Hilbert-Style Axiomatizations of Disjunctive and Implicative Finitely-Valued Logics with Equality Determinant

Alexej Pynko

August 19, 2020

Alexej P. Pynko

HILBERT-STYLE AXIOMATIZATIONS OF DISJUNCTIVE AND IMPLICATIVE FINITELY-VALUED LOGICS WITH EQUALITY DETERMINANT

Abstract

Here, we develop a unversal method of [effective] constructing a [finite] Hilbert-style axiomatization of the logic of a given finite disjunctive/implicative matrix with equality determinant (in particular, any/implicative four-valued expansion of Belnap's logic) [and finitely many connectives].

Keywords: [disjunctive/implicative] logic, [disjunctive/implicative] matrix, deduction theorem, Peirce Law, Belnap's four-valued logic, expansion, equality determinant, [{purely} single/multi-conclusion|premise] sequent (calculus).

1. Introduction

The general study [10] has suggested a universal method of [effective] constructing a multi-conclusion sequent calculus with structural rules and Cut Elimination Property for a given finite matrix with equality determinant [and finitely many connectives] (in particular, any four-valued expansion of Belnap's logic; cf. [1]). In this paper, providing the matrix involved is disjunctive (that equally covers four-valued expansions of Belnap's logic), we advance the mentioned study by [effective] transforming the calculus constructed therein to a [finite] Hilbert-style axiomatization of the logic of the matrix through intermediate equivalent axiomatic extensions of the single- and multi-conclusion sequent calculi constituted by merely structural rules and classical rules for disjunction (cf. [3]).

The rest of the paper is as follows. We entirely follow the standard conventions (as for Hilbert-style calculi) as well as those adopted in both [9] and [10] — as to sequent calculi. Section 2 is a concise summary of mainly those basic issues underlying the paper, which have proved beyond the scopes of the mentioned papers, those presented therein being normally (though not entirely) briefly summarized as well for the exposition to be properly self-contained. In Section 3 we present a uniform formalism for covering both Hilbert- and Gentzen-style calculi, and recall some key results concerning disjunctive logics (mainly belonging to a logical folklore) and sequent calculi with structural rules going back to [9]. Then, Section 4 is a preliminary study of minimal disjunctive Hilbert- as well as Gentzen-style (both multi- and single-conclusion) calculi to be used further. Section 5 then contains the main generic results of the paper. Finally, in Section 6 we apply it to disjunctive and implicative positive fragments of the classical logic as well as to four-valued expansions of Belnap's logic.

2. Basic issues

2.1. Set-theoretical background

We follow the standard set-theoretical convention, according to which natural numbers (including 0) are treated as finite ordinals (viz., sets of lesser natural numbers), the ordinal of all them being denoted by ω. The proper class of all ordinals is denoted by ∞. Likewise, functions are viewed as binary relations. In addition, singletons are often identified with their unique elements, unless any confusion is possible.

Given a set S, the set of all subsets of S [of cardinality $\in K \subseteq \infty$] is denoted by $\wp_{[K]}(S)$. Next, S-tuples (viz., functions with domain S) are often written in either sequence $\bar{t}$ or vector $\vec{t}$ forms, its s-th component (viz., the value under argument s), where $s \in S$, being written as either t_s or t^s. As usual, given two more sets A and B, any relation between them is identified with the equally-denoted relation between A^S and B^S defined point-wise. Further, elements of $S^* \triangleq (S^0 \cup S^+)$, where $S^+ \triangleq (\bigcup_{i\in(\omega\setminus 1)} S^i)$, are identified with ordinary finite tuples/[comma separated] sequences. Then, any binary operation $\diamond$ on S determines the equally-denoted mapping $\diamond : S^+ \to S$ as follows: by induction on the length $l = (\mathrm{dom}\,\bar{a})$ of any $\bar{a} \in S^+$, put:

$$\diamond\bar{a} \triangleq \begin{cases} a_0 & \text{if } l = 1, \\ (\diamond(\bar{a}{\restriction}(l-1))) \diamond a_{l-1} & \text{otherwise.} \end{cases}$$

Given any $f : S \to S$, put $f^1 \triangleq f$ and $f^0 \triangleq \Delta_S \triangleq \{\langle s, s\rangle \mid s \in S\}$, functions of the latter kind being said to be *diagonal*.

Let A be a set. A $U \subseteq \wp(A)$ is said to be *upward-directed*, provided, for every $S \in \wp_\omega(U)$, there is some $T \in U$ such that $(\bigcup S) \subseteq T$. An *operator over* A is any unary operation O on $\wp(A)$. This is said to be *(monotonic) [idempotent]* $\{$*transitive*$\}$ $\langle$*inductive/finitary/compact*$\rangle$, provided, for all $(B,)D \in \wp(A)\langle$resp., any upward-directed $U \subseteq \wp(A)\rangle$, it holds that $(O(B))[D]\{O(O(D)\} \subseteq O(D)\langle O(\bigcup U) \subseteq \bigcup O[U]\rangle$. A *closure operator over* A is any monotonic idempotent transitive operator C over A.

2.1.1. Disjunctivity versus multiplicativity

Fix any set A and any $\delta : A^2 \to A$. Given any $X, Y \subseteq A$, set $\delta(X, Y) \triangleq \delta[X \times Y]$. Then, a closure operator C over A is said to be *[K-]δ-multiplicative*, where $K \subseteq \infty$, provided

$$\delta(C(X \cup Y), a) \subseteq C(X \cup \delta(Y, a)), \tag{2.1}$$

for all $(X \cup \{a\}) \subseteq A$ and all $Y \in \wp_{[K]}(A)$.[1] Next, C is said to be *δ-disjunctive*, provided, for all $a, b \in A$ and every $Z \subseteq A$, it holds that

$$C(Z \cup \{\delta(a, b)\}) = (C(Z \cup \{a\}) \cap C(Z \cup \{b\})), \tag{2.2}$$

in which case the following clearly hold, by (2.2) with $Z = \varnothing$:

$$\begin{aligned}
\delta(a, b) &\in C(a), && (2.3)\\
\delta(a, b) &\in C(b), && (2.4)\\
a &\in C(\delta(a, a)), && (2.5)\\
\delta(b, a) &\in C(\delta(a, b)), && (2.6)\\
C(\delta(\delta(a, b), c)) &= C(\delta(a, \delta(b, c))), && (2.7)
\end{aligned}$$

for all $a, b, c \in A$.

LEMMA 2.1. *Let C be a[n inductive] closure operator over A. Then,* (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii)$\Leftarrow$ [$\Leftrightarrow$](iv)*, where:*

(i) *C is δ-disjunctive;*

(ii) (2.3)*,* (2.5) *and* (2.6) *(as well as* (2.7)*) hold and C is singularly-δ-multiplicative;*

[1] In this connection, "finitely-/singularly-" means "ω-/$\{1\}$-", respectively.

(iii) (2.3)*,* (2.5) *and* (2.6) *(as well as* (2.7)*) hold and C is finitely-δ-multiplicative;*

(iv) (2.3)*,* (2.5) *and* (2.6) *(as well as* (2.7)*) hold and C is δ-multiplicative.*

PROOF: First, (ii/iii) is a particular case of (iii/iv), respectively. [Next, (iii)$\Rightarrow$(iv) is by the inductivity of C.]

Further, assume (i) holds. Consider any $(X \cup \{a, b\}) \subseteq A$ and any $c \in C(X \cup \{b\})$, in which case $\delta(c, a) \in C(X \cup \{b\})$, by (2.3). Moreover, by (2.4), we also have $\delta(c, a) \in C(X \cup \{a\})$. Thus, by (2.2), we get $\delta(c, a) \in (C(X \cup \{b\}) \cup C(X \cup \{a\}) = C(X \cup \{\delta(b, a)\})$. In this way, (ii) holds.

Finally, assume (ii) without (2.7) holds.

In that case, both (2.3) and so, by (2.6), (2.4) hold, and so the inclusion from left to right in (2.2). Conversely, consider any $c \in (C(X \cup \{b\}) \cup C(X \cup \{a\})$. Then, by (2.6) and (2.1) with $Y = \{a\}$ and b instead of a, we have $\delta(b, c) \in C(X \cup \{\delta(a, b)\})$. Likewise, by (2.5) and (2.1) with $Y = \{b\}$ and c instead of a, we have $c \in C(X \cup \{\delta(b, c)\})$. Therefore, we eventually get $c \in C(X \cup \{\delta(a, b)\})$. Thus, (i) holds.

Now, assume (2.7) holds too. By induction on any $n \in \omega$, let us show that C is n-δ-multiplicative. For consider any $(X \cup \{a\}) \subseteq A$, any $Y \in \wp_n(A)$, in which case $n \neq 0$, and any $b \in C(X \cup Y)$. In case $Y = \varnothing$, (2.1) is by (2.3). Otherwise, take any $c \in Y$, in which case $Y' \triangleq (Y \setminus \{c\}) \in \wp_{n-1}(A)$, and put $X' \triangleq (X \cup \{c\}) \subseteq A$, in which case $(X' \cup Y') = (X \cup Y)$, and so $b \in C(X' \cup Y')$. Hence, by induction hypothesis, we get $\delta(b, a) \in C(X' \cup \delta(Y', a))$. Therefore, since C is singularly-δ-multiplicative, we then get $\delta(\delta(b, a), a) \in C(X \cup \delta(Y, a))$ as well as both $\delta(\delta(a, b), a) \in C(\delta(\delta(b, a), a))$, in view of (2.6), and $\delta(a, b) \in C(\delta(\delta(a, a), b))$, in view of (2.5). In this way, by (2.6) and (2.7), we eventually get $\delta(b, a) \in C(X \cup \delta(Y, a))$, as required. Thus, as $(\bigcup \omega) = \omega$, we conclude that C is finitely-δ-multiplicative, and so (iii) holds, as required. □

2.2. Algebraic background

Unless otherwise specified, throughout the paper, we deal with a fixed but arbitrary signature Σ of connectives of finite arity to be treated as function symbols.

Given any $\alpha \in \wp_{\infty \setminus 1}(\omega)$, $\mathfrak{Fm}^{\alpha}_{\Sigma}$ denotes the absolutely free Σ-algebra freely-generated by the set $V_\alpha \triangleq \{x_i \mid i \in \alpha\}$ of *variables*, its endomorphisms/elements of its carrier $\mathrm{Fm}^{\alpha}_{\Sigma}$ being called Σ-*substitutions/formulas*, in case $\alpha = \omega$. The finite set of all variables actually occurring in a $\varphi \in \mathrm{Fm}^{\omega}_{\Sigma}$ is denoted by $\mathrm{Var}(\varphi)$.

As usual, (logical) Σ-matrices (cf. [4]) are treated as first-order model structures (viz., algebraic systems; cf. [5]) of the first-order signature $\Sigma \cup \{D\}$ with unary *truth* predicate D,[2] any Σ-matrix $\mathcal{A}$ being traditionally identified with the couple $\langle \mathfrak{A}, D^{\mathcal{A}} \rangle$.

2.2.1. Equality determinants for matrices

According to [10], an *equality determinant for* a Σ-matrix $\mathcal{A}$ is any $\Upsilon \subseteq \mathrm{Fm}^{1}_{\Sigma}$ such that any $a, b \in A$ are equal, whenever, for each $\upsilon \in \Upsilon$, $\upsilon^{\mathfrak{A}}(a) \in D^{\mathcal{A}}$ iff $\upsilon^{\mathfrak{A}}(b) \in D^{\mathcal{A}}$.

3. Abstract propositional languages and calculi

A(n) *(abstract) Σ-[propositional]language* is any triple of the form $L = \langle \mathrm{Fm}_L, \Im_L, \mathrm{Var}_L \rangle$, where Fm_L is a set, whose elements are called *L-formulas*, $\Im_L : \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{Fm}^{\omega}_{\Sigma}) \to (\mathrm{Fm}_L)^{\mathrm{Fm}_L}$, preserving compositions and diagonality, any Σ-substitution σ being naturally identified with $\Im_L(\sigma)$, unless any confusion is possible, and $\mathrm{Var}_L : \mathrm{Fm}_L \to \wp_{\omega}(V_{\omega})$ (the language subscript is normally omitted, unless any confusion is possible) such that, for every $\Phi \in \mathrm{Fm}_L$ and any Σ-substitutions σ and ς such that $(\sigma \upharpoonright \mathrm{Var}_L(\Phi)) = (\varsigma \upharpoonright \mathrm{Var}_L(\Phi))$, it holds that $\sigma(\Phi) = \varsigma(\Phi)$.

Then, elements/subsets of $\mathrm{Ru}_L \triangleq (\wp_{\omega}(\mathrm{Fm}_L) \times \mathrm{Fm}_L)$ are referred to as *L-rules/calculi*, any L-rule $\mathcal{R} = \langle \Gamma, \Phi \rangle$ being normally written in the conventional fraction either displayed $\dfrac{\Gamma}{\Phi}$ or non-displayed Γ/Φ form, Φ/any element of Γ being called the/a *conclusion/premise of* $\mathcal{R}$, rules of the form Φ/Ψ, where $\Psi \in \Gamma$, being said to be *inverse to* $\mathcal{R}$. As usual, L-rules without premises are called *L-axioms* and are identified with their conclusions, calculi consisting of merely axioms being said to be *axiomatic*. In general, any function f with domain Fm_L (including Σ-substitutions) but Var_L determines the equally-denoted function with domain Ru_L as follows: for any $\mathcal{R} = \langle \Gamma, \Phi \rangle \in \mathrm{Ru}_L$, we set $f(\mathcal{R}) \triangleq \langle f[\Gamma], f(\Phi) \rangle$, whereas putting

[2] In general, [Σ-matrices are denoted by Calligraphic letters (possibly, with indices), their *underlying*] algebras [viz., Σ-reducts] being denoted by [corresponding] Fraktur letters (possibly, with [same] indices [if any]), their carriers being denoted by corresponding Italic letters (with same indices, if any).

$\mathrm{Var}_L(\mathcal{R}) \triangleq (\mathrm{Var}_L(\Phi) \cup \bigcup \mathrm{Var}_L[\Gamma]) \in \wp_\omega(V_\omega)$. (In this way, Ru_L actually forms a Σ-language.)

Next, an *L-logic* is any closure operator C on Fm_L that is *structural* in the sense that, for every Σ-substitution σ and all $\Gamma \subseteq \mathrm{Fm}_L$, it holds that $\sigma[C(\Gamma)] \subseteq C(\sigma[\Gamma])$. This is said to *satisfy* an L-rule Γ/Φ, whenever $\Phi \in C(\Gamma)$. Then, an L-logic C' is said to be an *extension of* C, provided $C \subseteq C'$. In that case, an L-calculus $\mathcal{C}$ is said to *axiomatize* C' *relatively to* C, provided C' is the least extension of C satisfying each rule in $\mathcal{C}$.

Further, an L-rule Γ/Φ is said to be *derivable in* an L-calculus $\mathcal{C}$, if there is a $\mathcal{C}$-*derivation of* it, i.e., a proof of Φ (in the conventional proof-theoretical sense) by means of axioms in Γ (as *hypotheses*) and rules in the set $\mathrm{SI}_\Sigma(\mathcal{C}) \triangleq \{\sigma(\mathcal{R}) \mid \mathcal{R} \in \mathcal{C}, \sigma \in \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{Fm}^\omega_\Sigma)\}$ of all *substitutional Σ-instances of* rules in $\mathcal{C}$. The extension $\mathrm{Cn}_\mathcal{C}$ of the diagonal Σ-logic relatively axiomatized by $\mathcal{C}$ is called the *consequence of* $\mathcal{C}$ and said to be *axiomatized by* $\mathcal{C}$, in which case it is inductive and satisfies any L-rule iff this is derivable in $\mathcal{C}$. (Conversely, any inductive L-logic is axiomatized by the set of all L-rules satisfied in it to be identified with the logic, in which case inductive L-logics become actually particular cases of L-calculi.) An $S \subseteq \mathrm{Fm}^\omega_\Sigma$ is said to be $\mathcal{C}$-*closed*, if, for every $(\Gamma/\Phi) \in \mathrm{SI}_\Sigma(\mathcal{C})$, it holds that $(\Gamma \subseteq S) \Rightarrow (\Phi \in S)$, in which case $\mathrm{Cn}_\mathcal{C}(\varnothing) \subseteq S$.

3.1. Hilbert-style calculi

The Σ-language $\mathcal{H}_\Sigma$ with first component $\mathrm{Fm}^\omega_\Sigma$, the diagonal second component and the third component Var is called the *Hilbert-style/sentential Σ-language*, $\mathcal{H}_\Sigma$-rules/axioms/calculi/logics being traditionally referred to as *(Hilbert-style/sentential) Σ-rules/axioms/calculi/logics* (cf., e.g., [4]).

From the model-theoretic point of view, any Σ-rule Γ/ϕ is viewed as the first-order basic Horn formula $(\bigwedge \Gamma) \to \phi$ under the standard identification of any Σ-formula ψ with the first-order atomic formula $D(\psi)$ we follow tacitly.

Given any class M of Σ-matrices, we have the Σ-logic Cn_M *of/defined by* it, given by

$$\mathrm{Cn}_\mathsf{M}(X) \triangleq (\mathrm{Fm}^\omega_\Sigma \cap \bigcap \{h^{-1}[D^\mathcal{A}] \supseteq X \mid \mathcal{A} \in \mathsf{M}, h \in \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{A})\}),$$

for all $X \subseteq \mathrm{Fm}^\omega_\Sigma$. (Due to [4], this is well known to be inductive, whenever both M and all members of it are finite.)

A Σ-matrix $\mathcal{A}$ is said to be $\diamond$-*disjunctive/implicative*, where $\diamond$ is a (possibly, secondary) binary connective of Σ, whenever, for all $a, b \in A$, it holds that $((a \in / \notin D^{\mathcal{A}})|(b \in D^{\mathcal{A}})) \Leftrightarrow ((a \diamond^{\mathfrak{A}} b) \in D^{\mathcal{A}})$, in which case it is $\veebar_\diamond$-disjunctive, where $(x_0 \veebar_\diamond x_1) \triangleq ((x_0 \diamond x_1) \diamond x_1)$.

3.1.1. Disjunctive sentential logics

Throughout the rest of the paper, unless otherwise specified, $\veebar$ is supposed to be any (possibly, secondary) binary connective of Σ.

LEMMA 3.1. *Let* M *be a class of* $\veebar$*-disjunctive* Σ*-matrices. Then, the logic of* M *is* $\veebar$*-multiplicative, and so* $\veebar$*-disjunctive.*

PROOF: Consider any $(X \cup Y \cup \{\psi\}) \subseteq \mathrm{Fm}^\omega_\Sigma$, any $\phi \in \mathrm{Cn}_\mathsf{M}(X \cup Y)$, any $\mathcal{A} \in \mathsf{M}$ and any $h \in \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{A})$ such that $(h(\phi) \veebar^{\mathfrak{A}} h(\psi)) = h(\phi \veebar \psi) \notin D^{\mathcal{A}}$, in which case $h(\phi) \notin D^{\mathcal{A}}$ and $h(\psi) \notin D^{\mathcal{A}}$, for $\mathcal{A}$ is $\veebar$-disjunctive, and so $h(\varphi) \notin D^{\mathcal{A}}$, for some $\varphi \in (X \cup Y)$, in which case $h(\varphi \veebar \psi) = (h(\phi) \veebar^{\mathfrak{A}} h(\psi)) \notin D^{\mathcal{A}}$, and so $(\phi \veebar \psi) \in \mathrm{Cn}_\mathsf{M}(X \cup (Y \veebar \psi)$, as required. Finally, Lemma 2.1(iv)$\Rightarrow$(i) completes the argument, for Cn_M clearly satisfies (2.3), (2.5) and (2.6). $\square$

Given a Σ-rule Γ/ϕ and a Σ-formula ψ, put $((\Gamma/\phi) \veebar \psi) \triangleq ((\Gamma \veebar \psi)/(\phi \veebar \psi))$. (This notation is naturally extended to Σ-calculi member-wise.)

THEOREM 3.2. *Let C be an inductive Σ-logic. Then, C is $\veebar$-disjunctive iff* (2.3), (2.5) *and* (2.6) *(as well as* (2.7)*) hold and, for any axiomatization $\mathcal{C}$ of C, every $(\Gamma \vdash \phi) \in \mathrm{SI}_\Sigma(\mathcal{C})$ and each $\psi \in \mathrm{Fm}^\omega_\Sigma$, it holds that $(\phi \veebar \psi) \in C(\Gamma \veebar \psi)$.*

PROOF: By Corollary 2.1(i)$\Leftrightarrow$(iv) and the structurality of C, with using (2.3) and the induction on the length of $\mathcal{C}$-derivations. $\square$

Let σ_{+1} be the Σ-substitution extending $[x_i/x_{i+1}]_{i\in\omega}$.

COROLLARY 3.3. *Let C be an inductive $\veebar$-disjunctive logic, $\mathcal{C}$ a Σ-calculus and $\mathcal{A} \subseteq \mathcal{C}$ an axiomatic Σ-calculus. Then, the extension C' of C relatively axiomatized by $\mathcal{C}' \triangleq (\mathcal{A} \cup (\sigma_{+1}[\mathcal{C} \setminus \mathcal{A}] \veebar x_0))$ is $\veebar$-disjunctive.*

PROOF: Then, C being inductive, is axiomatized by a finitary Σ-calculus $\mathcal{C}''$, in which case C' is axiomatized by the finitary Σ-calculus $\mathcal{C}'' \cup \mathcal{C}'$, and so is inductive. Moreover, C', being an extension of C, inherits (2.3), (2.5), (2.6) and (2.7) held for C. Then, we prove the $\veebar$-disjunctivity of C' with applying Theorem 3.2 to both C and C'. For consider any Σ-substitution σ and any $\psi \in \mathrm{Fm}^\omega_\Sigma$. First, consider any $\phi \in \mathcal{A}$. Then, by the structurality of C' and (2.3), we have $(\sigma(\phi) \veebar \psi) \in C'(\varnothing)$. Now, consider any $(\Gamma \vdash \phi) \in (\mathcal{C} \setminus \mathcal{A})$. Let ς be the Σ-substitution extending $(\sigma{\restriction}(V_\omega \setminus V_1)) \cup [x_0/(\sigma(x_0) \veebar \psi)]$, in which case $(\varsigma \circ \sigma_{+1}) = (\sigma \circ \sigma_{+1})$, and so, by (2.7) and the structurality of C', we eventually get $C'(\sigma[\sigma_{+1}[\Gamma] \veebar x_0] \veebar \psi) = C'((\varsigma[\sigma_{+1}[\Gamma]] \veebar \sigma(x_0)) \veebar \psi) \supseteq C'(\varsigma[\sigma_{+1}[\Gamma]] \veebar (\sigma(x_0) \veebar \psi)) = C'(\varsigma[\sigma_{+1}[\Gamma] \veebar x_0]) \supseteq C'(\varsigma(\sigma_{+1}(\varphi) \veebar x_0)) = C'(\varsigma(\sigma_{+1}(\varphi)) \veebar (\sigma(x_0) \veebar \psi)) \supseteq C'((\varsigma(\sigma_{+1}(\varphi)) \veebar \sigma(x_0)) \veebar \psi) = C'(\sigma(\sigma_{+1}(\varphi) \veebar x_0) \veebar \psi)$, as required. □

3.1.2. Implicative sentential logics

Throughout the rest of the paper, unless otherwise specified, $\rhd$ is supposed to be any (possibly, secondary) binary connective of Σ.

A Σ-logic C is said to be *$\rhd$-implicative*, whenever it has *Deduction Theorem* (*DT*, for short) *with respect to* $\rhd$ in the sense that:

$$(\psi \in C(\Gamma \cup \{\phi\})) \Rightarrow ((\phi \rhd \psi) \in C(\Gamma), \tag{3.1}$$

for all $(\Gamma \cup \{\phi, \psi\}) \subseteq \mathrm{Fm}_\Sigma^\omega$, as well as satisfies both the *Modus Ponens* rule:

$$\frac{x_0 \quad x_0 \rhd x_1}{x_1}, \tag{3.2}$$

and *Peirce Law* axiom (cf. [6]):

$$(((x_0 \rhd x_1) \rhd x_0) \rhd x_0). \tag{3.3}$$

(Clearly, the logic of any class of $\rhd$-implicative Σ-matrices is $\rhd$-implicative.) As it is well-known, C satisfies the following axioms:

$$x_0 \rhd (x_1 \rhd x_0) \tag{3.4}$$

$$(x_0 \rhd x_1) \rhd ((x_1 \rhd x_2) \rhd (x_0 \rhd x_2)) \tag{3.5}$$

whenever it has DT with respect to $\rhd$ and satisfies (3.2).

Lemma 3.4. *Any $\rhd$-implicative Σ-logic is $\veebar_\rhd$-disjunctive.*

Proof: With using Lemma 2.1(ii)⇒(i). First, (2.3) is by (3.2) and (3.1). Next, (2.5) is by (3.2) and (3.3)$[x_1/x_0]$. Further, by (3.2), (3.3) and (3.5), we have $x_0 \in C(\{x_0 \veebar_\rhd x_1, x_1 \rhd x_0\})$, in which case, by (3.1), we get $(x_1 \veebar_\rhd x_0) \in C(x_0 \veebar_\rhd x_1)$, and so (2.6) holds. Finally, consider any $(\Gamma \cup \{\phi, \psi\}) \subseteq \mathrm{Fm}_\Sigma^\omega$ and any $\varphi \in C(\Gamma \cup \{\phi\})$, in which case, by (3.1), we have $(\phi \rhd \varphi) \in C(\Gamma)$, and so, by (3.2) and (3.5), we get $\psi \in C(\Gamma \cup \{\phi \veebar_\rhd \psi, \varphi \rhd \psi\})$.

Hence, by (3.1), we eventually get $(\varphi \veebar_{\rhd} \psi) \in C(\Gamma \cup \{\phi \veebar_{\rhd} \psi\})$. Thus, C is singularly-$\veebar_{\rhd}$-multiplicative, as required. □

By $\mathcal{I}_{\rhd}^{[\mathrm{PL}]}$ we denote the Σ-calculus constituted by (3.2), (3.4) and (3.5) [as well as (3.3)]. Recall the following well-known observation proved by induction on the length of $(\mathcal{I}_{\rhd} \cup \mathcal{A})$-derivations:

LEMMA 3.5. *Let $\mathcal{A}$ be an axiomatic Σ-calculus. Then, $\mathrm{Cn}_{\mathcal{I}_{\rhd} \cup \mathcal{A}}$ has DT with respect to $\rhd$.*

Combining Lemmas 3.4 and 3.5, we eventually get:

THEOREM 3.6. *Let $\mathcal{A}$ be an axiomatic Σ-calculus. Then, $\mathrm{Cn}_{\mathcal{I}_{\rhd}^{\mathrm{PL}} \cup \mathcal{A}}$ is $\rhd$-implicative, and so $\veebar_{\rhd}$-disjunctive.*

COROLLARY 3.7. *Let $\mathcal{A} \cup \{\varphi, \phi, \psi\}$ be an axiomatic Σ-calculus and $v \in (V_\omega \setminus (\bigcup \mathrm{Var}[\{\varphi, \phi, \psi\}]))$. Then, the following hold:*

(i) *the Σ-axiom*

$$(\phi \veebar_{\rhd} \psi) \rhd \varphi \tag{3.6}$$

is derivable in $\mathcal{I}_{\rhd}^{\mathrm{PL}} \cup \mathcal{A}$, whenever the Σ-axioms:

$$\phi \rhd \varphi, \tag{3.7}$$
$$\psi \rhd \varphi \tag{3.8}$$

are so;

(ii) *the Σ-axiom*

$$\varphi \rhd (\phi \veebar_{\rhd} \psi) \tag{3.9}$$

is derivable in $\mathcal{I}_{\rhd}^{\mathrm{PL}} \cup \mathcal{A}$ iff the Σ-axiom

$$(\phi \rhd v) \rhd ((\psi \rhd v) \rhd (\varphi \rhd v)) \tag{3.10}$$

is so.

PROOF: In that case, by Theorem 3.6, $\mathrm{Cn}_{\mathcal{I}_{\rhd}^{\mathrm{PL}} \cup \mathcal{A}}$ is $\rhd$-implicative and $\veebar_{\rhd}$-disjunctive. In particular, by (2.2) with $Z = \varnothing$, (3.1) and (3.2), Σ-axioms:

$$\phi \rhd (\phi \veebar_{\rhd} \psi), \tag{3.11}$$
$$\psi \rhd (\phi \veebar_{\rhd} \psi), \tag{3.12}$$
$$(\phi \rhd \xi) \rhd ((\psi \rhd \xi) \rhd ((\phi \veebar_{\rhd} \psi) \rhd \xi)), \tag{3.13}$$

where $\xi \in \mathrm{Fm}_{\Sigma}^{\omega}$, are derivable in $\mathcal{I}_{\rhd}^{\mathrm{PL}} \cup \mathcal{A}$. In this way, (3.2), (3.7), (3.8) and (3.13) with $\xi = \varphi$ imply (3.6). Thus, (i) holds.

Next, assume (3.9) is derivable in $\mathcal{I}^{\mathrm{PL}}_{\triangleright} \cup \mathcal{A}$. Then, by (3.1), (3.2) and (3.13) with $\xi = v$, (3.10) is derivable in $\mathcal{I}^{\mathrm{PL}}_{\triangleright} \cup \mathcal{A}$. The converse is by (3.2), (3.11), (3.12) and (3.10)$[v/(\phi \veebar_{\triangleright} \psi)]$. Thus, (ii) holds, as required. $\square$

3.2. Gentzen-style calculi

Given any $(\alpha[\cup\beta]) \subseteq \omega$, elements of $\mathrm{Seq}_{\Sigma}^{[\beta\vdash]\alpha} \triangleq \{\langle\Gamma, \Delta\rangle \in ((\mathrm{Fm}_{\Sigma}^{\omega})^*)^2 \mid (\mathrm{dom}\,\Delta) \in \alpha\,[\&\,(\mathrm{dom}\,\Gamma) \in \beta]\}$ are called *α-conclusion [β-premise] Σ-sequents*. (In this connection, "[purely] single/multi" stands for "$(2/\omega)[\setminus 1]$", respectively.) Any sequent $\langle\Gamma, \Delta\rangle$ is normally written in the conventional form $\Gamma \vdash \Delta$. This is said to be *injective*, whenever both Γ and Δ are so. Likewise, it is said to be *disjoint*, whenever $((\mathrm{img}\,\Gamma) \cap (\mathrm{img}\,\Delta)) = \varnothing$. For any $\Phi = (\Gamma \vdash \Delta) \in \mathrm{Seq}_{\Sigma}^{[\beta\vdash]\alpha}$, set $\mathrm{Var}(\Phi) \triangleq (\bigcup \mathrm{Var}[\mathrm{img}(\Gamma, \Delta)]) \in \wp_{\omega}(V_{\omega})$ and $\sigma(\Phi) \triangleq ((\sigma \circ \Gamma) \vdash (\sigma \circ \Delta)) \in \mathrm{Seq}_{\Sigma}^{[\beta\vdash]\alpha}$, where σ is a Σ-substitution. In this way, $\mathrm{Seq}_{\Sigma}^{[\beta\vdash]\alpha}$ forms a Σ-language $\mathcal{S}_{\Sigma}^{[\beta\vdash]\alpha}$, called the *$\alpha$-conclusion [$\beta$-premise] Gentzen-style/sequent Σ-language*, $\mathcal{S}_{\Sigma}^{[\beta\vdash]\alpha}$-rules/axioms/calculi/logics being referred to as *α-conclusion [β-premise] (Gentzen-style/sequent) Σ-rules/axioms/calculi/logics.*

The following multi-conclusion sequent $\varnothing$-rules are said to be *structural*:

$$
\begin{array}{ccc}
\text{Reflexivity} & \multicolumn{2}{c}{x_0 \vdash x_0} \\[1ex]
\text{Cut} & \multicolumn{2}{c}{\dfrac{\Lambda, \Gamma \vdash \Delta, x_0 \quad \Gamma, x_0 \vdash \Delta, \Theta}{\Lambda, \Gamma \vdash \Delta, \Theta}} \\[1ex]
\text{Enlargement} & \dfrac{\Gamma \vdash \Delta}{x_0, \Gamma \vdash \Delta} & \dfrac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, x_0} \\[1ex]
\text{Contraction} & \dfrac{x_0, x_0, \Gamma \vdash \Delta}{x_0, \Gamma \vdash \Delta} & \dfrac{\Gamma \vdash \Delta, x_0, x_0}{\Gamma \vdash \Delta, x_0} \\[1ex]
\text{Permutation} & \dfrac{\Lambda, x_0, x_1, \Gamma \vdash \Delta}{\Lambda, x_1, x_0, \Gamma \vdash \Delta} & \dfrac{\Gamma \vdash \Delta, x_0, x_1, \Theta}{\Gamma \vdash \Delta, x_1, x_0, \Theta}
\end{array}
$$

where $\Lambda, \Gamma, \Delta, \Theta \in V_{\omega}^*$, Enlargement, Contraction and Permutation being referred to as *basic structural.*

Given two (purely) multi-conclusion [{purely} multi-premise] Σ-sequents $\Phi = (\Gamma \vdash \Delta)$ and $\Psi = (\Lambda \vdash \Theta)$, we have their *sequent disjunction/implication*:

$$
\begin{array}{rcl}
(\Phi \uplus \Psi) & \triangleq & (\Gamma, \Lambda \vdash \Delta, \Theta) \in \mathrm{Seq}_{\Sigma}^{[(\omega\{\setminus 1\})\vdash](\omega(\setminus 1))} / \\
(\Phi \sqsupset \Psi) & \triangleq & \{\phi, \Gamma \vdash \Delta \mid \phi \in (\mathrm{img}\,\Theta)\}
\end{array}
$$

$$\cup \quad \{\Gamma \vdash \Delta, \psi \mid \psi \in (\mathrm{img}\,\Lambda)\} \in \wp_\omega(\mathrm{Seq}_\Sigma^{[(\omega\{\setminus 1\})\vdash](\omega(\setminus 1))}).$$

Then, given any $X \in \wp_{\langle\omega\rangle}(\mathrm{Seq}_\Sigma^{[(\omega\{\setminus 1\})\vdash](\omega(\setminus 1))})$, set $(\Phi \sqsupset X) \triangleq (\bigcup\{\Phi \sqsupset \Psi \mid \Psi \in X\} \in \wp_{\langle\omega\rangle}(\mathrm{Seq}_\Sigma^{[(\omega\{\setminus 1\})\vdash](\omega(\setminus 1))})$. A (purely) multi-conclusion [{purely} multi-premise] sequent Σ-calculus $\mathcal{G}$ is said to be $\langle$*deductively*$\rangle$ *multiplicative*, provided, for every (purely) multi-conclusion [{purely} multi-premise] sequent Σ-rule X/Φ $\langle$derivable$\rangle$ in $\mathcal{G}$ and each multi-conclusion Σ-sequent Ψ, the rule $(X \uplus \Psi)/(\Phi \uplus \Psi)$ is derivable in $\mathcal{G}$. With using induction on the length of $\mathcal{G}$-derivations, it is routine checking that $\mathcal{G}$ is multiplicative iff it is deductively so.

THEOREM 3.8 (cf. the proof of Theorem 4.2 of [9]). *Let $\mathcal{G}$ be a $\langle$multiplicative$\rangle$ (purely) multi-conclusion [{purely} multi-premise] sequent Σ-calculus with basic structural rules and Cut$\langle$/Reflexivity$\rangle$ and $(X \cup \{\Phi, \Psi\}) \subseteq \mathrm{Seq}_\Sigma^{[(\omega\{\setminus 1\})\vdash](\omega(\setminus 1))}$. Then,*

$$\Psi \in \mathrm{Cn}_{\mathcal{G}}(X \cup \{\Phi\}) \Leftarrow \langle / \Rightarrow\rangle (\Phi \sqsupset \Psi) \subseteq \mathrm{Cn}_{\mathcal{G}}(X).$$

From the model-theoretic point of view, any Σ-sequent $\Gamma \vdash \Delta$ is treated as the first-order basic clause $\bigvee(\neg[\mathrm{img}\,\Gamma] \cup (\mathrm{img}\,\Delta))$ of the signature $\Sigma \cup \{D\}$ under the notorious identification of any Σ-formula φ with the first-order atomic formula $D(\varphi)$, any sequent Σ-rule being interpreted as implication of its premises (under the natural identification of any finite set X of first-order formulas with $\bigwedge X$ we follow tacitly as well) and its conclusion. (In this way, sequent disjunction/implication corresponds to the usual disjunction/implication.) This fits the standard matrix interpretation of sequents equally adopted in [9] and [10] and going back to [11].

4. Basic disjunctive calculi

4.1. The Hilbert-style calculus

By $\mathcal{D}_{\veebar}$ we denote the Σ-calculus constituted by the following Σ-rules:

$$\begin{array}{cccc} D_1 & D_2 & D_3 & D_4 \\ \dfrac{x_0 \veebar x_0}{x_0} & \dfrac{x_0}{x_0 \veebar x_1} & \dfrac{(x_0 \veebar x_1) \veebar x_2}{(x_1 \veebar x_0) \veebar x_2} & \dfrac{(x_0 \veebar (x_1 \veebar x_2)) \veebar x_3}{((x_0 \veebar x_1) \veebar x_2) \veebar x_3} \end{array}$$

Lemma 4.1. *Let* $\mathcal{C} \supseteq \mathcal{D}_{\veebar}$ *be a* Σ*-calculus,* $\mathcal{R} = (\Gamma/\phi)$ *a* Σ*-rule and* $v \in (V_\omega \setminus \mathrm{Var}(\mathcal{R}))$*. Suppose* $\mathcal{R} \veebar v$ *is derivable in* $\mathcal{C}$*. Then, so is* $\mathcal{R}$ *itself.*

Proof: First, for every $\psi \in \Gamma$, by $D_2[x_0/\psi, x_1/\phi]$, we have $(\psi \veebar \phi) \in \mathrm{Cn}_{\mathcal{C}}(\psi)$, and so we get $(\Gamma \veebar \phi) \in \mathrm{Cn}_{\mathcal{C}}(\Gamma)$. Then, applying $(\mathcal{R} \veebar v)[v/\phi]$, by the structurality of $\mathrm{Cn}_{\mathcal{C}}$, we conclude that $(\phi \veebar \phi) \in \mathrm{Cn}_{\mathcal{C}}(\Gamma)$. Finally, $D_1[x_0/\phi]$ completes the argument. $\square$

Applying Lemma 4.1 to both D_3 and D_4, we immediately get:

Corollary 4.2. *The following rules are derivable in* $\mathcal{D}_{\veebar}$*:*

$$\frac{x_0 \veebar x_1}{x_1 \veebar x_0}, \tag{4.1}$$

$$\frac{x_0 \veebar (x_1 \veebar x_2)}{(x_0 \veebar x_1) \veebar x_2}. \tag{4.2}$$

Now, we are in a position to prove the derivability of other useful rules in $\mathcal{D}_{\veebar}$.

Proposition 4.3. *The following rules are derivable in* $\mathcal{D}_{\veebar}$*:*

$$\frac{(x_0 \veebar x_1) \veebar x_2}{x_0 \veebar (x_1 \veebar x_2)}, \tag{4.3}$$

$$\frac{(x_0 \veebar x_0) \veebar x_1}{x_0 \veebar x_1}, \tag{4.4}$$

$$\frac{x_0 \veebar x_2}{(x_0 \veebar x_1) \veebar x_2}. \tag{4.5}$$

Proof: First, in view of Corollary 4.2, (4.3) is by the following $\mathrm{Cn}_{\mathcal{D}_{\veebar}}$-derivation:

1. $(x_0 \veebar x_1) \veebar x_2$ — hypothesis;
2. $(x_1 \veebar x_0) \veebar x_2$ — D_3: 1;
3. $x_2 \veebar (x_1 \veebar x_0)$ — $(4.1)[x_0/(x_1 \veebar x_0), x_1/x_2]$: 2;
4. $(x_2 \veebar x_1) \veebar x_0$ — $(4.2)[x_0/x_2, x_2/x_0]$: 3;
5. $(x_1 \veebar x_2) \veebar x_0$ — $D_3[x_0/x_2, x_2/x_0]$: 4;
6. $x_0 \veebar (x_1 \veebar x_2)$ — $(4.1)[x_0/(x_1 \veebar x_0), x_1/x_0]$: 5.

Then, in view of Corollary 4.2, (4.4) is by the following $\mathrm{Cn}_{\mathcal{D}_{\veebar}}$-derivation:

1. $(x_0 \veebar x_0) \veebar x_1$ — hypothesis;
2. $x_0 \veebar (x_0 \veebar x_1)$ — $(4.3)[x_1/x_0, x_2/x_1]$: 1;

3. $(x_0 \veebar x_1) \veebar x_0$ — (4.1)$[x_1/(x_0 \veebar x_1)]$: 2;
4. $((x_0 \veebar x_1) \veebar x_0) \veebar x_1$ — $D_2[x_0/((x_0 \veebar x_1) \veebar x_0)]$: 3;
5. $(x_0 \veebar x_1) \veebar (x_0 \veebar x_1)$ — (4.3)$[x_0/(x_0 \veebar x_1), x_1/x_0, x_1/x_2]$: 4;
6. $(x_0 \veebar x_1)$ — $D_1[x_0/(x_0 \veebar x_1)]$: 5.

Finally, in view of Corollary 4.2, (4.5) is by the following $\mathrm{Cn}_{\mathcal{D}_\veebar}$-derivation:

1. $x_0 \veebar x_2$ — hypothesis;
2. $(x_0 \veebar x_2) \veebar x_1$ — $D_2[x_0/(x_0 \veebar x_2)]$: 1;
3. $x_0 \veebar (x_2 \veebar x_1)$ — (4.3)$[x_1/x_2, x_2/x_1]$: 2;
4. $(x_2 \veebar x_1) \veebar x_0$ — (4.1)$[x_1/(x_2 \veebar x_1)]$: 3;
5. $x_2 \veebar (x_1 \veebar x_0)$ — (4.3)$[x_0/x_2, x_2/x_0]$: 4;
6. $(x_1 \veebar x_0) \veebar x_2$ — (4.1)$[x_0/x_2, x_1/(x_1 \veebar x_0)]$: 5;
7. $(x_0 \veebar x_1) \veebar x_2$ — $D_3[x_0/x_1, x_1/x_0]$: 6. $\square$

Corollary 4.4. *Let $\mathcal{R} = (\Gamma/\phi)$ be a Σ-rule, $\psi \in \mathrm{Fm}^\omega_\Sigma$, $\sigma \in \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{Fm}^\omega_\Sigma)$ and $v \in (V_\omega \setminus \mathrm{Var}(\mathcal{R}))$. Suppose $\mathcal{R} \veebar v$ is derivable in $\mathcal{D}_\veebar$. Then, so is $\sigma(\mathcal{R} \veebar v) \veebar \psi$.*

Proof: Then, by Corollary 4.2(4.2) and Proposition 4.3(4.3), (2.7) holds for $C \triangleq \mathrm{Cn}_{\mathcal{D}_\veebar}$. Let $\varsigma \in \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{Fm}^\omega_\Sigma)$ extend $(\sigma \restriction (V_\omega \setminus \{v\})) \cup [v/(\sigma(v) \veebar \varphi)]$, in which case $\sigma(\mathcal{R}) = \varsigma(\mathcal{R})$, for $v \notin \mathrm{Var}(\mathcal{R})$. Then, using (2.7) and the structurality of C, we eventually get $C(\sigma[\Gamma \veebar v] \veebar \varphi) = C((\sigma[\Gamma] \veebar \sigma(v)) \veebar \varphi) = C(\sigma[\Gamma] \veebar (\sigma(v) \veebar \varphi)) = C(\varsigma[\Gamma] \veebar \varsigma(v)) = C(\varsigma[\Gamma \veebar v]) \supseteq C(\varsigma(\phi \veebar v)) = C(\varsigma(\phi) \veebar \varsigma(v)) = C(\sigma(\phi) \veebar (\sigma(v) \veebar \varphi)) = C((\sigma(\phi) \veebar \sigma(v)) \veebar \varphi) = C(\sigma(\phi \veebar v) \veebar \varphi)$, as required. $\square$

Theorem 4.5. *$\mathrm{Cn}_{\mathcal{D}_\veebar}$ is $\veebar$-disjunctive.*

Proof: With using Theorem 3.2. First, by D_1, D_2 and Corollary 4.2(4.1), (2.3), (2.5) and (2.6) hold for $C \triangleq \mathrm{Cn}_{\mathcal{D}_\veebar}$.

Next, consider any $\sigma \in \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{Fm}^\omega_\Sigma)$, any $\varphi \in \mathrm{Fm}^\omega_\Sigma$ and any $i \in (5 \setminus 1)$. The case, when $i \notin 3$, is due to Corollary 4.4 well-applicable to D_i. Otherwise, we have $\mathrm{Var}(D_i) = V_i \not\ni x_i$. Then, by Proposition 4.3(4.4)/(4.5), $D_i \veebar x_i$ is derivable in $\mathcal{D}_\veebar$. Let $\varsigma \in \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{Fm}^\omega_\Sigma)$ extend $(\sigma \restriction V_{\omega \setminus \{i\}}) \cup [x_i/\varphi]$, in which case $\varsigma(D_i) = \sigma(D_i)$, and so, by the structurality of C, we eventually conclude that $(\sigma(D_i) \veebar \varphi) = (\varsigma(D_i) \veebar \varsigma(x_i)) = \varsigma(D_i \veebar x_i)$ is derivable in $\mathcal{D}_\veebar$, as required. $\square$

The following auxiliary observation has proved quite useful for reducing the number of rules of calculi to be constructed in Section 6 according to the universal method to be elaborated in Section 5:

Proposition 4.6. *Let* $\phi, \psi, \varphi \in \mathrm{Fm}^{\omega}_{\Sigma}$, $v \in (V_\omega \setminus (\bigcup \mathrm{Var}[\{\phi, \psi, \varphi\}]))$ *and* $\mathcal{C} \supseteq \mathcal{D}_{\veebar}$ *a* Σ*-calculus. Then, the rules* $\mathcal{R}_l = ((\phi \veebar v)/(\varphi \vee v))$ *and* $\mathcal{R}_r = ((\psi \veebar v)/(\varphi \vee v))$ *are both derivable in* $\mathcal{C}$ *iff the rule* $\mathcal{R} = (((\phi \veebar \psi) \vee v)/(\varphi \vee v))$ *is so.*

Proof: First, assume $\mathcal{R}$ is derivable in $\mathcal{C}$. Then, the derivability of $\mathcal{R}_l$ in $\mathcal{C}$ is by the following $\mathrm{Cn}_{\mathcal{C}}$-derivation:

1. $\phi \veebar v$ — hypothesis;
2. $v \veebar \phi$ — (4.1)$[x_0/\phi, x_1/v]$: 1;
3. $(v \veebar \phi) \veebar \psi$ — $D_2[x_0/(v \veebar \phi), x_1/\psi]$: 2;
4. $v \veebar (\phi \veebar \psi)$ — (4.3)$[x_0/v, x_1/\phi, x_2/\psi]$: 3;
5. $(\phi \veebar \psi) \veebar v$ — (4.1)$[x_0/v, x_1/(\phi \veebar \psi)]$: 4;
6. $\varphi \veebar v$ — $\mathcal{R}$: 5.

Likewise, the derivability of $\mathcal{R}_r$ in $\mathcal{C}$ is by the following $\mathrm{Cn}_{\mathcal{C}}$-derivation:

1. $\psi \veebar v$ — hypothesis;
2. $(\psi \veebar v) \veebar \phi$ — $D_2[x_0/(\psi \veebar v), x_1/\phi]$: 1;
3. $\phi \veebar (\psi \veebar v)$ — (4.1)$[x_0/(\psi \veebar v), x_1/\phi]$: 2;
4. $(\phi \veebar \psi) \veebar v$ — (4.2)$[x_0/\phi, x_1/\psi, x_2/v]$: 3;
5. $\varphi \veebar v$ — $\mathcal{R}$: 4.

Conversely, assume both $\mathcal{R}_l$ and $\mathcal{R}_r$ are derivable in $\mathcal{C}$. Then, the derivability of $\mathcal{R}$ in $\mathcal{C}$ is by the following $\mathrm{Cn}_{\mathcal{C}}$-derivation:

1. $(\phi \veebar \psi) \veebar v$ — hypothesis;
2. $\phi \veebar (\psi \veebar v)$ — (4.3)$[x_0/\phi, x_1/\psi, x_2/v]$: 1;
3. $\varphi \veebar (\psi \veebar v)$ — $\mathcal{R}_l[v/(\psi \veebar v)]$: 2;
4. $(\psi \veebar v) \veebar \varphi$ — (4.1)$[x_0/\varphi, x_1/(\psi \veebar v)]$: 3;
5. $\psi \veebar (v \veebar \varphi)$ — (4.3)$[x_0/\psi, x_1/v, x_2/\varphi]$: 4;
6. $\varphi \veebar (v \veebar \varphi)$ — $\mathcal{R}_r[v/(v \veebar \varphi)]$: 5;
7. $(v \veebar \varphi) \veebar \varphi$ — (4.1)$[x_0/\varphi, x_1/(v \veebar \varphi)]$: 6;
8. $v \veebar (\varphi \veebar \varphi)$ — (4.3)$[x_0/v, x_1/\varphi, x_2/\varphi]$: 7;
9. $(\varphi \veebar \varphi) \veebar v$ — (4.1)$[x_0/v, x_1/(\varphi \veebar \varphi)]$: 8;
10. $\varphi \veebar v$ — (4.4)$[x_0/(\varphi \veebar \varphi), x_1/v]$: 9. ◻

4.2. Single- versus multi-conclusion sequent calculi

Let $\mathcal{G}^{\alpha}_{\veebar}$, where $\alpha \subseteq \omega$, be the α-conclusion sequent Σ-calculus constituted by structural α-conclusion sequent rules and the following α-conclusion sequent Σ-rules:

$$\begin{array}{cc} G_l & G_r \\ \dfrac{\Gamma, x_0 \vdash \Delta \quad \Gamma, x_1 \vdash \Delta}{\Gamma, (x_0 \veebar x_1) \vdash \Delta} & \dfrac{\Gamma \vdash \Omega, x_k}{\Gamma \vdash \Omega, (x_0 \veebar x_1)} \end{array}$$

where $k \in 2$ and $\Gamma, \Delta, \Omega \in V_\omega^*$ such that $(\operatorname{dom} \Delta), ((\operatorname{dom} \Omega) + 1) \in \alpha$.

The set $\mathrm{Fm}^\omega_\veebar$ is defined in the obvious almost standard recursive manner as the least $S \subseteq \mathrm{Fm}^\omega_\Sigma$ such that $V_\omega \subseteq S$ and $(\phi \veebar \psi) \in S$, for all $\phi, \psi \in S$.

Lemma 4.7. *Let* $\psi \in \mathrm{Fm}^\omega_\veebar$ *and* $v \in \mathrm{Var}(\psi)$. *Suppose* $1 \in \alpha$. *Then,* $v \vdash \psi$ *is derivable in* $\mathcal{G}^\alpha_\veebar$.

Proof: By induction on construction of ψ. For consider the following complementary cases:

1. $\psi \in V_\omega$.
   Then, $\mathrm{Var}(\psi) = \{\psi\} \ni v$, in which case $\psi = v$, and so the Reflexivity axiom completes the argument.
2. $\psi \notin V_\omega$.
   Then, $\psi = (\varphi_0 \veebar \varphi_1)$, for some $\varphi_0, \varphi_1 \in \mathrm{Fm}^\omega_\veebar$, in which case $v \in \mathrm{Var}(\psi) = (\bigcup_{k \in 2} \mathrm{Var}(\varphi_k))$, and so $v \in \mathrm{Var}(\varphi_k)$, for some $k \in 2$. Hence, by induction hypothesis, $v \vdash \varphi_k$ is derivable in $\mathcal{G}^\alpha_\veebar$. In this way, G_r completes the argument. □

Corollary 4.8. *Let* $\phi, \psi \in \mathrm{Fm}^\omega_\veebar$. *Suppose* $\mathrm{Var}(\phi) \subseteq \mathrm{Var}(\psi)$ *and* $1 \in \alpha$. *Then,* $\phi \vdash \psi$ *is derivable in* $\mathcal{G}^\alpha_\veebar$.

Proof: By induction on construction of ϕ. For consider the following complementary cases:

1. $\phi \in V_\omega$.
   Then, $\mathrm{Var}(\psi) \supseteq \mathrm{Var}(\phi) = \{\phi\}$, in which case $\phi \in \mathrm{Var}(\psi)$, and so Lemma 4.7 completes the argument.
2. $\phi \notin V_\omega$.
   Then, $\phi = (\varphi_0 \veebar \varphi_1)$, for some $\varphi_0, \varphi_1 \in \mathrm{Fm}^\omega_\veebar$, in which case $\mathrm{Var}(\psi) \supseteq \mathrm{Var}(\phi) = (\bigcup_{k \in 2} \mathrm{Var}(\varphi_k))$, and so $\mathrm{Var}(\psi) \supseteq \mathrm{Var}(\varphi_k)$, for each $k \in 2$. Hence, by induction hypothesis, $\varphi_k \vdash \psi$ is derivable in $\mathcal{G}^\alpha_\veebar$, for every $k \in 2$. Thus, G_l completes the argument. □

Let $\tau_\veebar : \mathrm{Seq}^\omega_\Sigma \to \mathrm{Seq}^2_\Sigma$ be defined as follows:

$$\tau_\veebar(\Gamma \vdash \Delta) \triangleq \begin{cases} \Gamma \vdash \Delta & \text{if} \Delta = \varnothing, \\ \Gamma \vdash (\veebar \Delta) & \text{otherwise}, \end{cases}$$

for all $(\Gamma \vdash \Delta) \in \mathrm{Seq}^{\omega}_{\Sigma}$, in which case:

$$\sigma(\tau_{\veebar}(\Gamma \vdash \Delta)) = \tau_{\veebar}(\sigma(\Gamma \vdash \Delta)). \tag{4.6}$$

LEMMA 4.9. *For every* $\mathcal{R} \in \mathcal{G}^{\omega[\setminus 1]}_{\veebar}$, $\tau_{\veebar}(\mathcal{R})$ *is derivable in* $\mathcal{G}^{2[\setminus 1]}_{\veebar}$.

PROOF: Consider the following exhaustive cases:

1. $\mathcal{R}$ is either G_l or the Reflexivity axiom or a left-side basic structural rule or a Cut with $\Delta = \varnothing$.
   Then, $\tau_{\veebar}(\mathcal{R})$ is a substitutional Σ-instance of a rule in $\mathcal{G}^{2[\setminus 1]}_{\veebar}$, and so is derivable in it.
2. $\mathcal{R}$ is either G_r or a right-side basic structural rule.
   Then, $\tau_{\veebar}(\mathcal{R})$ is of the form

   $$\frac{\Lambda \vdash \phi}{\Lambda \vdash \psi},$$

   where $\Lambda \in V^*_{\omega}$ and $\phi, \psi \in \mathrm{Fm}^{\omega}_{\veebar}$, while $\mathrm{Var}(\phi) \subseteq \mathrm{Var}(\psi)$, in which case Corollary 4.8 and Cut complete the argument.
3. $\mathcal{R}$ is a Cut with $\Delta \neq \varnothing$.
   Then, $\tau_{\veebar}(\mathcal{R})$ is as follows:

   $$\frac{\Lambda, \Gamma \vdash (\phi \veebar x_0) \quad \Gamma, x_0 \vdash \psi}{\Lambda, \Gamma \vdash \psi},$$

   where $\phi \triangleq (\veebar\Delta) \in \mathrm{Fm}^{\omega}_{\veebar}$ and $\psi \triangleq (\veebar(\Delta, \Theta)) \in \mathrm{Fm}^{\omega}_{\veebar}$, in which case $\mathrm{Var}(\phi) \subseteq \mathrm{Var}(\psi)$, and so, by Corollary 4.8, $\phi \vdash \psi$ is derivable in $\mathcal{G}^{2[\setminus 1]}_{\veebar}$, and so is $\Gamma, \phi \vdash \psi$, by basic structural rules. Hence, by G_l, the rule $(\Gamma, x_0 \vdash \psi)/(\Gamma, (\phi \veebar x_0) \vdash \psi)$ is derivable in $\mathcal{G}^{2[\setminus 1]}_{\veebar}$. Thus, Cut completes the proof. ☐

Using induction on the length of $(\mathcal{G}^{\omega[\setminus 1]}_{\veebar} \cup \mathcal{A})$-derivations, by (4.6) and Corollary 4.9, we immediately get:

COROLLARY 4.10. *Let* $(\mathcal{A} \cup \{\Phi\}) \subseteq \mathrm{Seq}^{\omega[\setminus 1]}_{\Sigma}$. *Suppose* Φ *is derivable in* $\mathcal{G}^{\omega[\setminus 1]}_{\veebar} \cup \mathcal{A}$. *Then,* $\tau_{\veebar}(\Phi)$ *is derivable in* $\mathcal{G}^{2[\setminus 1]}_{\veebar} \cup \tau_{\veebar}[\mathcal{A}]$.[3]

[3]Although the converse holds as well, because Φ and $\tau_{\veebar}(\Phi)$ are interderivable in the [purely] multi-conclusion calculus including the [purely] single-conclusion one, this point is no matter for our further argumentation.

5. Main universal constructions

Fix any finite $\veebar$-disjunctive Σ-matrix $\mathcal{A}$ with a finite equality determinant $\Upsilon \ni x_0$. Given any $X \subseteq V_\omega$, put $\Upsilon[X] \triangleq \{\upsilon(x) \mid \upsilon \in \Upsilon, x \in X\}$.

First, consider any *complex* $\langle\Upsilon,\Sigma\rangle$*-type* in the sense of [10], that is, a couple of the form $\langle\upsilon, F\rangle$, where $\upsilon \in \Upsilon$ and $F \in \Sigma$ of arity $n \in (\omega \setminus 1)$ such that either $n \neq 1$ or $\upsilon(F(x_0)) \notin \Upsilon$. Then, according to the constructive proof of Theorem 1 of [10], there are some $\lambda_{\mathcal{T}}(\upsilon, F), \rho_{\mathcal{T}}(\upsilon, F) \in \wp_\omega((\Upsilon[V_n]^*)^2)$ with injective elements such that:

$$\mathcal{A} \models \langle\forall x_i\rangle_{i\in n}((\upsilon(F(x_i)_{i\in n}) \vdash) \leftrightarrow \lambda_{\mathcal{T}}(\upsilon, F)), \tag{5.1}$$

$$\mathcal{A} \models \langle\forall x_i\rangle_{i\in n}((\vdash \upsilon(F(x_i)_{i\in n})) \leftrightarrow \rho_{\mathcal{T}}(\upsilon, F)). \tag{5.2}$$

Then, $l \triangleq |\lambda_{\mathcal{T}}(\upsilon, F))| \in \omega$ and $r \triangleq |\rho_{\mathcal{T}}(\upsilon, F))| \in \omega$. Take any bijections $\overline{L} : l \to \lambda_{\mathcal{T}}(\upsilon, F)$ and $\overline{R} : r \to_{\mathcal{T}} \rho(\upsilon, F)$. By induction on any $(j/k) \in ((l/r) + 1))$, define $(\Lambda_j/\Xi_k) \in \wp_\omega(\mathrm{Seq}^\omega_\Sigma)$ as follows. In case $(j/k) = 0$, put $(\Lambda_j \triangleq \{\upsilon(F(x_i)_{i\in n}) \vdash\})/(\Xi_k \triangleq \{\vdash \upsilon(F(x_i)_{i\in n})\})$. Otherwise, set $(\Lambda/\Xi)_{j/k} \triangleq ((L/R)_{j/k} \sqsupset (\Lambda/\Xi)_{(j/k)-1}))$. Then, in view of (5.1)/(5.2), by induction, we conclude that $(\mathcal{A} \models \langle\forall x_i\rangle_{i\in n}((\mathrm{img}(\overline{(L/R)}\upharpoonright((l/r)\setminus(j/k)))) \to (\Lambda/\Xi)_{j/k})$. In particular, every element of $(\boldsymbol{\lambda}(\upsilon, F) \triangleq \Lambda_l)/(\boldsymbol{\rho}(\upsilon, F) \triangleq \Xi_r)$ is true in $\mathcal{A}$.

Example 5.1. When $\upsilon = x_0$ and $F = \veebar$, in which case $\veebar$ is a primary connective of Σ, one can always take $\lambda_{\mathcal{T}}(\upsilon, F) = \{x_0 \vdash; x_1 \vdash\}$ and $\rho_{\mathcal{T}}(\upsilon, F) = \{\vdash x_0, x_1\}$ to satisfy (5.1) and (5.2), in which case $\boldsymbol{\lambda}(\upsilon, F) = \{(x_0 \veebar x_1) \vdash x_0, x_1\}$ and $\boldsymbol{\rho}(\upsilon, F) = \{x_0 \vdash (x_0 \veebar x_1); x_1 \vdash (x_0 \veebar x_1)\}$, and so their elements are derivable in $\mathcal{G}^\omega_\veebar$. $\square$

In this way, let $\mathcal{A}'$ be the set of all elements of $\boldsymbol{\lambda}(\upsilon, F) \cup \boldsymbol{\rho}(\upsilon, F)$, for all complex $\langle\Upsilon,\Sigma\rangle$-types $\langle\upsilon, F\rangle$ but $\langle x_0, \veebar\rangle$, in case $\veebar \in \Sigma$ is primary.

Next, let $\mathcal{A}''$ be the set containing, for each nullary $c \in \Sigma$ and every $\upsilon \in \Upsilon$, the axiom $(\upsilon(c) \vdash)/(\vdash \upsilon(c))$, whenever this is true in $\mathcal{A}$.

Further, let $\mathcal{A}'''$ be the finite set of all those elements of $((\Upsilon)^*)^2$, which are both injective, disjoint and true in $\mathcal{A}$.

Finally, every element of $\mathcal{A} \triangleq (\mathcal{A}' \cup \mathcal{A}'' \cup \mathcal{A}''')$ is true in $\mathcal{A}$. Moreover, $\mathcal{A}$ is finite, whenever Σ is so.

Lemma 5.2. *Any multi-conclusion Σ-sequent is true in $\mathcal{A}$ iff it is derivable in $\mathcal{G}^\omega_\veebar \cup \mathcal{A}$.*

Proof: The "if" part is by the fact that every element of $\mathcal{A}$ is true in $\mathcal{A}$, while any $\underline{\vee}$-disjunctive Σ-matrix (in particular, $\mathcal{A}$) is a model of $\mathcal{G}^{\omega}_{\underline{\vee}}$.

Conversely, consider any complex $\langle\Upsilon,\Sigma\rangle$-type $\langle\upsilon,F\rangle$, following the notations adopted above. Then, every element of $(\Lambda/\Xi)_{l/r}$, being in $\mathcal{A}$, unless $\upsilon=x_0$ and $F=\underline{\vee}$, is derivable in $\mathcal{G}^{\omega}_{\underline{\vee}}\cup\mathcal{A}$, in view of Example 5.1. Therefore, by downward induction on any $(j/k)\in((l/r)+1)$, in view of Theorem 3.8, we conclude that the rule

$$\frac{\mathrm{img}(\overline{(L/R)}{\restriction}((l/r)\setminus(j/k)))}{(\Lambda/\Xi)_{j/k}}$$

is derivable in $\mathcal{G}^{\omega}_{\underline{\vee}}\cup\mathcal{A}$, and so is

$$\frac{(\lambda/\rho)_{\mathcal{T}}(\upsilon,F)}{(\upsilon(F(x_i)_{i\in n}\vdash)/(\vdash\upsilon(F(x_i)_{i\in n})}$$

when taking $(j/k)=0$. Moreover, $\mathcal{G}^{\omega}_{\underline{\vee}}\cup\mathcal{A}$ is clearly multiplicative. In this way, in view of the structurality of the consequence of any calculus, taking basic structural rules into account, we see that all rules with premises of the multi-conclusion Σ-calculus $\mathcal{S}^{(0,0)}_{\mathcal{A},\mathcal{T}}$ given by Definition 1 of [10] are derivable in $\mathcal{G}^{\omega}_{\underline{\vee}}\cup\mathcal{A}$. And what is more, in view of the structurality of the consequence of any calculus, taking basic structural rules and the Reflexivity axiom into account, we see that all axioms of $\mathcal{S}^{(0,0)}_{\mathcal{A},\mathcal{T}}$ are derivable in $\mathcal{G}^{\omega}_{\underline{\vee}}\cup\mathcal{A}$ too. Finally, Theorem 2 of [10], according to which any multi-conclusion Σ-sequent, being true in $\mathcal{A}$, is derivable in $\mathcal{S}^{(0,0)}_{\mathcal{A},\mathcal{T}}$, completes the argument. $\Box$

Given any $\mathcal{B}\subseteq\mathrm{Seq}^{\omega}_{\Sigma}$, set $\mathcal{B}_{\setminus 1}\triangleq((\mathcal{B}\cap\mathrm{Seq}^{\omega\setminus 1}_{\Sigma})\cup\{(\sigma_{+1}\circ\Gamma)\vdash x_0\mid\Gamma\in(\mathrm{Fm}^{\omega}_{\Sigma})^{*},(\Gamma\vdash)\in\mathcal{B}\})\subseteq\mathrm{Seq}^{\omega\setminus 1}_{\Sigma}$. Clearly, elements of $\mathcal{A}_{\setminus 1}$ are true in $\mathcal{A}$, for those of $\mathcal{A}$ are so.

Lemma 5.3. *Any purely multi-conclusion Σ-sequent is derivable in $\mathcal{G}^{\omega}_{\underline{\vee}}\cup\mathcal{A}$ iff it is derivable in $\mathcal{G}^{\omega\setminus 1}_{\underline{\vee}}\cup\mathcal{A}_{\setminus 1}$.*

Proof: The "if" part is by Lemma 5.2, for elements of $\mathcal{A}_{\setminus 1}$ are true in $\mathcal{A}$.

Conversely, consider any $\Phi=(\Gamma\vdash\Delta)\in\mathrm{Seq}^{\omega\setminus 1}_{\Sigma}$ and any $\mathcal{G}^{\omega}_{\underline{\vee}}\cup\mathcal{A}$-derivation $\overline{D}$ of it of length $n\in\omega$. Take any $\varphi\in(\mathrm{img}\,\Delta)\neq\varnothing$. Then, in view of left-side basic structural rules, $\langle\langle D_i\uplus(\vdash\varphi)\rangle_{i\in n},\Phi\rangle$ is a $\mathrm{Cn}_{\mathcal{G}^{\omega\setminus 1}_{\underline{\vee}}\cup\mathcal{A}_{\setminus 1}}$-derivation of Φ, as required. $\Box$

Combining Lemmas 5.2 and 5.3, we first get:

Corollary 5.4. *Any purely multi-conclusion Σ-sequent is true in $\mathcal{A}$ iff it is derivable in $\mathcal{G}_{\underline{\vee}}^{\omega\backslash 1} \cup \mathcal{A}_{\backslash 1}$.*

And what is more, we also have:

Corollary 5.5. *Any purely single-conclusion Σ-sequent is true in $\mathcal{A}$ iff it is derivable in $\mathcal{G}_{\underline{\vee}}^{2\backslash 1} \cup \tau_{\underline{\vee}}[\mathcal{A}_{\backslash 1}]$.*

Proof: The "if" part is by the fact that $\mathcal{A}$, being a $\underline{\vee}$-disjunctive model of $\mathcal{A}_{\backslash 1}$, is then a model of $\mathcal{G}_{\underline{\vee}}^{2\backslash 1} \cup \tau_{\underline{\vee}}[\mathcal{A}_{\backslash 1}]$. The converse is by Corollaries 4.10, 5.4 and the diagonality of $\tau_{\underline{\vee}} \restriction \mathrm{Seq}_{\Sigma}^{2}$. □

Given an axiomatic [finite] purely single-conclusion sequent Σ-calculus $\mathcal{G}$, we have the [finite] Hilbert-style Σ-calculus $(\mathcal{G}{\downarrow}) \triangleq \{(\mathrm{img}\,\Gamma)/\varphi \mid (\Gamma \vdash \varphi) \in \mathcal{G}\}$. Conversely, given a Hilbert-style Σ-calculus $\mathcal{C}$, we have the axiomatic purely single-conclusion sequent Σ-calculus $(\mathcal{C}{\uparrow}) \triangleq \{(\Gamma \vdash \varphi) \in \mathrm{Seq}_{\Sigma}^{2\backslash 1} \mid ((\mathrm{img}\,\Gamma)/\varphi) \in \mathcal{C}\}$, in which case $(\mathcal{C}{\uparrow}{\downarrow}) = \mathcal{C}$. Set $\mathcal{H} \triangleq ((\mathcal{D}_{\underline{\vee}} \cup (\tau_{\underline{\vee}}[\mathcal{A}] \cap \mathrm{Seq}_{\Sigma}^{0\vdash(2\backslash 1)}){\downarrow}) \cup (\sigma_{+1}[(\tau_{\underline{\vee}}[\mathcal{A}] \cap \mathrm{Seq}_{\Sigma}^{(\omega\backslash 1)\vdash(2\backslash 1)}){\downarrow}] \,\underline{\vee}\, x_0) \cup \{(\sigma_{+1}[\mathrm{img}\,\Gamma] \,\underline{\vee}\, x_0)/x_0 \mid \Gamma \in (\mathrm{Fm}_{\Sigma}^{\omega})^{*}, (\Gamma \vdash) \in \tau_{\underline{\vee}}[\mathcal{A}]\})$. This is finite, whenever Σ is finite, for $\mathcal{A}$ is finite in that case.

Theorem 5.6. *The logic of $\mathcal{A}$ is axiomatized by $\mathcal{H}$.*

Proof: First of all, recall that $C \triangleq \mathrm{Cn}_{\mathcal{D}_{\underline{\vee}}}$ is $\underline{\vee}$-disjunctive (cf. Theorem 4.5), in which case, in particular, it satisfies (2.3), (2.5), (2.6) and (2.7), and so, for any $\Gamma \in \wp_{\omega}(\mathrm{Fm}_{\Sigma}^{\omega})$, any extension of C satisfies $(\sigma_{+1}[\Gamma] \,\underline{\vee}\, x_0)/x_0$ iff it satisfies $(\sigma_{+1}[\sigma_{+1}[\Gamma]] \,\underline{\vee}\, x_0)/(x_1 \,\underline{\vee}\, x_0)$. Therefore, $C' \triangleq \mathrm{Cn}_{\mathcal{H}}$ is equally axiomatized by $\mathcal{C}' \triangleq (\mathcal{D}_{\underline{\vee}} \cup (\mathcal{C} \cap \mathrm{Fm}_{\Sigma}^{\omega}) \cup (\sigma_{+1}[\mathcal{C} \setminus \mathrm{Fm}_{\Sigma}^{\omega}] \,\underline{\vee}\, x_0))$, where $\mathcal{C} \triangleq (\tau_{\underline{\vee}}[\mathcal{A}_{\backslash 1}]{\downarrow})$.

Next, $\mathcal{A}$, being a $\underline{\vee}$-disjunctive model of $\mathcal{A}_{\backslash 1}$, is so of $\tau_{\underline{\vee}}[\mathcal{A}_{\backslash 1}]$, and so of $\mathcal{C}$, and so of $\mathcal{C}'$, in view of Lemma 3.1.

Conversely, consider any Σ-rule $\mathcal{R} = (X/\varphi)$ true in $\mathcal{A}$. Take any bijection $\Gamma : |X| \to X$. Then, the purely single-conclusion Σ-sequent $\Phi \triangleq (\Gamma \vdash \varphi)$ is true in $\mathcal{A}$, and so is derivable in $\mathcal{G}_{\underline{\vee}}^{2\backslash 1} \cup \tau_{\underline{\vee}}[\mathcal{A}_{\backslash 1}]$, in view of Corollary 5.5. On the other hand, by Corollary 3.3, C' is $\underline{\vee}$-disjunctive. Let S be the set of all rules satisfied in C' (viz., derivable in $\mathcal{H}$, i.e., in $\mathcal{C}'$), in which case $\mathcal{C} \subseteq S$, and so $\tau_{\underline{\vee}}[\mathcal{A}_{\backslash 1}] \subseteq T \triangleq (S{\uparrow})$. Therefore, in view of the structurality and $\underline{\vee}$-disjunctivity of C', T is $(\mathcal{G}_{\underline{\vee}}^{2\backslash 1} \cup \tau_{\underline{\vee}}[\mathcal{A}_{\backslash 1}])$-closed. Hence, T contains all those purely single-conclusion Σ-sequents, which are derivable in $\mathcal{G}_{\underline{\vee}}^{2\backslash 1} \cup \tau_{\underline{\vee}}[\mathcal{A}_{\backslash 1}]$ (in particular, Φ). Thus, $\mathcal{R} \in (T{\downarrow}) = S$, as required. □

5.1. Implicative case

Here, $\mathcal{A}$ is supposed to be a finite $\rhd$-implicative Σ-matrix with equality determinant $\Upsilon \ni x_0$, in which case it is $\veebar$-disjunctive, where $\veebar \triangleq \veebar_{\rhd}$ is *not* primary, and so is properly covered by the above discussion.

Let $\tau_{\rhd} : \mathrm{Seq}_{\Sigma}^{\omega\backslash 1} \to \mathrm{Fm}_{\Sigma}^{\omega}$ be defined as follows: by induction on the length of the left side Γ of any $(\Gamma \vdash \phi) \in \mathrm{Seq}_{\Sigma}^{\omega\backslash 1}$, set:

$$\begin{array}{rcl} \tau_{\rhd}(\vdash \phi) & \triangleq & \phi, \\ \tau_{\rhd}(\psi, \Gamma \vdash \phi) & \triangleq & (\psi \rhd \tau_{\rhd}(\Gamma \vdash \phi)), \end{array}$$

where $\psi \in \mathrm{Fm}_{\Sigma}^{\omega}$.

Example 5.7. When $\upsilon = x_0$ and $F = \rhd$, in which case $\rhd$ is a primary connective of Σ, one can always take $\lambda_{\mathcal{T}}(\upsilon, F) = \{\vdash x_0; x_1 \vdash\}$ and $\rho_{\mathcal{T}}(\upsilon, F) = \{x_0 \vdash x_1\}$ to satisfy (5.1) and (5.2), in which case $\boldsymbol{\lambda}(\upsilon, F) = \{x_0, (x_0 \rhd x_1) \vdash x_1\}$ and $\boldsymbol{\rho}(\upsilon, F) = \{\vdash (x_0 \rhd x_1), x_0; x_1 \vdash (x_0 \rhd x_1)\}$, and so elements of both $\tau_{\rhd}[\tau_{\veebar}[\boldsymbol{\lambda}(\upsilon, F)]] = \{x_0 \rhd ((x_0 \rhd x_1) \rhd x_1)\}$ and $\tau_{\rhd}[\tau_{\veebar}[\boldsymbol{\rho}(\upsilon, F)]] = (\{(3.4), (3.3)\}[x_0/x_1, x_1/x_0])$ are derivable in $\mathcal{I}_{\rhd}^{\mathrm{PL}}$, in view of Theorem 3.6, (3.1) and (3.2). □

In this way, let $\mathcal{A}'_{[\not\rhd]}$ be the set of all elements of $\boldsymbol{\lambda}(\upsilon, F) \cup \boldsymbol{\rho}(\upsilon, F)$, for all complex $\langle \Upsilon, \Sigma\rangle$-types $\langle \upsilon, F\rangle$ [but $\langle x_0, \rhd\rangle$, in case $\rhd \in \Sigma$ is primary]. Then, set $\mathcal{A}_{[\not\rhd]} \triangleq (\mathcal{A}'_{[\not\rhd]} \cup \mathcal{A}'' \cup \mathcal{A}''')$ and $\mathcal{I}_{[\not\rhd]} \triangleq (\mathcal{I}_{\rhd}^{\mathrm{PL}} \cup \tau_{\rhd}[\tau_{\veebar}[\mathcal{A}'_{[\not\rhd]\backslash 1}]])$.

Theorem 5.8. *The logic of $\mathcal{A}$ is axiomatized by $\mathcal{I}_{\not\rhd}$.*

Proof: First of all, note that, in view of Example 5.7, $C \triangleq \mathrm{Cn}_{\mathcal{I}_{\not\rhd}}$ is equally axiomatized by $\mathcal{I}$, and is $\veebar$-disjunctive, by Theorem 3.6.

Next, $\mathcal{A}$, being an $\rhd$-implicative (in particular, $\veebar$-disjunctive) model of $\mathcal{A}_{\backslash 1}$, is so of $\tau_{\veebar}[\mathcal{A}_{\backslash 1}]$, and so of $\mathcal{I}$.

Conversely, consider any Σ-rule $\mathcal{R} = (X/\varphi)$ true in $\mathcal{A}$. Take any bijection $\Gamma : |X| \to X$. Then, the purely single-conclusion Σ-sequent $\Phi \triangleq (\Gamma \vdash \varphi)$ is true in $\mathcal{A}$, and so is derivable in $\mathcal{G}_{\veebar}^{2\backslash 1} \cup \tau_{\veebar}[\mathcal{A}_{\backslash 1}]$, in view of Corollary 5.5. Let S be the set of all rules satisfied in C (viz., derivable in $\mathcal{I}_{\not\rhd}$, i.e., in $\mathcal{I}$), in which case $\mathcal{I} \subseteq S$, and so, by (3.2), $\tau_{\veebar}[\mathcal{A}_{\backslash 1}] \subseteq T \triangleq (S\uparrow)$. Therefore, in view of the structurality and $\veebar$-disjunctivity of C, T is $(\mathcal{G}_{\veebar}^{2\backslash 1} \cup \tau_{\veebar}[\mathcal{A}_{\backslash 1}])$-closed. Hence, T contains all those purely single-conclusion Σ-sequents, which are derivable in $\mathcal{G}_{\veebar}^{2\backslash 1} \cup \tau_{\veebar}[\mathcal{A}_{\backslash 1}]$ (in particular, Φ). Thus, $\mathcal{R} \in (T\downarrow) = S$, as required. □

6. Applications and examples

Here, we consider applications of the previous section, tacitly following notations adopted therein.

6.1. Disjunctive and implicative positive fragments of the classical logic

Here, we deal with the signature $\Sigma^{(\supset)}_{+[01]} \triangleq (\{\wedge,\vee\}[\cup\{\bot,\top\}](\cup\{\supset\}))$. By $\mathfrak{D}^{(\supset)}_{2[01]}$ we denote the $\Sigma^{(\supset)}_{+[01]}$-algebra such that $\mathfrak{D}^{(\supset)}_{2[01]}{\restriction}\Sigma_{+[01]}$ is the [bounded] distributive lattice given by the chain 2 ordered by inclusion (and $\supset^{\mathfrak{D}^{\supset}_{2[01]}}$ is the ordinary classical implication). Then, the logic of the $\vee$-disjunctive (and $\supset$-implicative) $\mathcal{D}^{(\supset)}_{2[01]} \triangleq \langle\mathfrak{D}^{(\supset)}_{2[01]},\{1\}\rangle$ with equality determinant $\Upsilon = \{x_0\}$ (cf. Example 1 of [10]) is the $\Sigma^{(\supset)}_{+[01]}$-fragment of the classical logic. Throughout the rest of this subsection, it is supposed that $\Sigma \subseteq \Sigma^{(\supset)}_{+,01}$ and $\mathcal{A} = (\mathcal{D}^{(\supset)}_{2,01}{\restriction}\Sigma)$, in which case $\mathcal{A}''' = \varnothing$.

First, in case $\Sigma = \{\supset\}$, both $\mathcal{A}'_{\not\supset}$ and $\mathcal{A}''$ are empty, and so is $\mathcal{A}_{\not\supset}$. In this way, we have the following well-known result:

Corollary 6.1. *The $\{\supset\}$-fragment of the classical logic is axiomatized by $\mathcal{I}^{\mathrm{PL}}_{\supset}$.*

Likewise, in case $\Sigma = \{\vee\}$, both $\mathcal{A}'$ and $\mathcal{A}''$ are empty, and so is $\mathcal{A}$. In this way, we get:

Corollary 6.2. *The $\{\vee\}$-fragment of the classical logic is axiomatized by $\mathcal{D}_{\vee}$.*

Next, let $\Sigma = \Sigma_+$. Then, $\mathcal{A}'' = \varnothing$, while one can take $\lambda_{\mathcal{T}}(x_0,\wedge) = \{x_0, x_1 \vdash\}$ and $\rho_{\mathcal{T}}(x_0,\wedge) = \{\vdash x_0; \vdash x_1\}$ to satisfy (5.1) and (5.2), in which case $\boldsymbol{\lambda}(x_0,\wedge) = \{(x_0\wedge x_1)\vdash x_0; (x_0\wedge x_1)\vdash x_1\}$ and $\boldsymbol{\rho}(x_0,\wedge) = \{x_0,x_1 \vdash (x_0\wedge x_1)\}$, and so $\mathcal{A} = \mathcal{A}' = \{(x_0\wedge x_1)\vdash x_0; (x_0\wedge x_1)\vdash x_1; x_0,x_1\vdash (x_0\wedge x_1)\}$. Thus, we get:

Corollary 6.3. *The Σ_+-fragment of the classical logic is axiomatized by the calculus $\mathcal{PC}_+$ resulted from $\mathcal{D}_\vee$ by adding the following rules:*

$$\begin{array}{ccc} C_1 & C_2 & C_3 \\ \dfrac{(x_1\wedge x_2)\vee x_0}{x_1\vee x_0} & \dfrac{(x_1\wedge x_2)\vee x_0}{x_2\vee x_0} & \dfrac{x_1\vee x_0; x_2\vee x_0}{(x_1\wedge x_2)\vee x_0} \end{array}$$

It is remarkable that the calculus $\mathcal{PC}_+$ consists of seven rules, while that which was found in [2] has nine rules. This demonstrates the practical applicability of our generic approach (more precisely, its factual ability to result in really "good" calculi to be enhanced a bit more by replacing appropriate pairs of rules/premises with single ones upon the basis of Proposition 4.6 and rules C_i, $i \in (4 \setminus 1)$, whenever it is possible, to be done below tacitly — "on the fly").

Likewise, let $\Sigma = \Sigma_+^{\supset}$. Then, $\mathcal{A}'' = \varnothing$, and so, taking Corollary 3.7(ii) and Example 5.1 into account, we have the following well-known result:

COROLLARY 6.4. *The $\Sigma_+^{\supset}$-fragment of the classical logic is axiomatized by the calculus $\mathcal{PC}_+^{\supset}$ resulted from $\mathcal{I}_{\supset}^{\mathrm{PL}}$ by adding the following axioms:*

$$(x_0 \wedge x_1) \supset x_i \qquad\qquad x_0 \supset (x_1 \supset (x_0 \wedge x_1))$$

$$x_i \supset (x_0 \vee x_1) \qquad (x_0 \supset x_2) \supset ((x_1 \supset x_2) \supset ((x_0 \vee x_1) \supset x_2))$$

where $i \in 2$.

Finally, let $\Sigma = \Sigma_{+,01}^{[\supset]}$, in which case $\mathcal{A}'$ is as above, while $\mathcal{A}'' = \{\vdash \top; \bot \vdash\}$, and so [taking Corollary 3.7(ii) into account] we get:

COROLLARY 6.5. *The $\Sigma_{+,01}^{[\supset]}$-fragment of the classical logic is axiomatized by the calculus $\mathcal{PC}_{+,01}^{[\supset]}$ resulted from $\mathcal{PC}_+^{[\supset]}$ by adding the following rules:*

$$\top \qquad\qquad \frac{\bot \vee x_0}{x_0}[\bot \supset x_0]$$

6.2. Miscellaneous four-valued expansions of Belnap's logic

From now on, it is supposed that $\Sigma \supseteq \Sigma_{\sim,+[01]} \triangleq (\Sigma_{+[01]} \cup \{\sim\})$, where $\sim$ is unary, $(\mathfrak{A}{\restriction}\Sigma_{\sim,+[01]}) = \mathfrak{DM}_{4[01]}$, where $(\mathfrak{DM}_{4[01]}{\restriction}\Sigma_{+[01]}) \triangleq \mathfrak{D}_{2[01]}^2$, while $\sim^{\mathfrak{DM}_{4[01]}}\langle i, j\rangle \triangleq \langle 1-j, 1-i\rangle$, for all $i, j \in 2$, in which case we use the following standard notations going back to [1]:

$$\mathsf{t} \triangleq \langle 1,1\rangle, \qquad \mathsf{f} \triangleq \langle 0,0\rangle, \qquad \mathsf{b} \triangleq \langle 1,0\rangle, \qquad \mathsf{n} \triangleq \langle 0,1\rangle,$$

and $\mathcal{A} \triangleq \langle \mathfrak{A}, \{\mathsf{b}, \mathsf{t}\}\rangle$, in which case it is $\vee$-disjunctive, while $\Upsilon = \{x_0, {\sim}x_0\}$ is an equality determinant for it (cf. Example 2 of [10]), whereas $\mathcal{A}''' = \varnothing$. (Since the logic $B_{4[01]}$ of $\mathcal{A}{\restriction}\Sigma_{\sim,+[01]}$ is the [bounded version of] Belnap's logic, the logic of $\mathcal{A}$ is a four-valued expansion of B_4.)

First, let $\Sigma = \Sigma_{\sim,+}$, in which case $\mathcal{A}'' = \varnothing$, while the case of the complex $\langle \Upsilon, \Sigma\rangle$-type $\langle x_0, \wedge\rangle$ is as in the previous subsection, whereas others but

$\langle x_0, \vee\rangle$ are as follows. First of all, one can take $\lambda_{\mathcal{T}}(\sim x_0, \vee) = \{\sim x_0, \sim x_1 \vdash\}$ and $\rho_{\mathcal{T}}(\sim x_0, \vee) = \{\vdash \sim x_0; \vdash \sim x_1\}$ to satisfy (5.1) and (5.2), in which case $\boldsymbol{\lambda}(\sim x_0, \vee) = \{\sim(x_0 \vee x_1) \vdash \sim x_0; \sim(x_0 \vee x_1) \vdash \sim x_1\}$ and $\boldsymbol{\rho}(\sim x_0, \vee) = \{\sim x_0, \sim x_1 \vdash \sim(x_0 \vee x_1)\}$. Likewise, one can take $\lambda_{\mathcal{T}}(\sim x_0, \wedge) = \{\sim x_0 \vdash ; \sim x_1 \vdash\}$ and $\rho_{\mathcal{T}}(\sim x_0, \wedge) = \{\vdash \sim x_0, \sim x_1\}$ to satisfy (5.1) and (5.2), in which case $\boldsymbol{\lambda}(\sim x_0, \wedge) = \{\sim(x_0 \wedge x_1) \vdash \sim x_0, \sim x_1\}$ and $\boldsymbol{\rho}(\sim x_0, \wedge) = \{\sim x_0 \vdash \sim(x_0 \wedge x_1); \sim x_1 \vdash \sim(x_0 \wedge x_1)\}$. Finally, one can take $\lambda_{\mathcal{T}}(\sim x_0, \sim) = \{x_0 \vdash\}$ and $\rho_{\mathcal{T}}(\sim x_0, \sim) = \{\vdash x_0\}$ to satisfy (5.1) and (5.2), in which case $\boldsymbol{\lambda}(\sim x_0, \sim) = \{\sim\sim x_0 \vdash x_0\}$ and $\boldsymbol{\rho}(\sim x_0, \sim) = \{x_0 \vdash \sim\sim x_0\}$. In this way, we get:

Corollary 6.6. *B_4 is axiomatized by the calculus $\mathcal{B}$ resulted from $\mathcal{PC}_+$ by adding the following rules as well as the inverse to these:*

$$\begin{array}{ccc} NN & ND & NC \\[1ex] \dfrac{x_1 \vee x_0}{\sim\sim x_1 \vee x_0} & \dfrac{(\sim x_1 \wedge \sim x_2) \vee x_0}{\sim(x_1 \vee x_2) \vee x_0} & \dfrac{(\sim x_1 \vee \sim x_2) \vee x_0}{\sim(x_1 \wedge x_2) \vee x_0} \end{array}$$

The calculus $\mathcal{B}$ has 13 rules, while the very first axiomatization of B_4 discovered in [8] (cf. Definition 5.1 and Theorem 5.2 therein) has 15 rules, "two rules win" being just to the advance of the present study with regard to [2] (cf. the previous subsection).

Now, let $\Sigma = \Sigma_{\sim,+,01}$, in which case $\mathcal{A}'$ is as above, while $\mathcal{A}'' = \{\top; \sim\bot; \bot \vdash; \sim\top \vdash\}$, and so we get:

Corollary 6.7. *$B_{4,01}$ is axiomatized by the calculus $\mathcal{B}_{01}$ resulted from $\mathcal{B} \cup \mathcal{PC}_{+,01}$ by adding the following axiom and rule:*

$$\sim\bot \qquad\qquad \frac{\sim\top \vee x_0}{x_0}$$

6.2.1. The classical expansion

Here, it is supposed that $\Sigma = \Sigma_{\simeq,+[01]} \triangleq (\Sigma_{\sim,+[01]} \cup \{\neg\})$, where $\neg$ is unary, while $\neg^{\mathfrak{A}}\langle i, j\rangle \triangleq \langle 1-i, 1-j\rangle$, for all $i, j \in 2$. Then, one can take $\lambda_{\mathcal{T}}(x_0, \neg) = \{\vdash x_0\}$ and $\rho_{\mathcal{T}}(x_0, \neg) = \{x_0 \vdash\}$ to satisfy (5.1) and (5.2), in which case $\boldsymbol{\lambda}(x_0, \neg) = \{\neg x_0, x_0 \vdash\}$ and $\boldsymbol{\rho}(x_0, \neg) = \{\vdash x_0, \neg x_0\}$. Likewise, one can take $\lambda_{\mathcal{T}}(\sim x_0, \neg) = \{\vdash \sim x_0\}$ and $\rho_{\mathcal{T}}(\sim x_0, \neg) = \{\sim x_0 \vdash\}$ to satisfy (5.1) and (5.2), in which case $\boldsymbol{\lambda}(\sim x_0, \neg) = \{\sim\neg x_0, \sim x_0 \vdash\}$ and $\boldsymbol{\rho}(\sim x_0, \neg) = \{\vdash \sim x_0, \sim\neg x_0\}$. Thus, we get:

COROLLARY 6.8. *The logic of $\mathcal{A}$ is axiomatized by the calculus $\mathcal{CB}_{[01]}$ resulted from $\mathcal{B}_{[01]}$ by adding the following rules:*

$$\begin{array}{cccc} N_1 & N_2 & N_3 & N_4 \\ \frac{(\neg x_1 \wedge x_1) \vee x_0}{x_0} & x_0 \vee \neg x_0 & \frac{(\sim\neg x_1 \wedge \sim x_1) \vee x_0}{x_0} & \sim x_0 \vee \sim\neg x_0 \end{array}$$

6.2.2. The bilattice expansions

Here, it is supposed that $\Sigma = \Sigma_{\sim/\simeq,2:+[01]} \triangleq (\Sigma_{\sim/\simeq,+[01]} \cup \{\sqcap, \sqcup\}[\cup\{\mathbf{0}, \mathbf{1}\}])$, where $\sqcap$ and $\sqcup$ (*knowledge* conjunction and disjunction, respectively) are binary [while $\mathbf{0}$ and $\mathbf{1}$ are nullary], whereas

$$(\langle i, j\rangle(\sqcap/\sqcup)^{\mathfrak{A}}\langle k, l\rangle) \triangleq \langle(\min/\max)(i, k), (\max/\min)(j, l)\rangle,$$

for all $i, j, k, l \in 2$ [while $\mathbf{0}^{\mathfrak{A}} \triangleq \mathsf{n}$ and $\mathbf{1}^{\mathfrak{A}} \triangleq \mathsf{b}$].

First, let $\Sigma = \Sigma_{\sim,2:+}$, in which case $\mathcal{A}'' = \varnothing$. Then, one can take $\lambda_{\mathcal{T}}(x_0, \sqcap) = \{x_0, x_1 \vdash\}$ and $\rho_{\mathcal{T}}(x_0, \sqcap) = \{\vdash x_0; \vdash x_1\}$ to satisfy (5.1) and (5.2), in which case $\boldsymbol{\lambda}(x_0, \sqcap) = \{(x_0 \sqcap x_1) \vdash x_0; (x_0 \sqcap x_1) \vdash x_1\}$ and $\boldsymbol{\rho}(x_0, \sqcap) = \{x_0, x_1 \vdash (x_0 \sqcap x_1)\}$. Likewise, one can take $\lambda_{\mathcal{T}}(x_0, \sqcup) = \{x_0 \vdash ; x_1 \vdash\}$ and $\rho_{\mathcal{T}}(x_0, \sqcup) = \{\vdash x_0, x_1\}$ to satisfy (5.1) and (5.2), in which case $\boldsymbol{\lambda}(x_0, \sqcup) = \{(x_0 \sqcup x_1) \vdash x_0, x_1\}$ and $\boldsymbol{\rho}(x_0, \sqcup) = \{x_0 \vdash (x_0 \sqcup x_1); x_1 \vdash (x_0 \sqcup x_1)\}$. Next, one can take $\lambda_{\mathcal{T}}(\sim x_0, \sqcap) = \{\sim x_0, \sim x_1 \vdash\}$ and $\rho_{\mathcal{T}}(\sim x_0, \sqcap) = \{\vdash \sim x_0; \vdash \sim x_1\}$ to satisfy (5.1) and (5.2), in which case $\boldsymbol{\lambda}(\sim x_0, \sqcap) = \{\sim(x_0 \sqcap x_1) \vdash \sim x_0; \sim(x_0 \sqcap x_1) \vdash \sim x_1\}$ and $\boldsymbol{\rho}(\sim x_0, \sqcap) = \{\sim x_0, \sim x_1 \vdash \sim(x_0 \sqcap x_1)\}$. Finally, one can take $\lambda_{\mathcal{T}}(\sim x_0, \sqcup) = \{\sim x_0 \vdash; \sim x_1 \vdash\}$ and $\rho_{\mathcal{T}}(\sim x_0, \sqcup) = \{\vdash \sim x_0, \sim x_1\}$ to satisfy (5.1) and (5.2), in which case $\boldsymbol{\lambda}(\sim x_0, \sqcup) = \{\sim(x_0 \sqcup x_1) \vdash \sim x_0, \sim x_1\}$ and $\boldsymbol{\rho}(\sim x_0, \sqcup) = \{\sim x_0 \vdash \sim(x_0 \sqcup \sim x_1); \sim x_1 \vdash \sim(x_0 \sqcup x_1)\}$. Thus, we get:

COROLLARY 6.9. *The logic of $\mathcal{A}$ is axiomatized by the calculus $\mathcal{BL}$ resulted from adding to $\mathcal{B}$ the following rules as well as the inverse to these:*

$$\begin{array}{cccc} KC & KD & NKC & NKD \\ \frac{(x_1 \wedge x_2) \vee x_0}{(x_1 \sqcap x_2) \vee x_0} & \frac{(x_1 \vee x_2) \vee x_0}{(x_1 \sqcup x_2) \vee x_0} & \frac{(\sim x_1 \wedge \sim x_2) \vee x_0}{\sim(x_1 \sqcap x_2) \vee x_0} & \frac{(\sim x_1 \vee \sim x_2) \vee x_0}{\sim(x_1 \sqcup x_2) \vee x_0} \end{array}$$

Likewise, let $\Sigma = \Sigma_{\sim,2+,01}$, in which case $\mathcal{A}'$ is as above, while $\mathcal{A}'' = (\{\bot \vdash; \top\} \cup \{\sim^i\mathbf{0} \vdash; \sim^i\mathbf{1} \mid i \in 2\})$, and so we have:

COROLLARY 6.10. *The logic of $\mathcal{A}$ is axiomatized by the calculus $\mathcal{BL}_{01}$ resulted from adding to $\mathcal{BL} \cup \mathcal{B}_{01}$ the following axioms and rules:*

$$\sim^i \mathbf{1} \qquad\qquad \frac{\sim^i \mathbf{0} \vee x_0}{x_0}$$

where $i \in 2$.

Finally, when $\Sigma = \Sigma_{\simeq,2:+[01]}$, we have:

COROLLARY 6.11. *The logic of $\mathcal{A}$ is axiomatized by the calculus $\mathcal{CB} \cup \mathcal{BL}_{[01]}$.*

6.2.3. Implicative expansions

Here, it is supposed that $\supset \in \Sigma$, while

$$(a \supset^{\mathfrak{A}} b) \triangleq \begin{cases} b & \text{if } \pi_0(a) = 1, \\ \mathsf{t} & \text{otherwise,} \end{cases}$$

for all $a, b \in 2^2$, in which case $\mathcal{A}$ is $\supset$-implicative.

First, let $\Sigma = (\Sigma_{\sim,+} \cup \{\supset\})$. Clearly, one can take $\lambda_{\mathcal{T}}(\sim x_0, \supset) = \{x_0, \sim x_1 \vdash\}$ and $\rho_{\mathcal{T}}(\sim x_0, \supset) = \{\vdash x_0; \vdash \sim x_1\}$ to satisfy (5.1) and (5.2), in which case $\boldsymbol{\lambda}(\sim x_0, \supset) = \{\sim(x_0 \wedge x_1) \vdash x_0; \sim(x_0 \wedge x_1) \vdash \sim x_1\}$ and $\boldsymbol{\rho}(\sim x_0, \supset) = \{x_0, \sim x_1 \vdash \sim(x_0 \wedge x_1)\}$. Therefore, taking Corollary 3.7(ii) and Example 5.1 into account, we get:

COROLLARY 6.12. *The logic of $\mathcal{A}$ is axiomatized by the calculus $\mathcal{B}^{\supset}$ resulted from $\mathcal{PC}_{+}^{\supset}$ by adding the following axioms:*

$$\begin{array}{lr} \sim\sim x_0 \supset x_0 & x_0 \supset \sim\sim x_0 \\ \sim(x_0 \vee x_1) \supset \sim x_i & \sim x_0 \supset (\sim x_1 \supset \sim(x_0 \vee x_1)) \\ \sim x_i \supset \sim(x_0 \wedge x_1) & (\sim x_0 \supset x_2) \supset ((\sim x_1 \supset x_2) \supset (\sim(x_0 \wedge x_1) \supset x_2)) \\ \sim(x_0 \supset x_1) \supset \sim^i x_i & x_0 \supset (\sim x_1 \supset \sim(x_0 \supset x_1)) \end{array}$$

where $i \in 2$.

It is remarkable that $\mathcal{B}^{\supset}$ is essentially the calculus Par introduced in [7] but regardless to any semantics. In this way, the present study provides a new (and quite immediate) insight into the issue of semantics of Par first being due to [9] but with using the intermediate equivalent (via $\tau_{\supset} \circ \tau_{\vee}$ and the diagonal mapping) purely multi-conclusion sequent calculus $GPar$ actually introduced in [7] and then studied semantically in [9].

Likewise, in case $\Sigma = (\Sigma_{\sim,+,01} \cup \{\supset\})$, we have:

COROLLARY 6.13. *The logic of $\mathcal{A}$ is axiomatized by the calculus $\mathcal{B}^{\supset}_{01}$ resulted from $\mathcal{B}^{\supset} \cup \mathcal{PC}^{\supset}_{+,01}$ by adding the following axioms:*

$$\sim\perp \qquad\qquad \sim\top \supset x_0$$

Now, let $(\Sigma = (\Sigma_{\sim,2:+} \cup \{\supset\})$. Then, we have:

COROLLARY 6.14. *The logic of $\mathcal{A}$ is axiomatized by the calculus $\mathcal{BL}^{\supset}$ resulted from $\mathcal{B}^{\supset}$ by adding the following axioms:*

$$\begin{array}{ll}
(x_0 \sqcap x_1) \supset x_i & x_0 \supset (x_1 \supset (x_0 \sqcap x_1)) \\
x_i \supset (x_0 \sqcup x_1) & (x_0 \supset x_2) \supset ((x_1 \supset x_2) \supset ((x_0 \sqcup x_1) \supset x_2)) \\
\sim(x_0 \sqcap x_1) \supset \sim x_i & \sim x_0 \supset (\sim x_1 \supset \sim(x_0 \sqcap x_1)) \\
\sim x_i \supset \sim(x_0 \sqcup x_1) & (\sim x_0 \supset x_2) \supset ((\sim x_1 \supset x_2) \supset (\sim(x_0 \sqcup x_1) \supset x_2))
\end{array}$$

where $i \in 2$.

Likewise, when $(\Sigma = (\Sigma_{\sim,2:+,01} \cup \{\supset\})$, we have:

COROLLARY 6.15. *The logic of $\mathcal{A}$ is axiomatized by the calculus $\mathcal{BL}^{\supset}_{01}$ resulted from $\mathcal{BL}^{\supset} \cup \mathcal{B}^{\supset}_{01}$ by adding the following axioms:*

$$\sim^i\mathbf{1} \qquad\qquad \sim^i\mathbf{0} \supset x_0$$

where $i \in 2$.

Further, let $\Sigma = (\Sigma_{\simeq,+[01]} \cup \{\supset\})$. Then, taking (3.2) and Corollary (3.7)(i) into account, we have:

COROLLARY 6.16. *The logic of $\mathcal{A}$ is axiomatized by the calculus $\mathcal{CB}^{\supset}_{[01]}$ resulted from $\mathcal{B}^{\supset}_{[01]}$ by adding the axioms N_2, N_4 and the following ones:*

$$\sim^i\neg x_1 \supset (\sim^i x_i \supset x_0),$$

where $i \in 2$.

Finally, when $\Sigma = (\Sigma_{\simeq,2:+[01]} \cup \{\supset\})$, we have:

COROLLARY 6.17. *The logic of $\mathcal{A}$ is axiomatized by the calculus $\mathcal{CB}^{\supset} \cup \mathcal{BL}^{\supset}_{[01]}$.*

Bibliography

[1] N. D. Belnap, Jr, *A useful four-valued logic*, [in:] J. M. Dunn, G. Epstein (eds.), **Modern uses of multiple-valued logic**, D. Reidel Publishing Company, Dordrecht (1977), pp. 8–37.

[2] K. Dyrda, T. Prucnal, *On finitely based consequence determined by a distributive lattice*, **Bulletin of the Section of Logic**, vol. 9 (1980), pp. 60–66.

[3] G. Gentzen, *Untersuchungen über das logische Schliessen*, **Mathematische Zeitschrift**, vol. 39 (1934), pp. 176–210, 405–431.

[4] J. Łoś, R. Suszko, *Remarks on sentential logics*, **Indagationes Mathematicae**, vol. 20 (1958), pp. 177–183.

[5] A. I. Mal'cev, **Algebraic systems**, Springer Verlag, New York (1965).

[6] C. Peirce, *On the Algebra of Logic: A Contribution to the Philosophy of Notation*, **American Journal of Mathematics**, vol. 7 (1885), pp. 180–202.

[7] V. M. Popov, *Sequential formulations of some paraconsistent logical systems*, [in:] V. A. Smirnov (ed.), **Syntactic and semantic investigations of non-extensional logics**, Nauka, Moscow (1989), pp. 285–289, in Russian.

[8] A. P. Pynko, *Characterizing Belnap's logic via De Morgan's laws*, **Mathematical Logic Quarterly**, vol. 41(4) (1995), pp. 442–454.

[9] A. P. Pynko, *Functional completeness and axiomatizability within Belnap's four-valued logic and its expansions*, **Journal of Applied Non-Classical Logics**, vol. 9(1/2) (1999), pp. 61–105, special Issue on Multi-Valued Logics.

[10] A. P. Pynko, *Sequential calculi for many-valued logics with equality determinant*, **Bulletin of the Section of Logic**, vol. 33(1) (2004), pp. 23–32.

[11] J. Zygmunt, *Entailment relations on logical matrices I*, **Bulletin of the Section of Logic**, vol. 8 (1979), pp. 112–115.

V.M. Glushkov Institute of Cybernetics
Glushkov prosp. 40, Kiev, 03680, Ukraine
pynko@i.ua