Exact solutions of third and fourth order obstacle problems using reduction-to-first-order method

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Abstract

In this dissertation, a method named Reduction-to-First-Order (RFO) method proposed by L. E. Nicholas Afima [1] in 1991 and first time presented at any forum in 2013 by A. M. Siddiqui and T. Haroon [2] is used to solve the system of third and fourth order boundary value problems. Several examples are presented to get the exact solutions, which illustrate the effectiveness and simplicity of the method.

Keywords: Reduction-to-First-Order Method, Obstacle Problems, System of boundary value problems.

1 Introduction

It is well known that a wide class of unrelated problems arising industrial, regional, economics, telecommunication, computer network, image reconstruction, pure and applied sciences can be studied in the general and unified frame work of variational inequalities. We note that the finite difference methods and splines techniques can not be applied directly to find the approximate solutions of variational inequalities. However, one can be characterized the variational inequalities associated with the third and fourth order obstacle problems by a system of third and fourth order differential equations using Reduction-to-First-Order technique. Reduction-to-First-Order is a mathematical technique used to solve the systems of boundary value problems. Boundary value problems manifest themselves in many branches of science. For example engineering, technology, control, optimization theory, draining and coating flows and various dynamic systems. In the solution of real-world problems ordinary differential equations (ODEs) are supposed to be basic tools.

In 1987, M. A. Noor and A. K. Khalifa solved the unilateral problems using cubic spline method in [12]. M. A. Noor and S. I. A. Tirmizi used the technique of finite difference to solve obstacle problems [13] in 1988. E. A. Al-Said used spline [14], smooth spline [15] and cubic spline [16] methods to find the numerical solutions of system of second order boundary value problems in 1996, 1999 and 2001 respectively. The same second order problems has been solved using a parametric cubic spline approach [17] in 2003 by A. Khan and T. Aziz. In 2010, the solution of same second order system of boundary value problems was solved numerically using Galerkins finite element formulation by S. Iqbal et al., [18], also, S. Iqbal solved the second order obstacle problem using cubic Lagrange polynomial in Galerkins finite element fashion [19]. The numerical solutions of obstacle problems of second order have been given by G. B. Loghmani et al., using B-spline function [20] in 2011.

A spatially Adaptive Grid Refinement Scheme was used for the Finite Element solution of a Second order Obstacle Problem in [21] by S. Iqbal et al., in 2013. In 2014 S. Iqbal et al., [22] used Galerkins Finite Element Formulation to find the numerical solution of second order and third order systems of boundary value problems. In 2006, F. Gao and C. M. Chi used quartic B-splines technique [23] to find the solution of third order system of boundary value problems. Variational iteration method is used in 2010 by F. Geng and M. Cui to solve the third order systems of boundary value problems [24]. Semi analytical methods are more suitable than numerical methods to solve nonlinear non-homogenous differential equations. The most powerful tool for the calculation of analytical solutions of the linear or nonlinear partial differential equation is Reduction-to-First-Order method. Reduction-to-First-Order method was firstly introduced by L. E. Nicholas Afima [1] in 1991. In 2013, Reduction-to-First-Order method was presented first time at any forum in a workshop held at 11th Conference on Frontiers of Information Technology by A. M. Siddiqui and T. Harron [2].

In this paper, the systems of third and fourth order boundary value problems are considered. Third-order boundary value problems arise in the study of draining and coating flows. Here, Reduction-to-First-Order method is used to find the exact solutions of third order systems of boundary value problems in its general form as:

$$u'''(x) = \begin{cases} f(x), & a \leq x < c, \\ g_1(x)u''(x) + g_2(x)u'(x) + g_3(x)u(x) + g_4(x), & c \leq x \leq d, \\ f(x), & d < x \leq b, \end{cases} \tag{1}$$

with the boundary and continuity conditions, $u(a) = \alpha_1$, $u'(a) = \alpha_2$ and $u'(b) = \alpha_3$, where α_1, α_2, α_3 are constants and the continuity condition of $u(x)$, $u'(x)$ and $u''(x)$ at c and d. Here f and g_i where $i = 1, 2, 3, 4$ are the continuous functions on $[a, b]$ and $[c, d]$ respectively. Fourth-order boundary value problems arise in the study of viscoelastic and inelastic flows [3], deformation of beams with nonlinear boundary conditions [4] and plate deflection theory [5]. Several numerical and analytical methods including finite difference method [6] for nonlinear boundary-value problems, Adomian decomposition method [7, 8] for numerical solution of fourth-order boundary value problems, differential transform method for the same problems [9], variational iteration technique for the solution of higher order boundary value problems [10] and Homotopy perturbation method [11], have been developed for solving general fourth-order boundary value problems. In this paper, Reduction-to-First-Order method is also used to find the exact solutions of fourth order systems of boundary value problems in its general form as:

$$u^{(iv)}(x) = \begin{cases} f(x), & a \leq x \leq c, \\ g(x)u(x) + f(x) + r, & c \leq x \leq d, \\ f(x), & d \leq x \leq b, \end{cases} \tag{2}$$

with the boundary and continuity conditions, $u(a) = u(b) = \alpha_1$, $u''(a) = u''(b) = \alpha_2$ and $u(c) = u(d) = \beta_1$, $u''(c) = u''(d) = \beta_2$, where α_1, α_2, β_1, β_2 and r are constants. Here f and g are the continuous functions on $[a, b]$ and $[c, d]$ respectively.

Usually, it is difficult to find the analytical solution for arbitrary choice of $f(x)$ and $g(x)$. Therefore, some numerical methods are opted to get approximate solutions of the problems. Such type of systems arise in the study of obstacle, unilateral and contact boundary value problems and have important applications in other branches of pure and applied sciences. Most of the nonlinear differential equations do not have analytical solution. However researchers used many numerical methods, but these methods require much time and more efficient computing devices.

Reduction-to-First-Order method requires a suitable tranformation to reduce the order of the linear nonhomogeneous differential equation with constant coefficients. This reduced order differential equation can then be solved using the integrating factor technique. To illustrate the utility of the method several examples of third and fourth order system of boundary value problems are presented.

2 Main Results of Third Order Obstacle Problems

2.1 Algorithm for Third Order Obstacle Problems

The formulation of working algorithm of Reduction-to-First-Order Method can be expressed in the following way [17, 18]:

(a) Write the governing differential equation

$$u''' + lu'' + mu' + nu = h(x). \tag{3}$$

(b) Suppose α, β and γ are the roots of polynomial $\lambda^3 + l\lambda^2 + m\lambda + n = 0$, at least one root will be real say α.

(c) Rearrange the Eq. (3) using α as

$$(u' - \alpha u)'' + p(u' - \alpha u)' + q(u' - \alpha u) = h(x), \tag{4}$$

so that β and γ are the roots of the quadratic equation $\lambda^2 + p\lambda + q = 0$.

(d) Introduce the new variable

$$w = u' - \alpha u, \tag{5}$$

so that, Eq. (3) will reduce to second order linear differential equation as

$$w'' + pw' + w = h(x). \tag{6}$$

(e) Factorize Eq. (6) as

$$(w' - \beta w)' - \gamma(w' - \beta w) = h(x). \tag{7}$$

(f) Introduce the new variable

$$z = w' - \beta w, \tag{8}$$

so that, Eq. (6) will reduce to first order linear differential equation as

$$z' - \gamma z = h(x), \tag{9}$$

with integration factor $e^{-\gamma x}$.

(g) Solve Eq. (9) for z

$$z = Ce^{\gamma x} + e^{\gamma x} \int e^{-\gamma x} h(x) dx. \tag{10}$$

(h) Restore the variable w by Eq. (5), once again first order linear differential equation will appear, which will be solved by the method of integrating factor to find the solution for w.

(i) Restore the original variable u by Eq. (5), once again first order linear differential equation will appear, which will be solved by the method of integrating factor to find the solution for u.

2.2 Illustrative Problem to Demonstrate (RFO) Method

In this section the solution of third order systems of boundary value problems is given by using the Reduction-to-First-Order method.

Problem: We consider the obstacle problem of the following form

$$u''' = \begin{cases} 1, & 0 \leq x < \frac{1}{4}, \\ 5u'' + u' - 5u + 1, & \frac{1}{4} \leq x \leq \frac{3}{4}, \\ 1, & \frac{3}{4} < x \leq 1, \end{cases} \tag{11}$$

with boundary conditions $u(0) = u'(0) = u'(1) = 0$. The problem is divided into three cases,

Case 1:$\left(0 \leq x < \frac{1}{4}\right)$

In this case, we have the following differential equation with boundary conditions,

$$u'''(x) = 1, \tag{12}$$

$$u(0) = u'(0) = 0, u'\left(\frac{1}{4}\right) = a,$$

where a is constant. After solving Eq. (12), we have

$$u_{[0,\frac{1}{4})}(x) = \frac{x^2}{48}(-3 + 96a + 8x). \tag{13}$$

Case 2: $\left(\frac{1}{4} \leq x \leq \frac{3}{4}\right)$

In this case, we have the following differential equation with boundary conditions,

$$u'''(x) - 5u''(x) - u'(x) + 5u(x) = 1, \tag{14}$$
$$u\left(\frac{1}{4}\right) = d, u'\left(\frac{1}{4}\right) = a, u'\left(\frac{3}{4}\right) = c.$$

Eq. (14) is rearranged as

$$\Big(u'(x) - u(x)\Big)'' - 4\Big(u'(x) - u(x)\Big)' - 5\Big(u'(x) - u(x)\Big) = 1. \tag{15}$$

A new variable z is introduce such that $z = u'(x) - u(x)$, so Eq. (15) becomes

$$z''(x) - 4z'(x) - 5z(x) = 1. \tag{16}$$

Eq. (16) is rearranged as

$$\Big(z'(x) + z(x)\Big)' - 5\Big(z'(x) + z(x)\Big) = 1. \tag{17}$$

Another variable w is introduced such that $w = z'(x) + z(x)$, so Eq. (17) becomes

$$w'(x) - 5w(x) = 1, \tag{18}$$

which is the first order linear differential equation. Now, by using integrating factor (IF) method the solution is

$$w(x) = -\frac{1}{5} + De^{5x}, \tag{19}$$

where D is a constant of integration. Restoring the variable z the differential equation becomes

$$z'(x) + z(x) = -\frac{1}{5} + De^{5x}, \tag{20}$$

which is again the first order linear differential equation, using (IF) method the obtained solution is

$$z(x) = -\frac{1}{5} + \frac{1}{6}De^{5x} + Ee^{-x}, \tag{21}$$

where E is constant of integration. Restoring the original variable u the differential equation becomes

$$u'(x) - u(x) = -\frac{1}{5} + \frac{1}{6}De^{5x} + Ee^{-x}, \tag{22}$$

which is linear first order differential equation, using (IF) method the obtained solution is

$$u(x) = \frac{1}{5} - \frac{1}{2}Ee^{-x} + \frac{1}{24}De^{5x} + Fe^{x}, \tag{23}$$

where F is a constant of integration. The boundary conditions are applied to obtain the values of D, E and F

$$D = -\frac{12(1 + 5a - 5d - 10c\sqrt{e} - e + 5ae + 5de)}{5e^{\frac{5}{4}}(-2 - 3e + 5e^3)}, \tag{24}$$

$$E = -\frac{4ce^{\frac{3}{4}} + e^{\frac{5}{4}} + ae^{\frac{5}{4}} - 5de^{\frac{5}{4}} - e^{\frac{13}{4}} - 5ae^{\frac{13}{4}} + 5de^{\frac{13}{4}}}{-2 - 3e + 5e^3}, \tag{25}$$

$$F = -\frac{-1 - a + 5d + 6c\sqrt{e} + e^3 - 5ae^3 - 5de^3}{2e^{\frac{1}{4}}(-2 - 3e + 5e^3)}. \tag{26}$$

Putting the values of D, E and F in Eq. (23) the solution is

$$\begin{aligned} u_{[\frac{1}{4},\frac{3}{4}]}(x) = \frac{1}{5} &- \frac{e^{\frac{-5}{4}+5x}(1 + 5a - 5d - 10c\sqrt{e} - e + 5ae + 5de)}{10(-2 - 3e + 5e^3)} \\ &- \frac{e^{\frac{-1}{4}+x}(-1 - a + 5d + 6c\sqrt{e} + e^3 - 5ae^3 - 5de^3)}{2(-2 - 3e + 5e^3)} \\ &+ \frac{e^{-x}\left(4ce^{\frac{3}{4}} + e^{\frac{5}{4}} + ae^{\frac{5}{4}} - 5de^{\frac{5}{4}} - e^{\frac{13}{4}} - 5ae^{\frac{13}{4}} + 5de^{\frac{13}{4}}\right)}{2(-2 - 3e + 5e^3)}. \end{aligned} \tag{27}$$

Case 3: $\left(\frac{3}{4} < x \leq 1\right)$

As in case 1, we have the following differential equation with boundary conditions

$$u'''(x) = 1, \tag{28}$$

$$u\left(\frac{3}{4}\right) = b, u'\left(\frac{3}{4}\right) = c, u'(1) = 0.$$

After solving Eq. (28), we have

$$u_{(\frac{3}{4},1]}(x) = b + \frac{1}{768}(-3 + 4x)\left(27 - 60x + 32x^2 - 96c(-5 + 4x)\right). \tag{29}$$

Now, to obtain continuous solution, for the given obstacle problem, following continuity conditions are used to find the values of a, b, c and d

$$\lim_{x\to\frac{1}{4}^-} u(x) = \lim_{x\to\frac{1}{4}^+} u(x), \tag{30}$$

$$\lim_{x\to\frac{3}{4}^-} u(x) = \lim_{x\to\frac{3}{4}^+} u(x), \tag{31}$$

$$\lim_{x\to\frac{1}{4}^-} u''(x) = \lim_{x\to\frac{1}{4}^+} u''(x), \tag{32}$$

$$\lim_{x\to\frac{3}{4}^-} u''(x) = \lim_{x\to\frac{3}{4}^+} u''(x), \tag{33}$$

we get

$$a = -\frac{1354 + 768\sqrt{e} - 4345e + 3759e^3}{96\gamma}, \tag{34}$$

$$b = \frac{1914 - 4045\sqrt{e} + 1575e - 11394e^{\frac{5}{2}} + 33015e^3 - 22025e^{\frac{7}{2}}}{360\gamma}, \tag{35}$$

$$c = -\frac{264 - 809\sqrt{e} + 180e - 5697e^{\frac{5}{2}} + 1860e^3 + 8810e^{\frac{7}{2}}}{288\gamma}, \tag{36}$$

$$d = -\frac{43 + 24\sqrt{e} - 135e + 132e^3}{24\gamma}, \tag{37}$$

where $\gamma = 22 + 25e + 465e^3$, by substituting the values of a, b, c and d we have the exact solution of this problem is

$$u(x) = \begin{cases} \frac{x^2}{48}\left(-3 + 8x - \frac{1354+768\sqrt{e}-4345e+3759e^3}{96\gamma}\right), & 0 \le x < \frac{1}{4}, \\ \frac{1}{5} + \frac{11e^{\frac{3}{4}-x}}{12\gamma} - \frac{4045e^{\frac{5}{4}-x}}{576\gamma} - \frac{1899e^{\frac{13}{4}-x}}{64\gamma} & \\ -\frac{809e^{\frac{-1}{4}+x}}{192\gamma} - \frac{5e^{\frac{1}{4}+x}}{8\gamma} - \frac{4405e^{\frac{11}{4}+x}}{64\gamma} & \\ -\frac{633e^{\frac{-5}{4}+5x}}{320\gamma} - \frac{31e^{\frac{-3}{4}+5x}}{24\gamma} + \frac{1405e^{\frac{-1}{4}+5x}}{576\gamma}, & \frac{1}{4} \le x \le \frac{3}{4}, \\ \frac{1}{11520\gamma}\Big(3(5(3-4x)^2(-9+8x)\gamma & \\ +88(307 - 160x + 80x^2) - 900e(-71 + 32x - 16x^2) & \\ +4045\sqrt{e}(-47 + 32x - 16x^2) - 44050e^{\frac{7}{2}}(1 + 32x - 16x^2) & \\ -1860e^3(-643 + 160x - 80x^2) + 5697e^{\frac{5}{2}}(-139 + 160x - 80x^2)\Big), & \frac{3}{4} < x \le 1. \end{cases} \tag{38}$$

3 Main Results of Fourth Order Obstacle Problems

3.1 Algorithm for Fourth Order Obstacle Problems

The formulation of working algorithm of Reduction-to-First-Order method can be expressed in the following way:

(a) Write the governing differential equation

$$u^{(iv)} + lu''' + mu'' + nu' + qu = h(x). \tag{39}$$

(b) Suppose α, β, γ and δ are the roots of polynomial $\lambda^4 + l\lambda^3 + m\lambda^2 + n\lambda + q = 0$, at least one root will be real say α.

(c) Rearrange the Eq. (39) using α as

$$(u' - \alpha u)''' + p(u' - \alpha u)'' + q(u' - \alpha u)' + r(u' - \alpha u) = h(x), \tag{40}$$

so that β , γ and δ are the roots of the cubic polynomial $\lambda^3 + p\lambda^2 + q\lambda + r = 0$.

(d) Introduce the new variable

$$w = u' - \alpha u, \tag{41}$$

so that, Eq. (39) will reduce to third order linear differential equation as

$$w''' + pw'' + qw' + rw = h(x). \tag{42}$$

(e) Factorize Eq. (42) as

$$(w' - \beta w)'' + \gamma(w' - \beta w)' + \delta(w' - \beta w) = h(x). \tag{43}$$

(f) Introduce the new variable

$$z = w' - \beta w, \tag{44}$$

so that, Eq. (42) will reduce to second order linear differential equation as

$$z'' + \gamma z' + \delta z = h(x). \tag{45}$$

(g) Factorize Eq. (45) as

$$(z' - \gamma z)' - \gamma(z' - \gamma z) = h(x). \tag{46}$$

(h) Introduce the new variable

$$t = z' - \gamma z, \tag{47}$$

so that, Eq. (45) will reduce to first order linear differential equation as

$$t' - \gamma t = h(x), \tag{48}$$

with integration factor $e^{-\gamma x}$.

(i) Solve Eq. (48) for t

$$t = Ce^{\gamma x} + e^{\gamma x} \int e^{-\gamma x} h(x) dx. \tag{49}$$

(j) Restore the variable z by Eq. (44), once again first order linear differential equation will appear, which will be solved by the method of integrating factor to find the solution for z.
(k) Restore the variable w by Eq. (42), once again first order linear differential equation will appear, which will be solved by the method of integrating factor to find the solution for w.
(l) Restore the original variable u by Eq. (42), once again first order linear differential equation will appear, which will be solved by the method of integrating factor to find the solution for u.

3.2 Illustrative Problem to Demonstrate (RFO) Method

In this section the solution of fourth order systems of boundary value problems is given by using the Reduction-to-First-Order method.

Problem: We consider the obstacle problem of the following form

$$u^{(iv)}(x) = \begin{cases} 1, & -1 \leq x \leq -\frac{1}{2}, \\ u+1, & -\frac{1}{2} \leq x \leq \frac{1}{2}, \\ 1, & \frac{1}{2} \leq x \leq 1, \end{cases} \tag{50}$$

with boundary and continuity conditions

$$u(-1) = u\left(-\frac{1}{2}\right) = u\left(\frac{1}{2}\right) = u(1) = 0,$$

$$u''(-1) = u''\left(-\frac{1}{2}\right) = u''\left(\frac{1}{2}\right) = u''(1) = 0.$$

The problem is divided into three cases

Case 1: $\left(-1 \leq x \leq -\frac{1}{2}\right)$

In this case, we have the following differential equation with boundary conditions

$$u^{(iv)}(x) = 1, \tag{51}$$

$$u(-1) = u\left(-\frac{1}{2}\right) = u''(-1) = u''\left(-\frac{1}{2}\right) = 0.$$

After solving Eq. (51), we have

$$u_{[-1,-\frac{1}{2}]}(x) = \frac{1}{24}x^4 + \frac{1}{8}(x^3 + x^2) + \frac{3}{64}x + \frac{1}{192}. \tag{52}$$

Case 2: $\left(-\frac{1}{2} \leq x \leq \frac{1}{2}\right)$

In this case, we have the following differential equation with boundary conditions

$$u^{(iv)}(x) - u(x) = 1, \tag{53}$$

$$u\left(\frac{1}{2}\right) = u\left(-\frac{1}{2}\right) = u''\left(\frac{1}{2}\right) = u''\left(-\frac{1}{2}\right) = 0.$$

Eq. (53) is rearranged as

$$\Big(u'(x) - u(x)\Big)''' + \Big(u'(x) - u(x)\Big)'' + \Big(u'(x) - u(x)\Big)' + \Big(u'(x) - u(x)\Big) = 1. \tag{54}$$

A new variable z is introduce such that $z = u'(x) - u(x)$, so Eq. (54) becomes

$$z'''(x) + z''(x) + z'(x) + z(x) = 1. \tag{55}$$

Eq. (55) is rearranged as

$$\Big(z'(x) + z(x)\Big)'' + \Big(z'(x) + z(x)\Big) = 1. \tag{56}$$

Another variable w is introduced such that $w = z'(x) + z(x)$, so Eq. (56) becomes

$$w''(x) + w(x) = 1. \tag{57}$$

Eq. (57) is rearranged as

$$(w' - iw)' + i(w' - iw) = 1. \tag{58}$$

Let $\beta = i$ and $\alpha = -i$, hence

$$(w' + \alpha w)' + \beta(w' + \alpha w) = 1. \tag{59}$$

Another variable t is introduced such that $t = w' + \alpha w$, so Eq. (59) becomes

$$t'(x) + \beta t(x) = 1, \tag{60}$$

which is the first order linear differential equation. Now by using (IF) method the solution is

$$t(x) = \frac{1}{\beta} + C_1 e^{-\beta x}, \tag{61}$$

where C_1 is a constant of integration. Restoring the variable w the differential equation becomes

$$w'(x) + \alpha w(x) = \frac{1}{\beta} + C_1 e^{-\beta x}, \tag{62}$$

which is again the first order linear differential equation, using (IF) method the obtained solution is

$$w(x) = \frac{1}{\alpha\beta} + \frac{C_1 e^{-\beta x}}{(\alpha - \beta)} + C_2 e^{-\alpha x}, \tag{63}$$

where C_2 is a constant of integration. Restoring the variable z the differential equation becomes

$$z'(x) + z(x) = \frac{1}{\alpha\beta} + \frac{C_1 e^{-\beta x}}{(\alpha - \beta)} + C_2 e^{-\alpha x}, \tag{64}$$

which is linear first order differential equation, using (IF) method the obtained solution is

$$z(x) = \frac{1}{\alpha\beta} + \frac{C_1 e^{-\beta x}}{(\alpha - \beta)(1 - \beta)} + \frac{C_2 e^{-\alpha x}}{1 - \alpha} + C_3 e^{-x}, \tag{65}$$

where C_3 is a constant of integration. Restoring the original variable u the differential equation becomes

$$u'(x) - u(x) = \frac{1}{\alpha\beta} + \frac{C_1 e^{-\beta x}}{(\alpha - \beta)(1 - \beta)} + \frac{C_2 e^{-\alpha x}}{1 - \alpha} + C_3 e^{-x}, \tag{66}$$

which is linear first order differential equation, using (IF) method the obtained solution is

$$u(x) = -\frac{1}{\alpha\beta} - \frac{C_1 e^{-\beta x}}{(-\alpha + \beta)(-1 + \beta^2)} + \frac{C_2 e^{-\alpha x}}{-1 + \alpha^2} - \frac{C_3 e^{-x}}{2} + C_4 e^{x}, \tag{67}$$

where C_4 is a constant of integration. The boundary conditions are applied to obtain the values of C_1, C_2, C_3 and C_4

$$C_1 = \frac{(-\alpha + \beta)\csc h(\frac{\alpha-\beta}{2})\sinh(\frac{\alpha}{2})}{\alpha\beta}, \tag{68}$$

$$C_2 = \frac{\csc h(\frac{\alpha-\beta}{2})\sinh\left(\frac{\beta}{2}\right)}{\alpha\beta}, \tag{69}$$

$$\begin{aligned} C_3 &= -2\sqrt{e}\left(\frac{(-(-1+e^{\alpha})(-1+e^{1+\beta})\beta^2) + \alpha^2(1+e^{\alpha}(-e+(-1+e)\beta^2)}{(-1+e^2)(e^{\alpha}-e^{\beta})\alpha(-1+\alpha^2)\beta(-1+\beta^2)}\right) \\ &- 2\sqrt{e}\left(\frac{e^{\beta}(-1+e^{1+\alpha}+\beta^2-e\beta^2)}{(-1+e^2)(e^{\alpha}-e^{\beta})\alpha(-1+\alpha^2)\beta(-1+\beta^2)}\right), \end{aligned}$$

$$C_4 = \left(\frac{e^{\frac{3}{2}}(-\alpha^2+\beta^2)+e^{\frac{1}{2}+\alpha}(\alpha^2+(-e+(-1+e)\alpha^2)\beta^2)}{(-1+e^2)(e^\alpha-e^\beta)\alpha(-1+\alpha^2)\beta(-1+\beta^2)}\right) + \left(\frac{e^{\frac{1}{2}+\beta}\left((-1+\alpha^2)\beta^2-e\alpha^2(-1+\beta^2)+e^\alpha(-\alpha^2+\beta^2)\right)}{(-1+e^2)(e^\alpha-e^\beta)\alpha(-1+\alpha^2)\beta(-1+\beta^2)}\right).$$

Putting the values of C_1, C_2, C_3 and C_4 in Eq. (67), the solution is

$$\begin{aligned} u_{[-\frac{1}{2},\frac{1}{2}]}(x) &= -\frac{1}{\alpha\beta} - \frac{e^{-x\beta}(-\alpha+\beta)\csc h(\frac{\alpha-\beta}{2})\sinh(\frac{\alpha}{2})}{\alpha\beta(-\alpha+\beta)(-1+\beta^2)} + \frac{e^{-x\alpha}\csc h(\frac{\alpha-\beta}{2})\sinh(\frac{\beta}{2})}{\alpha\beta(-1+\alpha^2)} \\ &+ e^{\frac{1}{2}-x}\left(\frac{(-(-1+e^\alpha)(-1+e^{1+\beta})\beta^2)+\alpha^2(1+e^\alpha(-e+(-1+e)\beta^2)}{(-1+e^2)(e^\alpha-e^\beta)\alpha(-1+\alpha^2)\beta(-1+\beta^2)}\right) \\ &+ e^{\frac{1}{2}-x}\left(\frac{e^\beta(-1+e^{1+\alpha}+\beta^2-e\beta^2)}{(-1+e^2)(e^\alpha-e^\beta)\alpha(-1+\alpha^2)\beta(-1+\beta^2)}\right) \\ &+ e^x\left(\frac{e^{\frac{3}{2}}(\alpha^2-\beta^2)+e^{\frac{1}{2}+\alpha}(e\beta^2+\alpha^2(-1-(-1+e)\beta^2)}{(-1+e^2)(e^\alpha-e^\beta)\alpha(-1+\alpha^2)\beta(-1+\beta^2)}\right) \\ &+ e^x\left(\frac{e^{\frac{1}{2}+\beta}(\beta^2+e^\alpha(\alpha-\beta)(\alpha+\beta)+\alpha^2(-e+(-1+e)\beta^2)}{(-1+e^2)(e^\alpha-e^\beta)\alpha(-1+\alpha^2)\beta(-1+\beta^2)}\right). \end{aligned}$$

Case 3: $\left(\frac{1}{2} \leq x \leq 1\right)$

As in case 1, we have the following differential equation with boundary conditions

$$u^{(iv)}(x) = 1, \tag{70}$$

$$u(1) = u\left(\frac{1}{2}\right) = u''(1) = u''\left(\frac{1}{2}\right) = 0.$$

After solving Eq. (70), we have

$$u_{[\frac{1}{2},1]}(x) = \frac{1}{24}x^4 - \frac{1}{8}\left(x^3 - x^2\right) - \frac{3}{64}x + \frac{1}{192}. \tag{71}$$

The exact solution of this problem is

$$u(x) = \begin{cases} \frac{1}{24}x^4 + \frac{1}{8}\left(x^3+x^2\right) + \frac{3}{64}x + \frac{1}{192}, & -1 \leq x \leq -\frac{1}{2}, \\ -\frac{1}{\alpha\beta} - \frac{e^{-x\beta}(-\alpha+\beta)\csc h(\frac{\alpha-\beta}{2})\sinh(\frac{\alpha}{2})}{\alpha\beta(-\alpha+\beta)(-1+\beta^2)} + \frac{e^{-x\alpha}\csc h(\frac{\alpha-\beta}{2})\sinh(\frac{\beta}{2})}{\alpha\beta(-1+\alpha^2)} & \\ +e^{\frac{1}{2}-x}\left(\frac{(-(-1+e^\alpha)(-1+e^{1+\beta})\beta^2)+\alpha^2(1+e^\alpha(-e+(-1+e)\beta^2)}{(-1+e^2)(e^\alpha-e^\beta)\alpha(-1+\alpha^2)\beta(-1+\beta^2)}\right) & \\ +e^{\frac{1}{2}-x}\left(\frac{e^\beta(-1+e^{1+\alpha}+\beta^2-e\beta^2)}{(-1+e^2)(e^\alpha-e^\beta)\alpha(-1+\alpha^2)\beta(-1+\beta^2)}\right) & \\ +e^x\left(\frac{e^{\frac{3}{2}}(\alpha^2-\beta^2)+e^{\frac{1}{2}+\alpha}(e\beta^2+\alpha^2(-1-(-1+e)\beta^2)}{(-1+e^2)(e^\alpha-e^\beta)\alpha(-1+\alpha^2)\beta(-1+\beta^2)}\right) & \\ +e^x\left(\frac{e^{\frac{1}{2}+\beta}(\beta^2+e^\alpha(\alpha-\beta)(\alpha+\beta)+\alpha^2(-e+(-1+e)\beta^2)}{(-1+e^2)(e^\alpha-e^\beta)\alpha(-1+\alpha^2)\beta(-1+\beta^2)}\right), & -\frac{1}{2} \leq x \leq \frac{1}{2}, \\ \frac{1}{24}x^4 - \frac{1}{8}\left(x^3-x^2\right) - \frac{3}{64}x + \frac{1}{192}, & \frac{1}{2} \leq x \leq 1. \end{cases} \tag{72}$$

4 CONCLUSION

In this dissertation Reduction-to-First-Order method has been presented to solve the system of third and fourth order boundary value problems. Several examples of third and fourth order obstacle, unilateral and contact problems demonstrated the exact solutions. We also noted that using Reduction-to-First-Order method, the higher order differential equations can be solved by first order linear method. So instead of learning the different methods like undetermined coefficient method and variation of parameters method, one have to learn only the first order linear method.

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