# Riemann Hypothesis on Grönwall's Function

Frank Vega



January 3, 2023

# Riemann Hypothesis on Grönwall's Function

Frank Vega[1*]

[1*]Software Department, CopSonic, 1471 Route de Saint-Nauphary, Montauban, 82000, Tarn-et-Garonne, France.

Corresponding author(s). E-mail(s): vega.frank@gmail.com;

**Abstract**

Grönwall's function $G$ is defined for all natural numbers $n > 1$ by $G(n) = \frac{\sigma(n)}{n \cdot \log \log n}$ where $\sigma(n)$ is the sum of the divisors of $n$ and $\log$ is the natural logaritm. We require the properties of extremely abundant numbers, that is to say left to right maxima of $n \mapsto G(n)$. We also use the colossally abundant and hyper abundant numbers. A number $n$ is said to be colossally abundant if, for some $\epsilon > 0$, $\frac{\sigma(n)}{n^{1+\epsilon}} \geq \frac{\sigma(m)}{m^{1+\epsilon}}$ for all $m > 1$. Let us call hyper abundant an integer $n$ for which there exists $u > 0$ such that $\frac{\sigma(n)}{n \cdot (\log n)^{u}} \geq \frac{\sigma(m)}{m \cdot (\log m)^{u}}$ for all $m > 1$. The Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. It is considered by many to be the most important unsolved problem in pure mathematics. There are several statements equivalent to the famous Riemann hypothesis. We state that the Riemann hypothesis is true if and only if there exist infinitely many consecutive colossally abundant numbers $3 \leq N < N'$ such that $G(N) \leq G(N')$. In addition, we prove that the Riemann hypothesis is true when there exist infinitely many hyper abundant numbers $n$ with any parameter $u \gtrapprox 1$.

**Keywords:** Riemann hypothesis, Extremely abundant numbers, Colossally abundant numbers, Hyper abundant numbers, Arithmetic functions

**MSC Classification:** 11M26 , 11A25

# 1 Introduction

As usual $\sigma(n)$ is the sum-of-divisors function of $n$

$$\sum_{d \mid n} d,$$

where $d \mid n$ means the integer $d$ divides $n$. In 1997, Ramanujan's old notes were published where it was defined the generalized highly composite numbers, which include the superabundant and colossally abundant numbers [1]. A natural number $n$ is called superabundant precisely when, for all natural numbers $m < n$

$$\frac{\sigma(m)}{m} < \frac{\sigma(n)}{n}.$$

A number $n$ is said to be colossally abundant if, for some $\epsilon > 0$,

$$\frac{\sigma(n)}{n^{1+\epsilon}} \geq \frac{\sigma(m)}{m^{1+\epsilon}} \ \ \mathit{for} \ \ (m > 1).$$

Every colossally abundant number is superabundant [2]. Let us call hyper abundant an integer $n$ for which there exists $u > 0$ such that

$$\frac{\sigma(n)}{n \cdot (\log n)^{u}} \geq \frac{\sigma(m)}{m \cdot (\log m)^{u}} \ \ \mathit{for} \ \ (m > 1),$$

where log is the natural logaritm. Every hyper abundant number is colossally abundant [3, pp. 255]. In 1913, Grönwall studied the function $G(n) = \frac{\sigma(n)}{n \cdot \log \log n}$ for all natural numbers $n > 1$ [4]. Next, we have the Robin's Theorem:

**Proposition 1** *Let $3 \leq N < N'$ be two consecutive colossally abundant numbers, then*

$$G(n) \leq \mathit{Max}\big(G(N), G(N')\big)$$

*when satisfying $N < n < N'$ [5, Proposition 1 pp. 192].*

There are champion numbers (i.e. left to right maxima) of the function $n \mapsto G(n)$:

$$G(m) \leq G(n)$$

for all natural numbers $10080 \leq m < n$. A positive integer $n$ is extremely abundant if either $n = 10080$, or $n > 10080$ is a champion number of the function $n \mapsto G(n)$ (Note that, in the reference paper it is defined the inequality as $G(m) < G(n)$ [6, Definition 3 pp. 5]. However, the Propositions 2 and 3 are still valid under the current definition with the inequality $G(m) \leq G(n)$). In 1859, Bernhard Riemann proposed his hypothesis [7]. Several analogues of the Riemann hypothesis have already been proved [7].

**Proposition 2** *The Riemann hypothesis is true if and only if there exist infinitely many extremely abundant numbers [6, Theorem 7 pp. 6].*

We use the following property for the extremely abundant numbers:

**Proposition 3** *Let $N < N'$ be two consecutive colossally abundant numbers and $n > 10080$ is some extremely abundant number, then $N'$ is also extremely abundant when satisfying $N < n < N'$ [6, Lemma 21 pp. 12].*

This is our main theorem

**Theorem 1** *The Riemann hypothesis is true if and only if there exist infinitely many consecutive colossally abundant numbers $3 \leq N < N'$ such that $G(N) \leq G(N')$.*

The following is a key Corollary.

**Corollary 1** *The Riemann hypothesis is true when there exist infinitely many hyper abundant numbers $N'$ with any parameter $u \gtrapprox 1$, where the symbol $\gtrapprox$ means "greater than or approximately to".*

Putting all together yields a new criterion for the Riemann hypothesis. Note also that, for all $u > 0$ [3, pp. 254]:

$$\lim_{n \to \infty} \frac{\sigma(n)}{n \cdot (\log n)^u} = 0$$

and so, we claim that there could be infinitely many hyper abundant numbers with any parameter $u \gtrapprox 1$.

## 2 Central Lemma

**Lemma 1** *For two large enough real numbers $y > x$:*

$$\frac{y}{x} \gg \frac{\log y}{\log x}$$

*where $\gg$ means "much greater than".*

*Proof* We have $y = x + \varepsilon$ for $\varepsilon > 0$. We obtain that

$$\begin{aligned} \frac{\log y}{\log x} &= \frac{\log(x + \varepsilon)}{\log x} \\ &= \frac{\log\left(x \cdot (1 + \frac{\varepsilon}{x})\right)}{\log x} \end{aligned}$$

$$= \frac{\log x + \log(1 + \frac{\varepsilon}{x})}{\log x}$$
$$= 1 + \frac{\log(1 + \frac{\varepsilon}{x})}{\log x}$$

and

$$\frac{y}{x} = \frac{x + \varepsilon}{x}$$
$$= 1 + \frac{\varepsilon}{x}.$$

We need to show that

$$\left(1 + \frac{\log(1 + \frac{\varepsilon}{x})}{\log x}\right) < \left(1 + \frac{\varepsilon}{x}\right)$$

which is equivalent to

$$\left(1 + \frac{\varepsilon}{x \cdot \log x}\right) < \left(1 + \frac{\varepsilon}{x}\right)$$

using the well-known inequality $\log(1+x) \leq x$ for $x > 0$. For $x$ large enough, we have

$$\frac{\varepsilon}{x} \gg \frac{\varepsilon}{x \cdot \log x}.$$

In conclusion, the inequality

$$\frac{y}{x} \gg \frac{\log y}{\log x}$$

holds on condition that $y > x$ are both large enough. □

## 3 Proof of Theorem 1

*Proof* Suppose there are not infinitely many consecutive colossally abundant numbers $3 \leq N < N'$ such that $G(N) \leq G(N')$. This implies that the inequality $G(N) \geq G(N')$ always holds for $N$ large enough when $3 \leq N < N'$ is a pair of consecutive colossally abundant numbers. That would mean the existence of a single colossally abundant number $N''$ such that $G(n) \leq G(N'')$ for all natural numbers $n > N''$ according to Proposition 1. We use the Proposition 3 to reveal that under these preconditions, then there are not infinitely many extremely abundant numbers. This implies that the Riemann hypothesis is false a consequence of Proposition 2. By contraposition, if the Riemann hypothesis is true, then there exist infinitely many consecutive colossally abundant numbers $3 \leq N < N'$ such that $G(N) \leq G(N')$.

Now, suppose that $N''$ is the greatest extremely abundant number such that $N'' < N'$ for a pair of consecutive colossally abundant numbers $3 \leq N < N'$ when $G(N) \leq G(N')$. We know that $N''$ must be a colossally abundant number by Proposition 3. By Proposition 1, we know that $G(N) \leq G(n) \leq G(N')$ when satisfying $N < n < N'$. So, if $n$ or $N$ is a extremely abundant number, then $N'$ would be extremely abundant as well by Proposition 3. Hence, we assume that there is a finite set of colossally abundant numbers $S$ such that $M \in S$ implies that $N'' < M < N'$. Let's take the greatest number $M''$ such that $M'' \in S$ and for each element $M \in S$ we have $G(M'') \geq G(M)$. Therefore, it is necessary that either $M''$ or $N'$ be an extremely abundant number. In any case, we obtain always another new extremely abundant number. Since we took the value of the colossally abundant number $N'$ into an arbitrary way, therefore if there exist infinitely many consecutive colossally abundant numbers $3 \leq N < N'$ such that $G(N) \leq G(N')$, then there exist infinitely many extremely abundant numbers. This implies that the Riemann hypothesis is true by Proposition 2 after using the modus ponens.

The result is done. □

# 4 Proof of Corollary 1

*Proof* Suppose there exists a large enough hyper abundant numbers $N'$ with a parameter $u \gtrapprox 1$. We know that $N'$ must be also a colossally abundant number. Let $N$ be the greatest colossally abundant number such that $3 \leq N < N'$, which means that $N$ and $N'$ is a pair of consecutive colossally abundant numbers. By definition of hyper abundant, we have

$$\frac{\sigma(N')}{N' \cdot (\log N')^u} \geq \frac{\sigma(N)}{N \cdot (\log N)^u}$$

which is the same as

$$\frac{\sigma(N') \cdot (\log N)^u}{N' \cdot (\log N')^u \cdot \log\log N} \geq \frac{\sigma(N)}{N \cdot \log\log N} = G(N).$$

Hence, it is enough to show that

$$G(N') = \frac{\sigma(N')}{N' \cdot \log\log N'} \geq \frac{\sigma(N') \cdot (\log N)^u}{N' \cdot (\log N')^u \cdot \log\log N}$$

which is equivalent to

$$\frac{(\log N')^u}{(\log N)^u} \geq \frac{\log\log N'}{\log\log N}.$$

Since $u \gtrapprox 1$, then we only need to show that the inequality

$$\frac{\log N'}{\log N} \gg \frac{\log\log N'}{\log\log N}.$$

holds on condition that $\log N' > \log N$ are both large enough by Lemma 1. Consequently, this arbitrary large enough hyper abundant numbers $N'$ with a parameter $u \gtrapprox 1$ reveals that $G(N) \leq G(N')$ holds on anyway. In this way, if there exist infinitely many hyper abundant numbers $N'$ with any parameter $u \gtrapprox 1$, then there are infinitely many consecutive colossally abundant numbers $3 \leq N < N'$ such that $G(N) \leq G(N')$.

Finally, the proof is complete by Theorem 1. □

# References

[1] J.L. Nicolas, G. Robin, Highly Composite Numbers by Srinivasa Ramanujan. The Ramanujan Journal **1**(2), 119–153 (1997). https://doi.org/10.1023/A:1009764017495

[2] L. Alaoglu, P. Erdős, On Highly Composite and Similar Numbers. Transactions of the American Mathematical Society **56**(3), 448–469 (1944). https://doi.org/10.2307/1990319

[3] J.L. Nicolas, Some Open Questions. The Ramanujan Journal **9**(1), 251–264 (2005). https://doi.org/10.1007/s11139-005-0836-2

[4] T.H. Grönwall, Some Asymptotic Expressions in the Theory of Numbers. Transactions of the American Mathematical Society **14**(1), 113–122 (1913). https://doi.org/10.2307/1988773

[5] G. Robin, Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann. J. Math. pures appl **63**(2), 187–213 (1984)

[6] S. Nazardonyavi, S.B. Yakubovich, Extremely Abundant Numbers and the Riemann Hypothesis. J. Integer Seq. **17**(2), 14–2 (2014)

[7] J.B. Conrey, The Riemann Hypothesis. Notices of the AMS **50**(3), 341–353 (2003)