Application of the Different Topology Models on a Finite Set to the Enumeration and Calculation of All Labeled T0-Topologies with a Given Weight

Anna Skryabina and Polina Stegantseva

December 5, 2023

Application of the Different Topology Models on a Finite Set to the Enumeration and Calculation of All Labeled T_0-Topologies With a Given Weight

Anna Skryabina[1], Polina Stegantseva[2]

[1] Zaporizhzhya National University, Zhukovskogo str. 66, building 1, office 21-A, Zaporizhzhya, Ukraine
[2] Zaporizhzhya National University, Zhukovskogo str. 66, building 1, office 21-A, Zaporizhzhya, Ukraine

Abstract

The tasks of studying topologies on a finite set are closely related to the tasks of studying bijunctive Boolean functions, partial orders, and graphs. This explains the possibility of using combinatorial methods for their research, and also makes such topologies important mathematical objects.

The problem of enumeration of homotopy types of finite topological spaces is as important as the problem of enumeration of homotopy types of finite simplicial complexes. These tasks are related to the digital processing of images based on finite sets of observations, that is, trying to understand the content of the image based on the concept of proximity of points.

The problem of computing topologies on a finite set remains unsolved at the moment, so interest in it remains, publications by researchers from different countries with new results have appeared.

This paper uses two models to study topology on a finite set. In the first model, each topology is assigned a nondecreasing sequence of non-negative integers (topology vector), in the second - the conjunctive normal form of the Boolean function.

Keywords

topology weight, topology vector, minimum element neighborhood, T_0-topologies, consistent topologies.

1. Introduction

A topological structure is often the base on which other mathematical structures are built. Topological spaces and their continuous mappings appear in many branches of mathematics. Studies of topology on the finite sets have numerical applications and are parallel to studies of partially ordered sets, finite graphs, and Boolean functions. Topologies on a finite set play a key role in the theory of pattern recognition [1], [2], the theory of molecular structures [3], [4], geometries on finite sets [5].

The first research results of topologies on finite sets appeared in the 1960s and 1970s [6], [7]. T_0-topologies play an important role in the study of topologies on a finite set. In the article by J.W. Evans., F. Harary, M.S. Lynn [6] a formula for the relationship between the number $T(n)$ of all topologies on an n-element set and the number $\tilde{T}(m)$ of all T_0-topologies on its m-element subsets is given:

$$T(n) = \sum_{m=1}^{n} S(n,m) \cdot \tilde{T}(m),$$

where $S(n,m)$ are Stirling numbers of the second kind.

Most early and modern publications describe the structure of topologies on a finite set and perform direct calculations of the number of topologies. There were also attempts to find patterns of transition from topologies on $(n-1)$-element to topologies on n -element set.

Research Twinning Conference on Digitalisation and Digital Transformation 2023, March 27-30, 2023, Liverpool, UK
anna_29_95@ukr.net (A. Skryabina); stegpol@gmail.com (P. Stegantseva)
0000-0001-5391-2001 (A. Skryabina); 0000-0001-8871-139X (P. Stegantseva)

2. Study of T_0-topologies with weight $2^{n-2} < m \le 2^{n-1}$ using the topology vector

A topology on an arbitrary n-element set X is said to belong to the m -class of topologies (or has weight m) if it consists of m elements ($m = 2,3,\dots,2^n$).

Let X be a finite set and let τ_X be a topology on it. A set A from τ_X is called maximal in τ_X if A is not contained in any other set from τ_X except the set X itself. The set X in this case is called ambient.

Assume that A, X, $A \subset X$ are given sets and τ_A is a given topology. We will restore all such topologies on the set X, for which the set A is maximal and which induce the topology τ_A on the set A (in this case the topology τ_A and the corresponding topologies on X will be called consistent).

The intersection of all neighborhoods of a point $x_i \in X$ is called the minimum neighborhood M_i of this point. The index of a point a in the topology is a number equal to the number of points different from a in its minimal neighborhood M_a [8].

Topologies with weight $m > 2^{n-1}$ are called close to discrete topology, they are fully investigated in [8] using the concept of a topology vector – a nondecreasing sequence of nonnegative integers that determine the minimal neighborhoods of the elements of a given finite set. In particular, it is shown that all of them are T_0-topologies with topology vectors of three types:

a) $v(\tau) = (0,\dots,0,\alpha_n)$, $1 \le \alpha_n \le n-1$, $|\tau| = 2^{n-1} + 2^{n-1-\alpha_n}$.

в) $v(\tau) = \left(\underbrace{0,\dots,0}_{k},1,\dots,1\right)$, $1 \le k \le n-2$ i $\bigcap_{m=k+1}^{n} M_m = \{y\}$, $|\tau| = 2^{n-1} + 2^{k-1}$.

c) $v(\tau) = (0,\dots,0,1,1)$ i $M_{n-1} \cap M_n = \emptyset$, $|\tau| = 2^{n-1} + 2^{n-4}$, where M_i - the minimal neighborhood of the element $x_i \in X$.

The sequence of the weights of the topologies close to the discrete topology on an n-element set has the form

$$(2^{n-1}+2^0),(2^{n-1}+2^1),\underbrace{\dots}_{2^1-1}(2^{n-1}+2^2),\dots,5\cdot 2^{n-3},\underbrace{\dots}_{2^{n-3}-1},3\cdot 2^{n-2},\underbrace{\dots}_{2^{n-2}-1},2^n,$$

where the numbers of classes that do not have any topology are indicated by curly brackets.

There are also published results about topologies with weight $2^{n-2} < m \le 2^{n-1}$. We note the works of R. Stanley [9] and Kolli [10], [11]. In [12] T_0-topologies with weight $2^{n-2} < m \le 2^{n-1}$ on an n-element set, consistent with topologies close to discrete on an $(n-1)$-element set with vectors of types a) and c) were found. Here we will add information about T_0-topologies, which are consistent with close to discrete topology of type b). They can be described by vectors $(\underbrace{0,\dots,0}_{k},\underbrace{1,\dots,1}_{n-k-1},\alpha_n)$, where $1 \le k \le n-3$, $2 \le \alpha_n \le n-1$ and $\bigcap_{m=k+1}^{n-1} M_m = \{y\}$. The non-homeomorphic topologies among them have different forms of the minimal neighborhood of the element x_n. For example, if the minimal neighborhood of an element x_n. has the form $M_n = \left\{x_1,\dots,y,\dots,x_d,\underbrace{x_{n-(\alpha_n-d)},\dots,x_{n-1}}_{\alpha_n-d},x_n\right\}$, i.e. $\bigcap_{m=k+1}^{n-1} M_m \cap M_n = \{y\}$, then the weight of such topologies is calculated by the formula $|\tau| = 2^{n-2} + 2^{k-1} + 2^{k-d} + 2^{k-d}(2^{n-k-(\alpha_n-d+1)} - 1)$. If the minimal neighborhood of an element x_n has the form $M_n = \{x_1,\dots,x_{\alpha_n+1},x_n\}$ and contains no element y, i.e. $\bigcap_{m=k+1}^{n-1} M_m \cap M_n = \emptyset$, then the weight of such topologies is calculated by the formula $|\tau| = 2^{n-2} + 2^{k-1} + 2^{k-\alpha_n} + 2^{k-(\alpha_n+1)}(2^{n-k-1} - 1)$. These are all possible cases in which T_0-topologies have weight $|\tau| \in [5\cdot 2^{n-4}, 2^{n-1}]$.

Let us now consider some classes of T_0-topologies on an n-element set with weight $2^{n-2} < m \le 2^{n-1}$, which are not consistent with close to discrete topology on an $(n-1)$-element set.

In work [12] it was shown that in classes of topologies with weight $|\tau| \in [13 \cdot 2^{n-5}, 2^{n-1}]$ all T_0-topologies are consistent with close to discrete or are dual to them. Among the classes of T_0-topologies with weight $|\tau| \in [5 \cdot 2^{n-4}, 13 \cdot 2^{n-5})$ there are such classes that are not exhausted by T_0-topologies consistent with close to discrete and dual to them. This is proven by comparing the total number of T_0-topologies in these classes with the number of T_0-topologies in them that are consistent with close to discrete and dual to them. There was also an example of a class in which there is not a single T_0-topology consistent with close to discrete.

Now we continue the list of T_0-topologies with weight $|\tau| \in (2^{n-2}, 2^{n-1}]$, which are not topologies consistent with close to discrete.

Consider the T_0-topologies with vectors $(0, \ldots, 0,1,3,3)$, $(0, \ldots, 0,1,2, \alpha_n)$, where $2 \leq \alpha_n \leq n-1$, $(0, \ldots, 0,1,1,1,2,2)$, $(0, \ldots, 0,1,1,1,2,3)$, $(0, \ldots, 0,1,1,1,1,2,2)$, $(0, \ldots, 0,1,2,2,2)$, $(0, \ldots, 0,2, \ldots, 2)$. Obviously, they are not topologies consistent with close to discrete. Let's find the conditions on the minimal boundaries of elements under which the weight of such T_0-topologies belongs to the interval $(2^{n-2}, 2^{n-1}]$.

For example, let us analyze T_0-topologies with vector $(0, \ldots, 0,1,3,3)$ for $n \geq 5$. It is sufficient to consider only the minimal neighborhoods of elements whose index in the topology vector is not equal to zero. In this case, these are M_{n-2}, M_{n-1}, and M_n, where M_{n-2} is a two-element set, and M_{n-1} and M_n are four-element sets. The weight of the topology will depend on the pairwise intersections of the specified minimal neighborhoods. We divided all possible cases into three classes, describing the powers of pairwise intersections of minimal neighborhoods. Let us formulate the obtained results.

1) If $|M_{n-2} \cap M_{n-1}| = 2$ and the power of at least one of the other two intersections is zero, then the weight of the topology is less than or equal to 2^{n-2}, that is, it does not belong to the set $(2^{n-2}, 2^{n-1}]$;

2) If $|M_{n-2} \cap M_{n-1}| = 1$ and at least one of the other two intersections is an empty set, then the weight of the topology does not belong to the set $(2^{n-2}, 2^{n-1}]$;

3) If $|M_{n-2} \cap M_{n-1}| = 0$, then the weight of the topology does not belong to the set $(2^{n-2}, 2^{n-1}]$.

In all other cases, T_0-topologies with vector $(0, \ldots, 0,1,3,3)$ for $n \geq 5$ have weight $|\tau| \in (2^{n-2}, 2^{n-1}]$.

In the case of T_0-topologies with the vector $(0, \ldots, 0,1,2, \alpha_n)$, where $2 \leq \alpha_n \leq n-1$, the analysis of the intersections of the minimal neighborhoods M_{n-2}, M_{n-1}, and M_n has made it possible to obtain the following conclusions:

1) If $|M_{n-2} \cap M_{n-1}| = 2$ or $|M_{n-2} \cap M_{n-1}| = 1$, then the weight of the topology belongs to the set $(2^{n-2}, 2^{n-1}]$;

2) If $|M_{n-2} \cap M_{n-1}| = 0$, then the weight of the topology does not belong to the set $(2^{n-2}, 2^{n-1}]$.

For the rest of the vectors, a similar analysis was carried out, conditions on minimal neighborhoods were found, and it was shown that among the corresponding T_0-topologies there are those that have a weight from the set $(2^{n-2}, 2^{n-1}]$.

3. Application of 2-CNF Boolean functions for the study of T_0-topologies

When modeling topologies with Boolean functions, a single conjunctive normal form of a certain form (maximal 2-CNF) is matched to each T_0-topology [13]. This tool turned out to be quite convenient for describing mutually dual and self-dual T_0-topologies.

We will show the application of this model for enumerating and counting all T_0-topologies with weight $25 \cdot 2^{n-6}$.

In work [10], table 6 indicates that there are 7 non-homeomorphic T_0-topologies in the class $25 \cdot 2^{n-6}$, and in work [11] it is shown that the number of all T_0-topologies of this class for $n > 6$ is equal to $\frac{n+14}{24} \cdot (n)_6 + \frac{1}{24} \cdot (n)_7$.

Theorem. A T_0-topology has the weight $25 \cdot 2^{n-6}$ if and only if its vector (under the specified conditions for the minimal neighborhoods of the elements) has one of the following forms:

1) $(0, \dots, 0,1,5)$, whenever either a) $M_{n-1} \subset M_n$ or b) $M_{n-1} \cap M_n$ is a one-element set;
2) $(0, \dots, 0,2,2)$, if $M_{n-1} \cap M_n = \emptyset$,
3) $(0, \dots, 0,1,1,1,1,2)$, if $\bigcap_{m=n-4}^{n-1} M_m = \{y\}$ is one-element set and either a) $M_{n-1} \subset M_n$ or b) $\bigcap_{m=n-4}^{n-1} M_m = \{y\}$;
4) $(0, \dots, 0,1,1,2)$, if the intersection $M_{n-2} \cap M_{n-1}$ is a one-element set and $\bigcap_{m=n-2}^{n} M_m = \emptyset$;
5) $(0, \dots, 0,1,1,1,1)$, if $M_{n-3} \cap M_{n-2} = \{y\}$ and $M_{n-1} \cap M_n = \{z\}$ are one-element sets with $y \neq z$.

With the help of 2-CNF, we showed the following:

- T_0-topology τ with vector $(0, \dots, 0,1,1,2)$ under conditions $M_{n-2} \cap M_{n-1} = \{x_i\}$ and $\bigcap_{m=n-2}^{n} M_m = \emptyset$ is self-dual;
- T_0-topologies with vectors $(0, \dots, 0,1,5)$ and $(0, \dots, 0,1,1,1,1,2)$, $(0, \dots, 0,2,2)$ and $(0, \dots, 0,1,1,1,1)$ are mutually dual.

Using the vector of T_0-topology in work [13], the number of all T_0-topologies with the specified vectors was calculated, and it was proved that this number coincides with the one obtained in Kolli's work, that is. So there are no other T_0-topologies with such a weight.

Another example of effective application of the topology model in the form of a maximal 2-CNF is the problem of enumerating all non-homeomorphic T_0-topologies on 4-element and 5-element sets. This problem is reduced to listing the maximal 2-CNFs of all non-homeomorphic T_0-topologies in each class.

Let's stop only at the case of a 4-element set. To solve the problem, we used the connection between the 2-CNF of the topology and 2-CNF of dual T_0-topology and the properties of the 2-CNF of the self-dual T_0-topology [13]:

- If some 2-CNF defines the topology τ, then the 2-CNF obtained from it by replacing $x_i \to \bar{x}_i$, defines a dual topology $C\tau$;
- Two 2-CNF $f_1$and f_2 of the topology on the set X are called equivalent if there exists a bijection $\varphi: X \to X$ such that $f_2 = \varphi(f_1)$. If f_τ and $f_{C\tau}$ are equivalent, then each of them will be called a self-dual 2-CNF, and the corresponding homeomorphic topologies will be called self-dual.

In the following list, each maximal 2-CNF specifies one of two mutually dual T_0-topologies or a self-dual T_0-topology. The complete list of maximal 2-CNF of all non-homeomorphic T_0-topologies on a 4-element set can be obtained by adding to each non-self-dual maximal 2-CNF the maximal 2-CNF that is dual to it.

List of maximal 2-CNF T_0-topologies on a 4-element set:

5- class: $(x_1 \vee \bar{x}_2)(x_1 \vee \bar{x}_3)(x_1 \vee \bar{x}_4)(x_2 \vee \bar{x}_3)(x_2 \vee \bar{x}_4)(x_3 \vee \bar{x}_4)$ – self-dual;
6- class: $(x_1 \vee \bar{x}_2)(x_1 \vee \bar{x}_3)(x_1 \vee \bar{x}_4)(x_2 \vee \bar{x}_3)(x_2 \vee \bar{x}_4)$;
$(x_1 \vee \bar{x}_2)(x_1 \vee \bar{x}_3)(x_1 \vee \bar{x}_4)(x_2 \vee \bar{x}_4)(x_3 \vee \bar{x}_4)$ – self-dual;
7- class: $(x_1 \vee \bar{x}_3)(x_1 \vee \bar{x}_4)(x_2 \vee \bar{x}_4)(x_3 \vee \bar{x}_4)$;
$(x_1 \vee \bar{x}_3)(x_1 \vee \bar{x}_4)(x_2 \vee \bar{x}_3)(x_2 \vee \bar{x}_4)$ – self-dual;
8- class: $(x_1 \vee \bar{x}_3)(x_1 \vee \bar{x}_4)(x_2 \vee \bar{x}_3)$ – self-dual;
$(x_1 \vee \bar{x}_3)(x_1 \vee \bar{x}_4)(x_3 \vee \bar{x}_4)$ – self-dual;
9- class: $(x_1 \vee \bar{x}_2)(x_1 \vee \bar{x}_3)(x_1 \vee \bar{x}_4)$;
$(x_1 \vee \bar{x}_3)(x_2 \vee \bar{x}_4)$ – self-dual;
10- class: $(x_1 \vee \bar{x}_3)(x_1 \vee \bar{x}_4)$;
12- class: $(x_1 \vee \bar{x}_4)$ – self-dual;
16- class: $f(x_1, x_2, x_3, x_4) \equiv 1$.

Conclusions

In this work, topologies on a finite set were modeled by a non-decreasing sequence of non-negative integers - a topology vector, as well as a conjunctive normal form of a certain form - a maximal 2-CNF. The results of the application of these models were obtained. Vectors of T_0-topologies with weight $2^{n-2} < m \leq 2^{n-1}$ on the n-element set are found, which are consistent with topologies close to discrete topology on the $(n-1)$-element set. Some classes of T_0-topologies with weight $2^{n-2} < m \leq 2^{n-1}$ are distinguished, which are not consistent with topologies close to discrete topology. The application of both models for enumeration and counting of all T_0-topologies with a weight of $25 \cdot 2^{n-6}$ is shown. The 2-CNFs of all non-homeomorphic T_0-topologies on the 4-element set are listed. There is an expectation that the study of the regularities of the transition from topologies on the $(n-1)$-element set to topologies on the n-element set can lead to qualitative results.

References

[1] Kovalevsky V.A. Finite topology as applied to image analysis. Computer Vision, Graphics, and Image Processing. 1989. Vol. 46. pp. 141–161.

[2] Kovalevsky V.A. Finite topology and image analysis. Advances in electronics and electron physics. 1992. Vol. 84. pp. 197–259.

[3] Merrifield R., Simmons H. The structures of molecular topological spaces. Theoretica Chimica Acta. 1980. Vol. 55. pp. 55–75.

[4] Merrifield R., Simmons H. Topological methods in chemistry. Wiley, New York, 1989. 230 p.

[5] Rosenfeld A., Kong T., Wu A., Graphical Models and Image Processing. Digital surfaces. 1991. Vol. 53. pp. 305-312.

[6] Evans J. W., Harary F., Lynn M. S. On the computer enumeration of finite topologies. Communications of the ACM. 1967. Vol.10. №5. pp. 295–297.

[7] Sharp, Jr. H. Cardinality of finite topologies. Journal Combinatorial Theory. 1968. Vol. 5. pp. 82–86.

[8] Velichko I.G., Stegantseva P.G., Bashova N.P. Enumeration of topologies close to discrete on finite sets. Izv. Vyssh. Uchebn. Zaved., Ser. Mat. 2015. No.11. pp. 23–31.

[9] Stanley R.P. On the number of open sets of finite topologies. Journal of combinatorial theory. 1971. Vol. 10. pp. 74–79.

[10] Kolli M. On the Cardinality of the T_0-Topologies on a Finite Set. International Journal of Combinatorics. 2014. Article ID 798074, pp. 1–7.

[11] Kolli M. Direct and elementary approach to enumerate topologies on a finite set. Journal of Integer Sequences. 2007. Vol. 10. Article 07.3.1. pp. 1–11.

[12] Stegantseva P.G., Skryabina A.V. Topologies on the n-element set consistent with topologies close to the discrete on an (n-1)-element set. Ukrainian Mathematical Journal. 2021. No. 2. Vol. 73. pp. 276-288, doi: 10.37863/umzh.v73i2.6174.

[13] Skryabina Anna, Stegantseva Polina, Bashova Nadia. The properties of 2-CNF of the mutually dual and self-dual T0-topologies on the finite set and the calculation of T0-topologies of a certain weight. Proceedings of the International Geometry Center. 2022. No. 1. Vol. 15. pp. 75–85, doi: 10.15673/tmgc.v15i1.2084.