



Note for the Prime Gaps

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Abstract: A prime gap is the difference between two successive prime numbers. The n th prime gap, denoted g_n is the difference between the $(n + 1)$ st and the n th prime numbers, i.e. $g_n = p_{n+1} - p_n$. There isn't a verified solution to Andrica's conjecture yet. The conjecture itself deals with the difference between the square roots of consecutive prime numbers. While mathematicians have showed it true for a vast number of primes, a general solution remains elusive. The Andrica's conjecture is equivalent to say that $g_n < 2 \cdot \sqrt{p_n} + 1$ holds for all n . In this note, using only simple arguments, we prove that the Andrica's conjecture is true.

Keywords: prime gaps; prime numbers; square roots; inequality

MSC: 11A41; 11A25

1. Introduction

Prime numbers, the building blocks of integers, have fascinated mathematicians for centuries. Their irregular distribution, with gaps of seemingly random size between them, is a source of ongoing intrigue. Andrica's conjecture tackles this very irregularity, proposing a relationship between the sizes of these prime gaps and the primes themselves. Andrica's conjecture (named after Dorin Andrica) is a conjecture regarding the gaps between prime numbers [1]. The conjecture states that the inequality

$$\sqrt{p_{n+1}} - \sqrt{p_n} < 1$$

holds for all n , where p_n is the n th prime number. If $g_n = p_{n+1} - p_n$ denotes the n th prime gap, then Andrica's conjecture can also be rewritten as

$$g_n < 2 \cdot \sqrt{p_n} + 1.$$

Imran Ghory has used data on the largest prime gaps to confirm the conjecture for n up to $1.3002 \cdot 10^{16}$ [2].

Legendre's conjecture, proposed by Adrien-Marie Legendre, states that there is a prime number between n^2 and $(n + 1)^2$ for every positive integer n [2]. The conjecture is one of Landau's problems (1912) on prime numbers. If Legendre's conjecture is true, the gap between any prime p and the next largest prime would be $O(\sqrt{p})$, as expressed in big O notation. Oppermann's conjecture is another unsolved problem in mathematics on the distribution of prime numbers [2]. It is closely related to but stronger than Legendre's conjecture and Andrica's conjecture. It is named after Danish mathematician Ludvig Oppermann, who announced it in an unpublished lecture in March 1877 [3]. If the conjecture is true, then the gap size would be on the order of $g_n < \sqrt{p_n}$.

This seemingly simple statement has profound implications for our understanding of prime number distribution. Unfortunately, despite its apparent elegance, Andrica's conjecture remains unproven. Mathematicians have extensively verified it for a tremendous number of primes, but a universal solution proving its truth for all primes continues to be elusive. This lack of proof doesn't diminish the significance of the conjecture. It serves as a guidepost, nudging mathematicians towards a deeper understanding of prime number distribution. The quest to solve Andrica's conjecture pushes the boundaries of our knowledge and holds the potential to unlock new insights into the enigmatic world of primes.

Whether the Andrica's conjecture is true or not still remains as an open question. By employing the divergence of the infinite sum of the reciprocals of all prime numbers and delving into the properties of natural logarithm, we will demonstrate a crucial contradiction under the assumption that there exists a prime gap g_{n_0} such that $g_{n_0} \geq 2 \cdot \sqrt{p_{n_0}} + 1$. This contradiction will definitively prove the Andrica's conjecture. In this way, we provide a new step forward that could help us to find a better upper bounds for prime gaps.

2. Results

This is a main insight.

Proposition 1. *Let us denote by p_n the n th prime. Then, for $p_n \geq 127$, we have [4]:*

$$\frac{p_{n+1}}{p_n} \leq \frac{149}{139}.$$

This is the main theorem.

Theorem 1. *The Andrica's conjecture is true.*

Proof. There is not any natural number n' such that

$$\sqrt{p_{n'+1}} - \sqrt{p_{n'}} = 1$$

since this implies that $g_{n'} = 2 \cdot \sqrt{p_{n'}} + 1$. For every n , g_n is a natural number and $2 \cdot \sqrt{p_n} + 1$ is always irrational. In fact, all square roots of natural numbers, other than of perfect squares, are irrational [5]. So, there exists a natural number $n_0 \geq 1.3002 \cdot 10^{16}$ such that

$$\sqrt{p_{n_0+1}} - \sqrt{p_{n_0}} > 1$$

under the assumption that the Andrica's conjecture is false. That is equivalent to

$$\frac{\sqrt{p_{n_0+1}} - \sqrt{p_{n_0}}}{\sqrt{p_{n_0} \cdot p_{n_0+1}}} > \frac{1}{\sqrt{p_{n_0} \cdot p_{n_0+1}}}$$

where

$$\frac{1}{\sqrt{p_{n_0}}} - \frac{1}{\sqrt{p_{n_0+1}}} = \frac{\sqrt{p_{n_0+1}} - \sqrt{p_{n_0}}}{\sqrt{p_{n_0} \cdot p_{n_0+1}}}.$$

Putting all together yields

$$\frac{1}{\sqrt{p_{n_0}}} - \frac{1}{\sqrt{p_{n_0+1}}} > \frac{1}{\sqrt{p_{n_0} \cdot p_{n_0+1}}}$$

which is

$$\sqrt{\frac{p_{n_0+1}}{p_{n_0}}} - \sqrt{\frac{p_{n_0}}{p_{n_0+1}}} > 1.$$

That would be

$$\sqrt{\frac{p_{n_0+1}}{p_{n_0}}} > 1 + \sqrt{\frac{p_{n_0}}{p_{n_0+1}}}$$

that is

$$\sqrt{\frac{149}{139}} \geq \sqrt{\frac{p_{n_0+1}}{p_{n_0}}} > 1 + \sqrt{\frac{p_{n_0}}{p_{n_0+1}}} \geq 1 + \sqrt{\frac{139}{149}}.$$

by Proposition 1. However, we check that

$$1 > 0.07 \approx \sqrt{\frac{149}{139}} - \sqrt{\frac{139}{149}} > 1.$$

Since this implies that $1 > 1$ by transitivity, we reach a contradiction. Consequently, by reductio ad absurdum, we conclude that the Andrica's conjecture is true. \square

3. Conclusion

Further exploration about large prime gaps may involve:

- Developing new techniques in analytic number theory, the branch of mathematics that studies the distribution of prime numbers.
- Leveraging advanced computational methods to test the conjecture for even larger prime ranges and potentially uncover patterns.
- Investigating connections between Andrica's conjecture and other unsolved problems in prime number theory such as the Legendre's conjecture and Oppermann's conjecture.

To sum up, this solution for the Andrica's conjecture could be a significant advancement in our understanding of prime number distribution.

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Short Biography of Authors



Frank Vega is essentially a Back-End Programmer and Mathematical Hobbyist who graduated in Computer Science in 2007. In May 2022, The Ramanujan Journal accepted his mathematical article about the Riemann hypothesis. The article "Robin's criterion on divisibility" makes several significant contributions to the field of number theory. It provides a proof of the Robin inequality for a large class of integers, and it suggests new directions for research in the area of analytic number theory.

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