)$, for all $i \in n$. Then, set $\mathcal{K}_n \triangleq \langle \mathfrak{K}_n, \{n-1\} \setminus 1\rangle$. In this way, $\mathcal{K}_{3/2}$ defines Kleene's three-valued logic K_3 [4]/ the classical logic PC, respectively.

A *filtered Kleene lattice* is any $F_{\sim}$-matrix, whose underlying algebra is a Kleene lattice and that satisfies the quasi-identities (6.8) and (6.9). This is said to be *resolutional*, whenever it satisfies the following notorious *Resolution* rule:

$$\{D(x_0 \vee x_1), D(\sim x_0 \vee x_1)\} \to D(x_1), \tag{6.23}$$

in which case it is non-paraconsistent. The quasivariety of all resolutional filtered Kleene lattices is denoted by RFKL. In view of Theorem 5.3 (used tacitly throughout the rest of this subsection) and the following preliminary result, to find extensions of K_3 is to find subprevarieties of RFKL relatively axiomatized without equality.

Proposition 6.18. *Any resolutional filtered Kleene lattice is discriminated by* $\mathcal{K}_3 \in \mathsf{RFKL}$.

Proof. Clearly, $\mathcal{K}_3 \in \mathsf{RFKL}$. Conversely, consider any $\mathcal{A} \in \mathsf{RFKL}$, any $a \in (A \setminus D^{\mathcal{A}})$ and the following complementary cases:

(1) $D^{\mathcal{A}} = \varnothing$.
Then, since $\{\langle 0, 1\rangle\}$ is an embedding of $\mathcal{K}_1$ into $\mathcal{K}_3$, the following claim completes the argument:

Claim 6.19. *Any truth-empty* $\mathcal{A} \in \mathsf{RFKL}$ *is discriminated by* $\mathcal{K}_1$.

Proof. Consider any $a \in A = (A \setminus D^{\mathcal{A}})$. Then, $h \triangleq (A \times \{0\}) \in \hom(\mathcal{A}, \mathcal{K}_1)$ and $a \notin \varnothing = h^{-1}[\varnothing] = h^{-1}[D^{\mathcal{K}_1}]$, as required. □

(2) $D^{\mathcal{A}} \neq \varnothing$,
in which case $D^{\mathcal{A}}$ is a filter of the Kleene lattice $\mathfrak{A}$. Moreover, as it is well-known, in view of (6.22), $\mathcal{I}' \triangleq \{b \in A \mid b \leqslant^{\mathfrak{A}} \sim^{\mathfrak{A}} b\}$ is an ideal of $\mathfrak{A}$, in which case, by (6.23), $\mathcal{I} \triangleq \{b \in A \mid b \leqslant^{\mathfrak{A}} (a \vee^{\mathfrak{A}} c), c \in \mathcal{I}'\}$ is an ideal of $\mathfrak{A}$ disjoint with $D^{\mathcal{A}}$. Hence, by the Prime Ideal Theorem, there is a prime filter $\mathcal{F} \supseteq D^{\mathcal{A}}$ of $\mathfrak{A}$ disjoint with $\mathcal{I}$, and so with $\mathcal{I}' \subseteq \mathcal{I}$. Then, by (6.1), (6.2) and (6.3), $\mathcal{F}' \triangleq (\sim^{\mathfrak{A}})^{-1}[A \setminus \mathcal{F}]$ is a prime filter of $\mathfrak{A}$. Moreover, as $\mathcal{F}$ is disjoint with $\mathcal{I}'$, we have $\mathcal{F} \subseteq \mathcal{F}'$. Therefore, $h \triangleq \chi^{\langle A, \mathcal{F}', \mathcal{F}\rangle} \in \hom(\mathfrak{A}{\restriction}F_+, \mathfrak{D}_3)$, in which case $D^{\mathcal{A}} \subseteq \mathcal{F} = h^{-1}[\{2\}] \not\ni a$, for $\mathcal{F}$ is disjoint with $\mathcal{I} \ni a$. And what is more, using (6.1), it is routine checking that $h(\sim^{\mathfrak{A}} b) = (2 - h(b))$, for all $b \in A$. Thus, $h \in \hom(\mathcal{A}, \mathcal{K}_3)$, as required. □

Note that (3.6) is satisfied in $\mathcal{K}_2$ but is not satisfied in $\mathcal{K}_3$ under $[x_0/1]$. Moreover, $\{1\}$ and $\{0, 2\}$ are the only subsets of 3 forming subalgebras of $\mathfrak{K}_3$, $\mathcal{K}_3{\restriction}\{0, 2\}$ being isomorphic to $\mathcal{K}_2$, while $\mathcal{K}_3{\restriction}\{1\}$ being truth-empty, and so satisfying no axiom. In this way, combining (3.2) and Lemma 5.1 with Propositions 5.6 and 6.18, we first get:

Corollary 6.20. *A resolutional filtered Kleene lattice is non-paracomplete iff it is discriminated by* $\mathcal{K}_2$. *In particular, PC is the only proper consistent axiomatic extension of* K_3 *and is relatively axiomatized by* (3.6).

Next, we have:

Corollary 6.21. *A resolutional filtered Kleene lattice satisfies the rule:*

$$D(x_0) \to D(x_1 \vee \sim x_1) \tag{6.24}$$

iff it is discriminated by $\{\mathcal{K}_2, \mathcal{K}_1\}$. *In particular,* PC_{+0}, *being defined by* $\{\mathcal{K}_2, \mathcal{K}_1\}$, *is the extension of* K_3 *relatively axiomatized by* (6.24).

Proof. With using Remark 3.3. Clearly, both $\mathcal{K}_2$, being non-paracomplete, and $\mathcal{K}_1$, being truth-empty, satisfy (6.24). Then, the "if" part is by Lemma 5.1. Conversely, consider any $\mathcal{A} \in \mathsf{RFKL}$ satisfying (6.24) and the following complementary cases:

(1) $D^{\mathcal{A}} = \varnothing$.
Then, Claim 6.19 completes the argument.

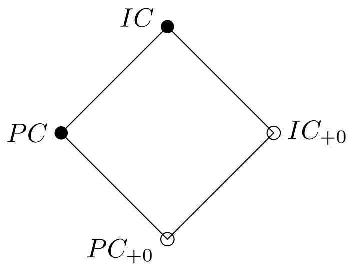

FIGURE 2. Proper [non-seudo-axiomatic] extensions of K_3 [with solely solid circles].

(2) $D^{\mathcal{A}} \neq \varnothing$.
Then, $\mathcal{A}$ is not paracomplete, and so Corollary 6.20 completes the argument. □

By IC we denote the inconsistent $F_\sim$-logic that is defined by $\varnothing$ and is the axiomatic extension of K_3 relatively axiomatized by $D(x_0)$.

Proposition 6.22. *Any $\mathcal{A} \in \mathsf{RFKL}$ satisfies the rule:*

$$D(x_0) \to D(x_1) \tag{6.25}$$

iff it is discriminated by $\mathcal{K}_1$. In particular, IC_{+0}, being defined by $\mathcal{K}_1$, is the extension of K_3 relatively axiomatized by (6.25)*.*

Proof. With using Remark 3.3. Clearly, $\mathcal{K}_1$, being truth-empty, satisfies (6.25). Then, the "if" part is by Lemma 5.1. Conversely, consider any $a \in (A \setminus D^{\mathcal{A}})$, in which case, by (6.25), $\mathcal{A}$ is truth-empty, and so Claim 6.19 completes the argument. □

Theorem 6.23. *Proper [non-pseudo-axiomatic] extensions of K_3 form the four [two]-element diamond[chain] depicted at Figure 2.*

Proof. We use Remark 3.2, Propositions 3.4, 6.18, 6.22 and Corollaries 6.20 and 6.21 tacitly. First, (6.24) is not satisfied in $\mathcal{K}_3$ under $[x_0/2, x_1/1]$. Next, $\mathcal{K}_1$, being truth-empty, does not satisfy any axiom, and so is paracomplete. Further, (6.25) is not satisfied in $\mathcal{K}_2$ under $[x_0/1, x_1/0]$. Finally, $\mathcal{K}_{1/2}$ are both consistent, while any logic satisfying both (6.25) and any axiom (in particular, (3.6)) is inconsistent. Thus, the four logics $PC_{[+0]}$ and $IC_{[+0]}$ are proper extensions of K_3 and form the diamond depicted at Figure 2. After all, consider any extension C' of K_3, in which case it is defined by $\mathsf{S} \triangleq (\mathrm{Mod}(C') \cap \mathsf{RFKL})$, and the following five exhaustive cases:

(1) $IC \subseteq C'$.
Then, $C' = IC$.

(2) $PC \nsubseteq C'$ but $IC_{+0} \subseteq C'$,
in which case $IC \nsubseteq C'$, and so, by the following claim, C' is theorem-less:

Claim 6.24. *Let C'' and C''' be F-logics. Suppose $C'' \nsubseteq C'''$ is non-pseudo-axiomatic and $C''_{+0} \subseteq C'''$. Then, C''' has no theorem.*

Proof. By contradiction and with using Remark 3.2 tacitly. For suppose C''' has a theorem, in which case it is non-pseudo-axiomatic, and so $C'''_{-0} = C'''$. In this way, we get $C'' = (C''_{+0})_{-0} \subseteq C'''_{-0} = C'''$. This contradiction completes the argument. □

Therefore, as $C'_{-0} \subseteq IC$, we have $C' = (C'_{-0})_{+0} \subseteq IC_{+0}$, and so we get $C' = IC_{+0}$.

(3) $IC_{+0} \nsubseteq C'$ but $PC \subseteq C'$,
in which case $IC \nsubseteq C'$, and so there is some consistent non-paracomplete $\mathcal{A} \in \mathsf{S}$. Then, there is some $a \in A$ such that $b \triangleq (a \vee^{\mathfrak{A}} \sim^{\mathfrak{A}} a) \in D^{\mathcal{A}}$, in which case $b \geqslant^{\mathfrak{A}} \sim^{\mathfrak{A}} b \notin D^{\mathcal{A}}$, for $\mathcal{A}$, being resolutional, is not paraconsistent but is consistent, and so $\{\langle 0, \sim^{\mathfrak{A}} b\rangle, \langle 1, b\rangle\}$ is an embedding of $\mathcal{K}_2$ into $\mathcal{A}$. Hence, by (3.2), $\mathcal{K}_2 \in \mathsf{S}$, in which case $C' \subseteq PC$, and so $C' = PC$.

(4) $PC_{+0} \subseteq C'$ but both $PC \nsubseteq C'$ and $IC_{+0} \nsubseteq C'$.
Then, by Claim 6.24, C' has no theorem. Moreover, (6.24), being satisfied in PC_{+0}, is so in C', in which case, by the structurality of C', $(x_0 \vee \sim x_0) \in (\bigcap_{\beta \in \kappa} C'(x_\beta)) = C'_{-0}(\varnothing)$, and so $PC \subseteq C'_{-0}$. On the other hand, $IC = (IC_{+0})_{-0} \nsubseteq C'_{-0}$, so $IC_{+0} \nsubseteq C'_{-0}$. Hence, by Case (3), $C'_{-0} = PC$. In this way, $C' = (C'_{-0})_{+0} = PC_{+0}$.

(5) $PC_{+0} \nsubseteq C'$.
Then, there is some $\mathcal{A} \in \mathsf{S}$ not satisfying (6.24), in which case there are some $a \in D^{\mathcal{A}}$ and some $b \in A$ such that $c \triangleq (b \vee^{\mathfrak{A}} \sim^{\mathfrak{A}} b) \notin D^{\mathcal{A}}$, and so $c \neq d \triangleq (a \vee^{\mathfrak{A}} c) \in D^{\mathcal{A}}$. Clearly, $B \triangleq \{\sim^{\mathfrak{A}} d, \sim^{\mathfrak{A}} c, c, d\}$, forming a chain sublattice of $\mathfrak{A} \upharpoonright F_+$, forms a subalgebra of $\mathfrak{A}$, by (6.1), while $(\{\langle (\sim^{\mathfrak{A}})^i d, 2 \cdot (1-i)\rangle \mid i \in 2\} \cup \{\langle (\sim^{\mathfrak{A}})^i c, 1\rangle \mid i \in 2\})$ is a strict surjective homomorphism from $(\mathcal{A} \upharpoonright B) = \langle \mathfrak{A} \upharpoonright B, \{d\}\rangle$ onto $\mathcal{K}_3$, in which case $\mathcal{K}_3 \in \mathsf{S}$, by (3.2), and so $C' = K_3$. □

It is remarkable that K_3 is not covered by Section 4, for, otherwise, by Theorem 4.3, ε would be empty for K_3 to satisfy $\tau_\varepsilon(3.7)$, since K_3 has no theorem, in which case $\tau_\varepsilon(3.11)$ would be equal to (6.25) that is not satisfied in $\mathcal{K}_3$ under $[x_0/2, x_1/0]$. Likewise, since $1 \notin \{2\} = D^{\mathcal{K}_3}$ forms a subalgebra of $\mathfrak{K}_3$, $\mathcal{K}_3$ is not covered by the *purely*-algebraic approach of [14]. Thus, meanwhile, Section 5 remains a unique generic approach applicable to K_3 that highlights its value.

7. Conclusions

As a matter of fact, in view of Theorem 4.3(ii)⇔(iii), the general approach developed here is equally applicable to to arbitrary *equivalential* equality-free universal Horn theories in the sense of [18] (in particular, to all sequent calculi constructed in [17]; cf. Proposition 10 of [18]). However, this issue deserves a particular emphasis and, for this reason, is going to be eventually presented elsewhere.

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