Approach to Partial Eigenvalue Assignment Using Sylvester Equation in System-Second Order

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***Abstract*—The paper considers an approach to partial eigenvalue assignment in second-order descriptor systems via proportional plus derivative plus output feedback controller. It is shown that the problem is closely related to a so-called second-order Sylvester matrix equation. This study presents an approach to partial the eigenstructure assignment for the descriptor system where an algorithm is presented for calculated the output feedback matrix by equation de Sylvester. Two complete parametric methods for the proposed approach to partial eigenstructure assignment problems are presented. Both methods give simple complete parametric expressions for the feedback gains and the closed-loop eigenvector matrices. The first one mainly depends on a series of singular value decompositions. The second one utilizes the system's factorization and allows the closed-loop eigenvalues to be set undetermined and sought via specific optimization procedures. The theorems are presented using the Sylvester equations. Two algorithms are implemented using the Sylvester equation, and examples are presented with their conclusions.**

Keywords: Descriptor System, Second-order system, Sylvester equation.

I. Introduction

Second-order linear systems capture the dynamic behavior of many natural phenomena, and have found applications in many fields, such as vibration and structural analysis, spacecraft control and robotics control and, hence, have attracted much attention , [1], [2], [3], [4], [5].

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The solution of a generalized Sylvester equation associated to a linear descriptor system and subject to some rank and regional pole-placement constraints. Under the hypothesis of strong-detectability of the descriptor system, a sequence of coordinate transformations is proposed such that the considered problem can be solved through a Sylvester equation associated to a detectable reduced-order normal system [6].

The control of the following second-order descriptor dynamical linear system:

$$\begin{aligned} M\ddot{x} + D\dot{x} + Nx &= Bu \\ y_0 &= C_0 x \\ y_1 &= C_1 \dot{x} \\ y_2 &= C_2 \dot{x} \end{aligned} \tag{1}$$

where $x \in R^n$ and $u \in R^m$ are the state vector and the control vector, respectively, and $M, D, N \in R^{n\times n}$, and $B \in R^{n\times m}$, $C_0, C_1, C_2 \in R^{p\times n}$ are the system coefficient matrices. In certain applications, the matrices M, D, and N being called the mass matrix, the structural damping matrix and the stiffness matrix, respectively. These coefficient matrices satisfy the following assumptions.

Assumption 1.1: A_1: rank(M) = q, $0 < q \leq n$, rank(B) = m, and $rank(C_0) = rank(C_1) = rank(C_2) = p$.

Concerning the control of the second-order linear system (1) most of the results are focused on stabilization (see, for example, pole assignment [1], [2], and partial pole assignment [3], [4]. Furthermore, many theoretical results for second-order systems have been developed via the corresponding extended first-order descriptor state space model.

Eigenstructure assignment is a very important problem in linear control systems design, the solution of a generalized Sylvester equation associated with a linear descriptor system where the proposed results are motivated from its use for designing minimal-order observers and for computing output feedback control laws by a particular technique in [6], [7], [8]. The design degree of freedom provided by eigenstructure assignment is utilized to minimize the condition number of the closed-loop system. The article present in section II the problem formulation, in section III present the solution to problem SSE, in section IV present the solution to problem ESA, in section V present the eigenstructure assignment and the section VI the conclusions.

II. PROBLEM FORMULATION

For the second-order descriptor dynamical system (1), by choosing the following control law:

$$u(t) = -F_0 y_0(t) - F_1 y_1(t) - F_2 y_2(t) \quad (2)$$

with $F_0, F_1, F_2 \in R^{p\times n}$. It obtain the closed-loop system as follows:

$$(M + BF_2C_2)\ddot{x} + (D + BF_1C_1)\dot{x} + (N + BF_0C_0)x = 0 \quad (3)$$

System (3) can be written in the first-order state-space form

$$E_c\dot{z} = A_c z; \quad (4)$$

with

$$\begin{aligned} E_c &= \begin{bmatrix} I_n & 0 \\ 0 & (M + BF_2C_2) \end{bmatrix} \; and \\ A_c &= \begin{bmatrix} 0 & I \\ -(N + BF_0C_0) & -(D + BF_1C_1) \end{bmatrix} \end{aligned} \quad (5)$$

, We here require the closed-loop matrix pair $(E_c; A_c)$ to be nondefective, that is, the Jordan form of the matrix pair $(E_c; A_c)$ possesses a diagonal form. Further, following the pole assignment theory for first-order descriptor linear systems, under the controllability of system (1), $n+q$ finite eigenvalues can be assigned to the closed-loop system (4), (5). Therefore, the desired Jordan form of the matrix pair $(E_c; A_c)$ takes the form

$$\Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_{n+q}) \quad (6)$$

where $\lambda_i, i = 1, 2, \ldots n+q$, are clearly the eigenvalues of the matrix pair $(E_c; A_c)$. Based on lemma in [14] we have the following lemma (2.1)

Lemma 2.1: Let E_c, A_c be given by (4), (5) , and by (6) Then, the following hold. 1) There exist matrices $V_1, V_2 \in R^{n\times n+q}$ satisfying

$$A_c \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = E_c \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \Lambda \quad (7)$$

if and only if

$$\begin{aligned} &(M + BF_2C_2)V_1\Lambda^2 + (D + BF_1C_1)V_1\Lambda + \\ &\qquad (N + BF_0C_0)V_1 = 0 \end{aligned} \quad (8)$$

and

$$V_2 = V_1\Lambda \quad (9)$$

2) There exist matrices V_∞, $V_\infty^{'} \in R^{n\times q}$ satisfying

$$E_c \begin{bmatrix} V_\infty^{'} \\ V_\infty \end{bmatrix} = 0, and, rank(\begin{bmatrix} V_\infty^{'} \\ V_\infty \end{bmatrix}) = n - q \quad (10)$$

if and only if $V_\infty^{'} = 0$ and consider the matrix $M_1 = (M + BF_2C_2)$

$$M_1V_\infty = 0 \; rank(V_\infty) = n - q \quad (11)$$

III. SOLUTION TO PROBLEM SSE

Consider the following linear time-invariant descriptor system

$$E\dot{x}(t) = Ax(t) + Bu(t); y(t) = Cx(t) \quad (12)$$

where $E, A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{p\times n}$ and rank$E = q \le n$. Assume that the matrix pencil$(\lambda E - A)$ is nonsingular, i.e., rank$(\lambda E - A) = n$.

Proposition 3.1: [9] The system (12) is controllable at infinity if and only if

$$rank[E, B] = rank[E, A, B] \quad (13)$$

Proposition 3.2: [9] The system 12 is C-controllable if and only if condition (13) is satisfied together with

$$rank[\lambda E - A, B] = rank[E, A, B], \forall\lambda \in C. \quad (14)$$

Proposition 3.3: [9] The system (12) is I-controllable if and only if

$$\begin{bmatrix} E & 0 & 0 \\ A & E & B \end{bmatrix} = rank[E, A, B] + rankE. \quad (15)$$

Proposition 3.4: [9] The system (12) is S-controllable if and only if both the conditions (14) and (15) are satisfied.

Letting

$$\begin{aligned} &W = F_1C_1V\lambda + F_0C_0V = \\ &\begin{bmatrix} F_0C_0 & F_1C_1 \end{bmatrix} \begin{bmatrix} V \\ V\Lambda \end{bmatrix} \end{aligned} \quad (16)$$

then (8) becomes

$$\begin{aligned} &(M + BF_2C_2)V\Lambda^2 + DV\Lambda + NV = \\ &\qquad BF_1C_1V\Lambda + (BF_0C_0)V \end{aligned} \quad (17)$$

$$(M + BF_2C_2)V\Lambda^2 + DV\Lambda + NV = BW \quad (18)$$

Denote

$$V = \begin{bmatrix} v_1 & v_2 & \cdots & v_q \end{bmatrix} \quad (19)$$

$$W = \begin{bmatrix} w_1 & w_2 & \cdots & w_q \end{bmatrix} \quad (20)$$

We can convert the second-order Sylvester matrix equation (18) into the following column form:

$$(\lambda_i^2(M + BF_2C_2) + \lambda_i D + N)v_i = Bw_i \; i = 1, 2, \cdots, q. \quad (21)$$

The equations in (21), can be further written in the following form:

$$\Pi_i \begin{bmatrix} v_i \\ w_i \end{bmatrix} = 0 \; i = 1, 2, \cdots, q. \quad (22)$$

where

$$\Pi_i = \begin{bmatrix} \lambda_i^2 M + \lambda_i D + N & -B \end{bmatrix}, \; i = 1, 2, \cdots, q. \quad (23)$$

This states that

$$\begin{bmatrix} v_i \\ w_i \end{bmatrix} \in Ker\,\Pi_i \; i=1,2,\cdots,q. \tag{24}$$

The following algorithm produces two sets of constant matrices T_i and U_i, $i = 1,2,\cdots,q$, to be used in the representation of the solution to the matrix equation (18), the following simple procedure can also be used and is based in [14].

Algorithm new P_1

Solving T_i and U_i, $i = 1,2,\cdots,q$,

Step 1) Through applying SVD to the matrix Π_i, $i = 1,2,\cdots,q$, obtain two sets of matrices $P_i \in C^{n\times n}$ and $Q_i \in C^{n+m\times n+m}$, $i = 1,2,\cdots,q$, satisfying

$$P_i\Pi_iQ_i = \begin{bmatrix} diag(\sigma_1,\sigma_2,\cdots,\sigma_q) & 0 \\ 0 & 0 \end{bmatrix},\; i=1,2,\cdots,q. \tag{25}$$

where, $\sigma_i > 0$, $i = 1,2,\cdots,q$, are the singular values of Π_i

$$q = rank\begin{bmatrix} \lambda^2(M+BF_2C_2)+\lambda D+N & B \end{bmatrix}, \tag{26}$$

Step 2) Obtain the matrices $T_i \in R^{n\times n+m-q}$ and $U_i \in R^{m\times n+m-q}$, $i = 1,2,\cdots,q$, by partitioning the matrix Q_i as follows

$$Q_i = \begin{bmatrix} * & T_i \\ * & U_i \end{bmatrix},\; i=1,2,\cdots,q. \tag{27}$$

As a result of (25) and (27), the matrices $T_i \in R^{n\times n+m-q}$ and $U_i \in R^{m\times n+m-q}$, $i = 1,2,\cdots,q$, obtained through Algorithm P_1 satisfy

$$\Pi_i\begin{bmatrix} T_i \\ U_i \end{bmatrix} = 0,$$
$$rank\begin{bmatrix} T_i \\ U_i \end{bmatrix} = n+m-q \; i=1,2,\cdots,q. \tag{28}$$

Therefore, the columns of $\begin{bmatrix} T_i \\ U_i \end{bmatrix}$ form a set of basis for kerΠ_i. i. The previous deduction clearly yields the following result, where the theorem 3.1 is based in [14].

Theorem 3.1: Let (1) S-controllability with the condition (26), and the conditions (14) and (15) are satisfied.

$T_i \in R^{n\times n+m-q}$ and $U_i \in R^{m\times n+m-q}$, $i = 1,2,\cdots,q$, be obtained via Algorithm new P_1. Then, all the matrices V and W satisfying the second-order Sylvester matrix equation (18) can be parameterized by columns as follows

$$\begin{bmatrix} v_i \\ w_i \end{bmatrix} = \begin{bmatrix} T_i \\ U_i \end{bmatrix} f_i \; i=1,2,\cdots,q. \tag{29}$$

$f_i \in R^{n+m-q}$, $i = 1,2,\cdots,q$, are a set of arbitrary parameter vectors.

Regarding the S-controllability of (1), we have the following basic result which is a general extension of the well-known PHB criterion [10].

Lemma 3.1: The second-order dynamical system (1), is S-controllable if and only if

$$rank\begin{bmatrix} \lambda_i^2(M+BF_2C_2)+\lambda_iD+N & B \end{bmatrix} = n,\; \forall\lambda\in C. \tag{30}$$

The solution for this case depends on a pair of polynomial matrices $T(\lambda) \in R^{n\times n+m-q}$ and $U(\lambda) \in R^{m\times n+m-q}$, satisfying

$$\begin{bmatrix} \lambda_i^2(M+BF_2C_2)+\lambda_iD+N \end{bmatrix} T(\lambda) = BU(\lambda). \tag{31}$$

In the case where (1), is regular, that is, det$(\lambda_i^2(M+BF_2C_2)+\lambda_iD+N)$ is not identically zero, the aforementioned equation can be written as

$$\begin{bmatrix} \lambda_i^2(M+BF_2C_2)+\lambda_iD+N \end{bmatrix}^{-1} B = T(\lambda)U^{-1}(\lambda). \tag{32}$$

which can be viewed as the right factorization of the following transfer function

$$G(\lambda) = \begin{bmatrix} \lambda_i^2(M+BF_2C_2)+\lambda_iD+N \end{bmatrix}^{-1} B.$$

For simplicity, we also call (31) the factorization of (1), where the theorem 3.2 is based in [14]

Theorem 3.2: Let (1), be S-controllable, and $T(\lambda) \in R^{n\times n+m-q}$ and $U(\lambda) \in R^{m\times n+m-q}$ satisfy the factorization (31). Then, the following hold.

1) The matrices V and W given by (19), (20)

$$\begin{bmatrix} v_i \\ w_i \end{bmatrix} = \begin{bmatrix} T(\lambda_i) \\ U(\lambda_i) \end{bmatrix} f_i \; i=1,2,\cdots,q. \tag{33}$$

satisfy the second-order Sylvester matrix equation (18) for $f_i \in C^{n+m-q}$, $i = 1,2,\cdots,q$,

2) When

$$rank\begin{bmatrix} T(\lambda_i) \\ U(\lambda_i) \end{bmatrix} = n+m-q \; i=1,2,\cdots,q. \tag{34}$$

The factorization (31) performs a fundamental role in the solution (33). When (1), is regular and $\lambda_i, i = 1,2,\cdots,q$, are chosen to be different from the zeros of det$(\lambda^2M+\lambda D+N)$,$M_1 = (M+BF_2C_2)$ we can take

$$\begin{cases} T(\lambda_i) = Adj(\lambda^2M_1+\lambda D+N)B \\ U(\lambda_i) = det(\lambda^2M_1+\lambda D+N)I_m \end{cases}$$

For general numerical algorithms solving such factorizations, one can refer to [12], [13]. The following simple procedure can also be used and is based in [14].

Algorithm new P_2 (new coprime factorization)

Step 1) Under the S-controllability of system (1), find a pair of unimodular matrices $P(\lambda)$ and $Q(\lambda)$, of appropriate dimensions, satisfying

$$P(\lambda)\begin{bmatrix} \lambda^2M_1+\lambda D+N & B \end{bmatrix} Q(\lambda) = \begin{bmatrix} I_n & 0 \end{bmatrix}$$

Step 2) Obtain the pair of polynomial matrices $T(\lambda) \in R^{n\times m}$ and $U(\lambda) \in R^{m\times m}$ by partitioning the unimodular matrix $Q(\lambda)$

$$Q(\lambda) = \begin{bmatrix} * & T(\lambda) \\ * & U(\lambda) \end{bmatrix}$$

It is worth pointingout that the pair of polynomial matrices $T(\lambda) \in R^{n\times m}$ and $U(\lambda) \in R^{m\times m}$
satisfying the right factorization (31) obtained from the Algorithm new P_2 are coprime

$$rank \begin{bmatrix} T(\lambda) \\ U(\lambda) \end{bmatrix} = m \ \forall \lambda \in C \tag{35}$$

IV. SOLUTION TO PROBLEM ESA

Following from the results in Section III, we can obtain the following two theorems regarding the solution to Problem ESA. Where the theorems 4.1 and 4.2 is based in [14].

Theorem 4.1: Let n_i $i = 1, 2, \cdots, n + q$ be given by , and the condition (26), and

$T_i \in R^{n\times n+m-q}$ and $U_i \in R^{m\times n+m-q}$, $i = 1, 2, \cdots, n+q$, be given by Algorithm P_1. Then, the following hold.

1) Problem ESA has solutions if and only if there exist a group of parameters $f_i \in C^{n+m-q}$ $i = 1, 2, \cdots, n + q$, satisfying the following constraints.

Constraint Co_1 : $f_i = \bar{f}_j$ if $\lambda_i = \bar{\lambda}_j$.

Constraint $C2_a$: det$V_c \neq 0$, with

$$V_c = \begin{bmatrix} T_1 f_1 & T_2 f_2 \cdots & T_{n+q} f_{n+q} & 0 \\ \lambda_1 T_1 f_1 & \lambda_2 T_2 f_2 \cdots & \lambda_{n+q} T_{n+q} f_{n+q} & V_\infty \end{bmatrix} \tag{36}$$

2) When this condition is met, all the solutions to problem ESA are given by

$$V = \begin{bmatrix} T_1 f_1 & T_2 f_2 & \cdots & T_{n+q} f_{n+q} \end{bmatrix} \tag{37}$$

and

$$\begin{bmatrix} F_0 C_0 & F_1 C_1 \end{bmatrix} = \begin{bmatrix} U_1 f_1 & U_2 f_2 & \cdots & U_{n+q} f_{n+q} & W_\infty \end{bmatrix} V_c^{-1} \tag{38}$$

where $f_i \in R^{n+m-q}$, $i = 1, 2, \cdots, n + q$, are arbitrary parameter vectors satisfying Constraints Co_1 and C_{2a} and $W_\infty \in C^{m\times(n-q}$ is an arbitrary parameter matrix.

Theorem 4.2:

Let (1), be S-controllable, and $T(\lambda) \in R^{n\times m}$ and $U(\lambda) \in R^{m\times m}$ be a pair of polynomial matrices satisfying the right factorization (31) and condition (34). Then

1) Problem ESA has solutions if and only if there exist a group of parameters $f_i \in C^m$ $i = 1, 2, \cdots, n + q$, satisfying Co_1 and Constraint C_{2b}: det$V_{cb} \neq 0$ with

$$V_{cb} = \begin{bmatrix} T(\lambda_1) f_1 & T(\lambda_2) f_2 \cdots & T(\lambda_{n+q}) f_{n+q} & 0 \\ \lambda_1 T(\lambda_1) f_1 & \lambda_2 T(\lambda_2) f_2 \cdots & \lambda_{n+q} T(\lambda_{n+q}) f_{n+q} & V_\infty \end{bmatrix} \tag{39}$$

2) When this condition is met, all the solutions to problem ESA are given by

$$V = \begin{bmatrix} T(\lambda_1) f_1 & T(\lambda_2) f_2 \cdots & T(\lambda_{n+q}) f_{n+q} \end{bmatrix} \tag{40}$$

and

$$\begin{bmatrix} F_0 C_0 & F_1 C_1 \end{bmatrix} = \begin{bmatrix} U(\lambda_1) f_1 & U(\lambda_2) f_2 & \cdots & U(\lambda_{n+q}) f_{n+q} & W_\infty \end{bmatrix} V_{cb}^{-1} \tag{41}$$

where $f_i \in R^m$, $i = 1, 2, \cdots, n + q$, are arbitrary parameter vectors satisfying Constraints Co_1 and C_{2b} and $W_\infty \in C^{m\times(n-q)}$ is an arbitrary parameter matrix.

V. PROBLEM EIGENSTRUCTURE ASSIGNMENT:APPROACH

Strong stability can be interpreted in terms of the closed-loop system's self-structure: 1. Asymptotic stability is equivalent to that all finite poles are inside the left semi plane of the complex plane. 2.The absence of impulsive modes is equivalent to having q finite poles in a closed-loop. 3. Regularity is guaranteed if the system is free from impulsive modes. Based on this interpretation, necessary and sufficient conditions for the existence of S-stabilizable output feedback are established from a set of generalized coupled Sylvester equations, [6], [7], [8].

A. Eigenstructure by equation Sylvester

The system (3) can be written in the first-order state-space form (4) and (5). Thus for obtained the Output feedback K $\sigma(E_d, A_d + B_d K C_d) \in C^-$, is used the Silvester equation in [8], [7].

Consider the following linear time-invariant descriptor system in [8], [7].

$$\begin{aligned} E_d \dot{x}(t) &= A_d x(t) + B_d u(t) \\ y(t) &= C_d x(t) \end{aligned} \tag{42}$$

The Sylvester equations in [8], [7].

$$A_d V_d - E_d V_d H_V = -B_d W_d, \qquad \sigma(H_V) \in \mathcal{C}^- \tag{43}$$

$$P_d A_d - H_P P_d E_d = -U_d C_d, \qquad \sigma(H_P) \in \mathcal{C}^- \tag{44}$$

The theorem 5.1 is based in [14], [8].

Theorem 5.1: Let (1), be S-controllable, and $V_d \in R^{2n\times p}$ and $W_d \in R^{m\times p}$ satisfy the equation (43). Then, the following hold.

1) The matrices V_d and W_d given by (45),

$$\begin{bmatrix} A_d - \lambda_i E_d & B_d \end{bmatrix} \begin{bmatrix} v_i \\ w_i \end{bmatrix} = \ i = 1, 2, \cdots, q. \tag{45}$$

satisfy Sylvester matrix equation (43) for, $i = 1, 2, \cdots, q$,

2) When

$$rank(\begin{bmatrix} V_i \\ W_i \end{bmatrix}) = m \ i = 1, 2, \cdots, q. \tag{46}$$

hold, (45) gives all the solutions.

Proof Based in [7], [8].

The following basic procedure is proposed to calculate the feedback controller that stabilizes the closed loop system, when $m + p > q$. Closed loop eigenvalues are positioned arbitrarily close to the set ; they are symmetric sets of pre-specified eigenvalues. The (E_d, A_d, B_d, C_d) system is considered to be strongly controllable and strongly detectable.

Algorithm $S1$

Step 1: Choose an array $H_P \in \Re^{q-p\times q-p}$ such that $\sigma(H_P) = \Lambda_P \in \mathcal{C}^-$ and Sylvester's equation (44) is solved to find a matrix

$P_d \in \Re^{n+q-p\times n+q}$ such that

$$rank\left(\left[\begin{array}{c} P_d E_d \\ C_d \end{array}\right]\right) = q \tag{47}$$

Step 2: Sylvester's equation (43) is solved, for some $H_V \in \Re^{p\times p}$ matrix such that $\sigma(H_V) = \Lambda_V \in \mathcal{C}^-$ taking into account that the matrix V_d taking into account that $rank\,(E_d V_d) = p$ (or $Ker\,(P_d E_d) = Ker\,(E_d) \oplus Im\,(V_d)$, where $\oplus$ represents the direct sum).

Step 3: By construction, the matrix V_d must verify that rank $(C_d V_d) = p$ and the matrix K can be calculated by:

$$K = W_d (C_d V_d)^{-1} \tag{48}$$

$\circ$

In step 1, under the condition that the system is strongly observable (detectable). As will be seen later, degrees of freedom existing in the choice of V_d, can also be used to guarantee obtaining K such that $K C_d V_d = W_d$ in [8].

B. Example

Consider a simple linear dynamical system (1) in [14]

$$M = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$D = \left[\begin{array}{ccc} 2.5 & -0.5 & 0 \\ -0.5 & 2.5 & -2 \\ 0 & -2 & 2 \end{array}\right]$$

$$N = \left[\begin{array}{ccc} 10 & -5 & 0 \\ -5 & 25 & -20 \\ 0 & -20 & 20 \end{array}\right] B = \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{array}\right]$$

Considered the system in (4), (5) $E_d = \left[\begin{array}{cc} I_n & 0 \\ 0 & M \end{array}\right]$;

$$A_d = \left[\begin{array}{cc} 0 & I_n \\ -N & -D \end{array}\right];$$

$$B_d = \left[\begin{array}{c} 0 \\ B \end{array}\right] C_d = \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right]$$

$$C_0 = \left[\begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right]; C_1 = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 1 \end{array}\right]$$

$$C_2 = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right].$$

Considered the system in the equations (7), (8)

$$E_d = \left[\begin{array}{cc} I_n & 0 \\ 0 & M \end{array}\right]; A_d = \left[\begin{array}{cc} 0 & I_n \\ -N & -D \end{array}\right];$$

$$B_d = \left[\begin{array}{c} 0 \\ B \end{array}\right] C_d = \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right]$$

Algorithm S1

Resolved the equation (43) for calculate the matrices W_d, V_d, satisfies the equation (46) and the matrix K, such that $K C_d V_d = W_d$:

$$V_d = \left[\begin{array}{cc} 0.1095906 & 0.0695801 \\ 0.0973067 & 0.0550873 \\ 0.1567901 & 0.1249138 \\ -0.3287717 & -0.2783205 \\ -0.2919201 & -0.2203492 \\ -0.4703702 & -0.4996553 \end{array}\right]$$

$$W_d = \left[\begin{array}{c} 0.91971790.9480201 \\ 0.8327670.8379183 \end{array}\right]$$

$$K = \left[\begin{array}{cc} -36.816339 & 19.498243 \\ -30.793252 & 16.558322 \end{array}\right]$$

$\lambda_1 = -3$, $\lambda_2 = -4$, $\lambda_3 = -1.0795178 + 6.1019018j$, $\lambda_4 = -1.0795178 - 6.1019018j$, $\lambda_5 = -4.4381969$.

C. Numerical algorithm

We first present an approach to the general solutions for F_0, F_1 and F_2. Let Q denote the matrix that its rows are comprised of orthonormal basis vectors of the null space, we have

$$[F_0, F_1, F_2] = VQ \tag{49}$$

where the parametric matrix V is to be determined based in the sylvester equation (43).

The matrices F_0, F_1 and F_2 must also implement some given eigenvalues assignment.

The theorem 5.2 is based in , [8] [11] .

Theorem 5.2: Let (1), be S-controllable, and $V_d \in R^{2n\times p}$ and $W_d \in R^{m\times p}$ satisfy the equation (43). Then, the following hold.

1) The matrices V_d and W_d given by (50),

$$\left[\begin{array}{cc} A_d - \lambda_i E_d & B_d \end{array}\right] \left[\begin{array}{c} v_i \\ w_i \end{array}\right] = \; i = 1, 2, \cdots, q. \tag{50}$$

satisfy Sylvester matrix equation (43) for, $i = 1, 2, \cdots, q$,

2) When

$$rank(\left[\begin{array}{c} V_i \\ W_i \end{array}\right]) = m \; i = 1, 2, \cdots, q. \tag{51}$$

3) F_0, F_1, F_2 is such that it satisfies

$$(M + BF_2C_2)\ddot{q}(t) + (D + BF_1C_1)\dot{q}(t) + (N + BF_0C_0)q(t) = 0 \tag{52}$$

Proof Based in [7], [8], and [11].

The following basic procedure is proposed to calculate the feedback controller that stabilizes the closed loop system, when $m+p > q$. Closed loop eigenvalues are positioned arbitrarily close to the set ; they are symmetric sets of pre-specified eigenvalues. The (E_d, A_d, B_d, C_d) system is considered to be strongly controllable and strongly detectable.

Algorithm Z1 Input: M, D, N, B, C_0, C_1, C_2 Output: F_0, F_1, F_2

Step (1) Compute the left null space Q of the coefficient matrix in Equation (49).

Step (2) Compute v_i, w_i, $i =, 01, 2$ by Equations (43), (50), (51) to form V_0, V_1, V_2 and W_0, W_1, W_2.

Step(3) With the matrices W_0, W_1, W_2, V_0, V_1, V_2, satisfies the equation (50),(51) and the matrix F_0, F_1,F_2 such that $F_2C_2V_2 = W_2$, $F_1C_1V_1 = W_1$, and $F_0C_0V_0 = W_0$,

Step (4) Substitute V back into Equation (49) with (52) to give F_0, F_1, F_2.

$\circ$

D. Example

Consider a simple linear dynamical system (1) in [14]

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \; D = \begin{bmatrix} 2.5 & -0.5 & 0 \\ -0.5 & 2.5 & -2 \\ 0 & -2 & 2 \end{bmatrix}$$

$$N = \begin{bmatrix} 10 & -5 & 0 \\ -5 & 25 & -20 \\ 0 & -20 & 20 \end{bmatrix} \; B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Considered the system in (4), (5) $E_d = \begin{bmatrix} I_n & 0 \\ 0 & M \end{bmatrix}$; $A_d = \begin{bmatrix} 0 & I_n \\ -N & -D \end{bmatrix}$; $B_d = \begin{bmatrix} 0 \\ B \end{bmatrix}$

$$C_0 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \; C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Considered the system in the equations (4), (5)

$$E_d = \begin{bmatrix} I_n & 0 \\ 0 & M \end{bmatrix}; \; A_d = \begin{bmatrix} 0 & I_n \\ -N & -D \end{bmatrix};$$

$$B_d = \begin{bmatrix} 0 \\ B \end{bmatrix}$$

$$C_d = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \; C_0 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix};$$

$$C_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}; \; C_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Algorithm Z1

Step (1) Compute the left null space Q of the coefficient matrix in Equation (21).

Step (2) Compute v_i, w_i, $i =, 01, 2$ by Equations (43), (50), (51) to form V_0, V_1, V_2 and W_0, W_1, W_2.

Step(3) With the matrices W_0, W_1, W_2, V_0, V_1, V_2, satisfies the equation (50),(51) and the matrix F_0, F_1,F_2 such that $F_2C_2V_2 = W_2$, $F_1C_1V_1 = W_1$, and $F_0C_0V_0 = W_0$,

Step (4) Substitute V back into Equation (49) with (52) to give F_0, F_1, F_2.

$$V_0 = V_1 = V_2 = \begin{bmatrix} 0.1095906 & 0.0695801 \\ 0.0973067 & 0.0550873 \\ 0.1567901 & 0.1249138 \\ -0.3287717 & -0.2783205 \\ -0.2919201 & -0.2203492 \\ -0.4703702 & -0.4996553 \end{bmatrix}$$

$$W_0 = W_1 = W_2 = \begin{bmatrix} 0.9197179 & 0.9480201 \\ 0.832767 & 0.8379183 \end{bmatrix}$$

$$F_0 = \begin{bmatrix} 51.199353 & -50.054933 \\ 42.823231 & -41.616079 \end{bmatrix}$$

where the eigenvalues are

$\lambda_1 = -3$, $\lambda_2 = -4$, $\lambda_3 = -1.9669794 + 4.0571187j$, $\lambda_4 = -1.9669794 - 4.0571187j$, $\lambda_5 = -9.3303829$.

$$F_1 = \begin{bmatrix} -36.816339 & 19.498243 \\ -30.793252 & 16.558322 \end{bmatrix}$$

$\lambda_1 = -3$, $\lambda_2 = -4$, $\lambda_3 = -1.0795178 + 6.1019018j$, $\lambda_4 = -1.0795178 - 6.1019018j$, $\lambda_5 = -4.4381969$.

$$F_2 = \begin{bmatrix} 56.687413 & -54.391796 \\ 48.612123 & -46.190671 \end{bmatrix}$$

where the eigenvalues are $\lambda_1 = -3$, $\lambda_2 = -4$, $\lambda_3 = -10$, $\lambda_4 = -2 + 4.1836895j$, $\lambda_5 = -2 - 4.1836895j$.

VI. Conclusions

The paper has presented an approach to partial eigenvalue assignment in second-order descriptor linear systems via proportional plus derivative plus output feedback controller was presented. The approach to partial eigenstructure assignment is shown using a proportional plus-derivative plus output feedback in the second-order linear system. This study presented an approach to partial the eigenstructure assignment for the descriptor system where an algorithm is presented for calculating the output feedback matrix by equation de Sylvester. Two theorems were presented using Sylvester's equations. Two algorithms were implemented using the Sylvester equation as a basis, and examples are presented with their conclusions.

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