Properties of the Robin's Inequality

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ABSTRACT. In mathematics, the Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. Many consider it to be the most important unsolved problem in pure mathematics. The Robin's inequality consists in $\sigma(n) < e^{\gamma} \times n \times \ln\ln n$ where $\sigma(n)$ is the divisor function and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. The Robin's inequality is true for every natural number $n > 5040$ if and only if the Riemann hypothesis is true. We prove the Robin's inequality is true for every natural number $n > 5040$ when n is not divisible by 3. More precisely: every possible counterexample $n > 5040$ of the Robin's inequality must comply that n should be divisible by $2^{20} \times 3^{13}$.

1. Introduction

In mathematics, the Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. Many consider it to be the most important unsolved problem in pure mathematics [2]. It is of great interest in number theory because it implies results about the distribution of prime numbers [2]. It was proposed by Bernhard Riemann (1859), after whom it is named [2]. It is one of the seven Millennium Prize Problems selected by the Clay Mathematics Institute to carry a US 1,000,000 prize for the first correct solution [2].

The divisor function $\sigma(n)$ for a natural number n is defined as the sum of the powers of the divisors of n

$$\sigma(n) = \sum_{k \mid n} k$$

where $k \mid n$ means that the natural number k divides n [5]. In 1915, Ramanujan proved that under the assumption of the Riemann hypothesis, the inequality

$$\sigma(n) < e^{\gamma} \times n \times \ln\ln n$$

holds for all sufficiently large n, where $\gamma \approx 0.57721$ is the Euler-Mascheroni constant [2]. The largest known value that violates the inequality is $n = 5040$. In 1984, Guy Robin proved that the inequality is true for all $n > 5040$ if and only if the Riemann hypothesis is true [2]. Using this inequality, we show an interesting result.

2. Results

Theorem 2.1. *Given a natural number*

$$n = p_1^{a_1} \times p_2^{a_2} \times \ldots \times p_m^{a_m}$$

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such that $p_1, p_2, \ldots, p_m$ are prime numbers, then we obtain the following inequality

$$\frac{\sigma(n)}{n} < \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{p_i + 1}{p_i}.$$

Proof. From the article reference [1], we know that

$$\frac{\sigma(n)}{n} < \prod_{i=1}^{m} \frac{p_i}{p_i - 1}. \tag{2.1}$$

We can easily prove that

$$\prod_{i=1}^{m} \frac{p_i}{p_i - 1} = \prod_{i=1}^{m} \frac{1}{1 - p_i^{-2}} \times \prod_{i=1}^{m} \frac{p_i + 1}{p_i}.$$

However, we know that

$$\prod_{i=1}^{m} \frac{1}{1 - p_i^{-2}} < \prod_{j=1}^{\infty} \frac{1}{1 - p_j^{-2}}$$

where p_j is the j^{th} prime number and

$$\prod_{j=1}^{\infty} \frac{1}{1 - p_j^{-2}} = \frac{\pi^2}{6}$$

as a consequence of the result in the Basel problem [5]. Consequently, we obtain that

$$\frac{\sigma(n)}{n} < \prod_{i=1}^{m} \frac{p_i}{p_i - 1} < \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{p_i + 1}{p_i}.$$

□

Theorem 2.2. *For $x \geq 11$, we have*

$$\sum_{p \leq x} \frac{1}{p} < \ln \ln x + \gamma - 0.12$$

where $p \leq x$ means all the primes lesser than or equal to x.

Proof. For $x > 1$, we have

$$\sum_{p \leq x} \frac{1}{p} < \ln \ln x + B + \frac{1}{\ln^2 x}$$

where

$$B = 0.2614972128\ldots$$

is the (Meissel-)Mertens constant, since this is a proven result from the article reference [4]. This is the same as

$$\sum_{p \leq x} \frac{1}{p} < \ln \ln x + \gamma - (C - \frac{1}{\ln^2 x})$$

where $\gamma - B = C > 0.31$, because of $\gamma > B$. If we analyze $(C - \frac{1}{\ln^2 x})$, then this complies with

$$(C - \frac{1}{\ln^2 x}) > (0.31 - \frac{1}{\ln^2 11}) > 0.12$$

for $x \geq 11$ and thus, we finally prove that

$$\sum_{p \leq x} \frac{1}{p} < \ln \ln x + \gamma - (C - \frac{1}{\ln^2 x}) < \ln \ln x + \gamma - 0.12.$$

$\square$

Definition 2.3. We recall that an integer n is said to be squarefree if for every prime divisor p of n we have $p^2 \nmid n$, where $p^2 \nmid n$ means that p^2 does not divide n [1].

Theorem 2.4. *Given a squarefree number*

$$n = q_1 \times \ldots \times q_m$$

such that $q_1, q_2, \ldots, q_m$ *are odd prime numbers, the greatest prime divisor of* n *is greater than* 7 *and* $3 \nmid n$*, then we obtain the following inequality*

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \sigma(n) \leq e^{\gamma} \times n \times \ln \ln(2^{19} \times n).$$

Proof. This proof is very similar with the demonstration in Theorem 1.1 from the article reference [1]. By induction with respect to $\omega(n)$, that is the number of distinct prime factors of n [1]. Put $\omega(n) = m$ [1]. We need to prove the assertion for those integers with $m = 1$. From a squarefree number n, we obtain that

$$\sigma(n) = (q_1 + 1) \times (q_2 + 1) \times \ldots \times (q_m + 1) \tag{2.2}$$

when $n = q_1 \times q_2 \times \ldots \times q_m$ [1]. In this way, for every prime number $p_i \geq 11$, then we need to prove that

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (1 + \frac{1}{p_i}) \leq e^{\gamma} \times \ln \ln(2^{19} \times p_i). \tag{2.3}$$

For $p_i = 11$, we have that

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (1 + \frac{1}{11}) \leq e^{\gamma} \times \ln \ln(2^{19} \times 11)$$

is actually true. For another prime number $p_i > 11$, we have that

$$(1 + \frac{1}{p_i}) < (1 + \frac{1}{11})$$

and

$$\ln \ln(2^{19} \times 11) < \ln \ln(2^{19} \times p_i)$$

which clearly implies that the inequality (2.3) is true for every prime number $p_i \geq 11$. Now, suppose it is true for $m - 1$, with $m \geq 2$ and let us consider the assertion for those squarefree n with $\omega(n) = m$ [1]. So let $n = q_1 \times \ldots \times q_m$ be a squarefree number and assume that $q_1 < \ldots < q_m$ for $q_m \geq 11$.

Case 1: $q_m \geq \ln(2^{19} \times q_1 \times \ldots \times q_{m-1} \times q_m) = \ln(2^{19} \times n)$.

By the induction hypothesis we have

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (q_1 + 1) \times \ldots \times (q_{m-1} + 1) \leq e^{\gamma} \times q_1 \times \ldots \times q_{m-1} \times \ln \ln(2^{19} \times q_1 \times \ldots \times q_{m-1})$$

and hence

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (q_1 + 1) \times \ldots \times (q_{m-1} + 1) \times (q_m + 1) \leq$$
$$e^{\gamma} \times q_1 \times \ldots \times q_{m-1} \times (q_m + 1) \times \ln \ln(2^{19} \times q_1 \times \ldots \times q_{m-1})$$

when we multiply the both sides of the inequality by $(q_m + 1)$. We want to show that

$$e^{\gamma} \times q_1 \times \ldots \times q_{m-1} \times (q_m + 1) \times \ln\ln(2^{19} \times q_1 \times \ldots \times q_{m-1}) \leq$$

$$e^{\gamma} \times q_1 \times \ldots \times q_{m-1} \times q_m \times \ln\ln(2^{19} \times q_1 \times \ldots \times q_{m-1} \times q_m) = e^{\gamma} \times n \times \ln\ln(2^{19} \times n).$$

Indeed the previous inequality is equivalent with

$$q_m \times \ln\ln(2^{19} \times q_1 \times \ldots \times q_{m-1} \times q_m) \geq (q_m + 1) \times \ln\ln(2^{19} \times q_1 \times \ldots \times q_{m-1})$$

or alternatively

$$\frac{q_m \times (\ln\ln(2^{19} \times q_1 \times \ldots \times q_{m-1} \times q_m) - \ln\ln(2^{19} \times q_1 \times \ldots \times q_{m-1}))}{\ln q_m} \geq$$

$$\frac{\ln\ln(2^{19} \times q_1 \times \ldots \times q_{m-1})}{\ln q_m}.$$

From the reference [1], we have that if $0 < a < b$, then

$$\frac{\ln b - \ln a}{b - a} = \frac{1}{(b-a)} \int_a^b \frac{dt}{t} > \frac{1}{b}. \tag{2.4}$$

We can apply the inequality (2.4) to the previous one just using $b = \ln(2^{19} \times q_1 \times \ldots \times q_{m-1} \times q_m)$ and $a = \ln(2^{19} \times q_1 \times \ldots \times q_{m-1})$. Certainly, we have that

$$\ln(2^{19} \times q_1 \times \ldots \times q_{m-1} \times q_m) - \ln(2^{19} \times q_1 \times \ldots \times q_{m-1}) =$$

$$\ln \frac{2^{19} \times q_1 \times \ldots \times q_{m-1} \times q_m}{2^{19} \times q_1 \times \ldots \times q_{m-1}} = \ln q_m.$$

In this way, we obtain that

$$\frac{q_m \times (\ln\ln(2^{19} \times q_1 \times \ldots \times q_{m-1} \times q_m) - \ln\ln(2^{19} \times q_1 \times \ldots \times q_{m-1}))}{\ln q_m} >$$

$$\frac{q_m}{\ln(2^{19} \times q_1 \times \ldots \times q_m)}.$$

Using this result we infer that the original inequality is certainly satisfied if the next inequality is satisfied

$$\frac{q_m}{\ln(2^{19} \times q_1 \times \ldots \times q_m)} \geq \frac{\ln\ln(2^{19} \times q_1 \times \ldots \times q_{m-1})}{\ln q_m}$$

which is trivially true for $q_m \geq \ln(2^{19} \times q_1 \times \ldots \times q_{m-1} \times q_m)$ [1].

Case 2: $q_m < \ln(2^{19} \times q_1 \times \ldots \times q_{m-1} \times q_m) = \ln(2^{19} \times n)$.

We need to prove that

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \frac{\sigma(n)}{n} \leq e^{\gamma} \times \ln\ln(2^{19} \times n).$$

We know that $\frac{3}{2} < 1.503 < \frac{4}{2.66}$. Nevertheless, we could have that

$$\frac{3}{2} \times \frac{\sigma(n)}{n} \times \frac{\pi^2}{6} < \frac{4 \times \sigma(n)}{3 \times n} \times \frac{\pi^2}{2 \times 2.66}$$

and therefore, we only need to prove that

$$\frac{\sigma(3 \times n)}{3 \times n} \times \frac{\pi^2}{5.32} \leq e^{\gamma} \times \ln\ln(2^{19} \times n)$$

where this is possible because of $3 \nmid n$. If we apply the logarithm to the both sides of the inequality, then we obtain that

$$\ln(\frac{\pi^2}{5.32}) + (\ln(3+1) - \ln 3) + \sum_{j=i}^{m}(\ln(q_j + 1) - \ln q_j) \leq$$

$$\gamma + \ln\ln\ln(2^{19} \times n).$$

From the reference [1], we note that

$$\ln(p_1 + 1) - \ln p_1 = \int_{p_1}^{p_1+1} \frac{dt}{t} < \frac{1}{p_1}.$$

In addition, note that $\ln(\frac{\pi^2}{5.32}) < \frac{1}{2} + 0.12$. It is enough to prove that

$$0.12 + \frac{1}{2} + \frac{1}{3} + \frac{1}{q_1} + \ldots + \frac{1}{q_m} \leq 0.12 + \sum_{p \leq q_m} \frac{1}{p} \leq \gamma + \ln\ln\ln(2^{19} \times n)$$

where $q_m \geq 11$. However, we know that

$$\gamma + \ln\ln q_m < \gamma + \ln\ln\ln(2^{19} \times n)$$

since $q_m < \ln(2^{19} \times n)$ and therefore, we only need to prove that

$$\sum_{p \leq q_m} \frac{1}{p} \leq \gamma + \ln\ln q_m - 0.12$$

which is true according to the Theorem 2.2 when $q_m \geq 11$. In this way, we finally show the Theorem is indeed satisfied. $\square$

Theorem 2.5. *Given a natural number*

$$n = 2^{a_1} \times 3^{a_2} \times 5^{a_3} \times 7^{a_4} > 5040$$

such that $a_1, a_2, a_3, a_4 \geq 0$ are integers, then the Robin's inequality is true for n.

Proof. Given a natural number $n = p_1^{a_1} \times p_2^{a_2} \times \ldots \times p_m^{a_m} > 5040$ such that $p_1, p_2, \ldots, p_m$ are prime numbers, we need to prove that

$$\frac{\sigma(n)}{n} < e^{\gamma} \times \ln\ln n$$

that is true when

$$\prod_{i=1}^{m} \frac{p_i}{p_i - 1} < e^{\gamma} \times \ln\ln n$$

according to the inequality (2.1). Given a natural number $n = 2^{a_1} \times 3^{a_2} \times 5^{a_3} > 5040$ such that $a_1, a_2, a_3 \geq 0$ are integers, we have that

$$\prod_{i=1}^{m} \frac{p_i}{p_i - 1} \leq \frac{2 \times 3 \times 5}{1 \times 2 \times 4} = 3.75 < e^{\gamma} \times \ln\ln(5040) \approx 3.81.$$

However, we know for $n > 5040$ that

$$e^{\gamma} \times \ln\ln(5040) < e^{\gamma} \times \ln\ln n$$

and therefore, the proof is completed for that case. Hence, we only need to prove the Robin's inequality for every natural number $n = 2^{a_1} \times 3^{a_2} \times 5^{a_3} \times 7^{a_4} > 5040$ such that $a_1, a_2, a_3 \geq 0$ and $a_4 \geq 1$ are integers. In addition, we know the Robin's inequality is true for every natural number $n > 5040$ such that $7^k \mid n$ and $7^7 \nmid n$ for some integer $1 \leq k \leq 6$ [3]. Therefore, we need to prove this case for those natural numbers $n > 5040$ such that $7^7 \mid n$. In this way, we have that

$$\prod_{i=1}^{m} \frac{p_i}{p_i - 1} \leq \frac{2 \times 3 \times 5 \times 7}{1 \times 2 \times 4 \times 6} = 4.375 < e^{\gamma} \times \ln\ln(7^7) \approx 4.65.$$

However, we know for $n > 5040$ and $7^7 \mid n$ that

$$e^{\gamma} \times \ln\ln(7^7) \leq e^{\gamma} \times \ln\ln n$$

and as a consequence, the proof is completed. □

Theorem 2.6. *The Robin's inequality is true for every natural number $n > 5040$ when $3 \nmid n$. More precisely: every possible counterexample $n > 5040$ of the Robin's inequality must comply that $(2^{20} \times 3^{13}) \mid n$.*

Proof. We will check the Robin's inequality for every natural number $n = p_1^{a_1} \times p_2^{a_2} \times \ldots \times p_m^{a_m} > 5040$ such that $p_1, p_2, \ldots, p_m$ are prime numbers and $3 \nmid n$. We know this is true when the greatest prime divisor of $n > 5040$ is lesser than or equal to 7 according to the Theorem 2.5. Therefore, the remaining case is when the greatest prime divisor of $n > 5040$ is greater than 7. We need to prove that

$$\frac{\sigma(n)}{n} < e^{\gamma} \times \ln\ln n$$

that is true when

$$\frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{p_i + 1}{p_i} < e^{\gamma} \times \ln\ln n$$

according to Theorem 2.1. Using the equation (2.2), we obtain that will be equivalent to

$$\frac{\pi^2}{6} \times \frac{\sigma(n')}{n'} < e^{\gamma} \times \ln\ln n$$

where $n' = q_1 \times \ldots \times q_m$ is the squarefree kernel of n [1]. However, the Robin's inequality has been proved for all integers n not divisible by 2 (which are bigger than 10) [1]. Hence, we only need to prove the Robin's inequality when $2 \mid n'$. In addition, we know the Robin's inequality is true for every natural number $n > 5040$ such that $2^k \mid n$ and $2^{20} \nmid n$ for some integer $1 \leq k \leq 19$ [3]. Consequently, we only need to prove the Robin's inequality for all $n > 5040$ such that $2^{20} \mid n$ and thus,

$$e^{\gamma} \times n' \times \ln\ln(2^{19} \times \frac{n'}{2}) < e^{\gamma} \times n' \times \ln\ln n$$

because of $2^{19} \times \frac{n'}{2} < n$ when $2^{20} \mid n$ and $2 \mid n'$. In this way, we only need to prove that

$$\frac{\pi^2}{6} \times \sigma(n') \leq e^{\gamma} \times n' \times \ln\ln(2^{19} \times \frac{n'}{2}).$$

According to the equation (2.2) and $2 \mid n'$, we have that

$$\frac{\pi^2}{6} \times 3 \times \sigma(\frac{n'}{2}) \leq e^{\gamma} \times 2 \times \frac{n'}{2} \times \ln\ln(2^{19} \times \frac{n'}{2})$$

which is the same as

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \sigma(\frac{n'}{2}) \leq e^{\gamma} \times \frac{n'}{2} \times \ln\ln(2^{19} \times \frac{n'}{2})$$

that is true according to the Theorem 2.4. In addition, we know the Robin's inequality is true for every natural number $n > 5040$ such that $3^k \mid n$ and $3^{13} \nmid n$

for some integer $1 \leq k \leq 12$ [3]. Consequently, we only need to prove the Robin's inequality for all $n > 5040$ such that $2^{20} \mid n$ and $3^{13} \mid n$. To sum up, the proof is completed. $\square$

3. Conclusions

The practical uses of the Riemann hypothesis include many propositions known true under the Riemann hypothesis, and some that can be shown equivalent to the Riemann hypothesis [2]. Certainly, the Riemann hypothesis is close related to various mathematical topics such as the distribution of prime numbers, the growth of arithmetic functions, the Lindelöf hypothesis, the large prime gap conjecture, etc [2]. Indeed, a proof of the Riemann hypothesis could spur considerable advances in many mathematical areas, such as the number theory and pure mathematics [2]. In this way, this work represents a new step forward in the efforts of trying to prove the Riemann hypothesis.

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