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ABSTRACT

In this work a very important theorem about the separation of a vector space convex parts, consequence of the Hahn-Banach Theorem, is presented.

Keywords: Hahn-Banach Theorem, separation, convex sets, convex functionals.

1. INTRODUCTION

The definition of separation, see (1), to be considered in this work is:

Definition 1.1

Be M and N two subsets of a real vector space L.A linear functional *f*, defined in L separates M and N if and only if exists a number c such that

$$f(x) \geq c, \text{for } x \in M \text{ and } f(x) \leq c, \text{for } x \in N$$

that is, if

$$\inf_{x \in M} f(x) \geq \sup_{x \in N} f(x).$$

The functional *f* separates strictly the sets M and N if and only if

$$\inf_{x \in M} f(x) > \sup_{x \in N} f(x).$$

Proposition 1.1

a) A linear functional separate the sets M and N if and only if separates the sets $M - N = \{x - y : x \in M \ \wedge y \in N\}$ and $\{0\}$.

b) A linear functional f separates the sets M and N if and only if separates the sets M-x and N-x for any $x \in L$.

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2. THE HAHN-BANACH THEOREM

Definition 2.1

Consider a vector space L and a respective subspace L_0. Suppose that in L_0 is defined a linear functional f_0.

A linear functional f defined in L is an extension of the functional f_0 if and only if

$$f(x) = f_0(x), \underset{x \in L_0}{\forall}.$$

The Hahn-Banach Theorem, see (2), plays an important role in the resolution of the problem of finding an extension of a linear functional.

Theorem 2.1(Hahn-Banach)

Be p a positive homogeneous convex functional defined in a real vector space L and L_0 a subspace of L. If f_0 is a linear functional defined in L_0 , fulfilling the condition

$$f_0(x) \leq p(x), \underset{x \in L_0}{\forall} \qquad (2.1),$$

there is an extension f of f_0 defined in L, linear, and such that $f(x) \leq p(x), \underset{x \in L}{\forall}$.

Dem.:

Begin by showing that if $L_0 \neq L$, there is an extension of f_0, f', defined in a subspace L' such that $L \subset L'$, to fulfill the condition (2.1).

Be z any element of L not belonging to L_0; if L' is the subspace generated by L_0 and z, each element of L' is expressed in the form $tz + x$, being $x \in L_0$. If f' is an extension (linear) of the functional f_0 to L', it will be $f'(tz + x) = tf'(z) + f_0(x)$ or, imposing $f'(z) = c$,

$$f'(tz + x) = tc + f_0(x).$$

Choose now c in a way that respects on L' the condition (2.1), that is: fulfilling the inequality $f_0(x) + tc \leq p(x + tz)$, for any $x \in L_0$ and any real number t.

For t>0 that inequality is equivalent to the condition $f_0\left(\frac{x}{t}\right) + c \leq p\left(\frac{x}{t} + z\right)$ or

$$c \leq p\left(\frac{x}{t} + z\right) + f_0\left(\frac{x}{t}\right) \qquad (2.2).$$

For t<0 it is equivalent to the condition $f_0\left(\frac{x}{t}\right) + c \geq -p\left(-\frac{x}{t} - z\right)$, or

$$c \geq -p\left(-\frac{x}{t} - z\right) - f_0\left(\frac{x}{t}\right) \qquad (2.3).$$

It will be shown now that there is always a number c that satisfies simultaneously the conditions (2.2) and (2.3).

Given any two elements y´ $and\ y´´$ fromL_0,

$$-f_0(y'') + p(y'' + z) \geq -f_0(y') - p(-y'' - z) \qquad (2.4)$$

since $f_0(y'') - f_0(y') \leq p(y'' - y') = p\big((y'' + z) - (y' + z)\big) \leq p(y'' + z) + p(-y' - z)$.

Be $c'' = \inf_{y''}(-f_0(y'') + p(y'' + z))$ and $c' = \sup_{y'}(-f_0(y') - p(-y' - z))$. As y' and y'' are arbitrary, it results from (2.4) that $c'' \geq c'$. Choosing c so that $c'' \geq c \geq c'$, it is defined the functional f' on L' as

$$f'(tz + x) = tc + f_0(x).$$

This functional satisfies the condition (2.1). So any functional f_0 defined in a subspace $L_0 \subset L$ and having to fulfill in L_0 the condition (2.1), may be extended to a subspace L'. The extension f' satisfies the condition

$$f'(x) \leq p(x), \underset{x \in L'}{\forall}.$$

If L has a numerable algebraic base $(x_1, x_2, \dots, x_n, \dots)$, the functional is built in L by finite induction, considering an increasing sequence of subspaces

$$L^{(1)} = (L_0, x_1),\ L^{(2)} = (L^{(1)}, x_2), \dots$$

(Calling $(L^{(k)}, x_{k+1})$ the L subspace generated by $L^{(k)}$ and x_{k+1}).

In the general case, that is, when L has not a numerable algebraic base, it is mandatory to use a transfinite induction process, for instance, the Haudsdorf maximal chain Theorem. Be $\mathfrak{I}$ the set of the whole pairs (L', f'), where L' is a L subspace that contains L_0 and f' is an extension of f_0 to L' that verifies (2.1). Order partially $\mathfrak{I}$ in the following way:

$$(L', f') \leq (L'', f'') \text{ if and only if } L' \subset L'' \text{ and } f''/_{L'} = f'.$$

By the Haudsdorf maximal chain Theorem, there is a chain (that is: a subset of $\mathfrak{I}$ totally ordered) maximal (that is: not strictly contained in another chain). Call it Ω. Be Φ a family of the whole L' such that $(L', f') \in \Omega$. Φ is totally ordered by the inclusion of sets; so, the union T of the whole elements belonging to Φ is a subspace of L. If $x \in T$, so $x \in L'$ for some $L' \in \Phi$; define $\tilde{f}(x) = f'(x)$, where f' is the extension of f_0 that is in the pair (L', f') —the definition of $\tilde{f}$ is obviously coherent. It is easy to check that $\tilde{L}$=L and that f=$\tilde{f}$ fulfills the condition (2.1). □

It follows the Hahn- Banach Theorem complex case that corresponds to the contribution of Hahn to the theorem, see (3). Begin with

Definition 2.2

A linear functional p, assuming only positive values, defined in a complex vector space L, is called homogeneous convex if and only if, for any $x, y \in L$ and and any complex number λ,

$$p(x + y) \leq p(x) + p(y), p(\lambda x) = |\lambda| p(x).$$

Theorem 2.2 (Hahn-Banach)

Be p a homogeneous convex functional defined in a vector space L and f_0 a linear functional, defined in a subspace $L_0 \subset L$, fulfilling the condition

$$|f_0(x)| \leq p(x), x \in L_0.$$

So, there is a linear functional f defined in L, satisfying the condition

$$|f(x)| \leq p(x), x \in L;\ f(x) = f_0(x), x \in L_0.$$

Dem.:

Call L_R and L_{OR} the real vector spaces underlying, respectively the spaces L and L_0. Evidently, p is a homogeneous convex functional in L_R and $f_{OR}(x) = Re[f_0(x)]$ a real linear functional in L_{OR}, satisfying the condition $|f_{OR}(x)| \leq p(x)$ and so

$$|f_{OR}(x)| \leq p(x).$$

So, from Theorem 2.1, there is a real linear functional f_R, defined in the whole space L_R, that satisfies the conditions

$$f_R(x) \leq p(x), x \in L_R\ ; f_R(x) = f_{OR}(x), x \in L_{OR}.$$

But $-f_R(x) = f_R(-x) \leq p(-x) = p(x)$ and so

$$|f_R(x)| \leq p(x), x \in L_R \qquad (2.5).$$

Define in L the functional f setting

$$f(x) = f_R(x) - if_R(ix).$$

Obviously, f is a complex linear functional in L such that

$$f(x) = f_0(x), x \in L_0\ ;\ Re[f(x)] = f_R(x), x \in L.$$

Finally, it must be shown that $|f(x)| \leq p(x), x \in L$.

Proceed by absurd. Suppose that there is $x_0 \in L$ such that $|f(x_0)| > p(x_0)$.So, $f(x_0)$=$\rho e^{-i\omega}, \rho > 0$ and making $y_0 = \rho e^{-i\omega} x_0$, it would happen that $f_R(y_0) = Re[f(y_0)] = Re\left[e^{-i\omega} f(x_0)\right] = \rho > p(x_0) = p(y_0)$ that is contrary of (2.5). □

3. SEPARATION OF VECTOR SPACE CONVEX PARTS

The theorem main objective of this work is

Theorem 3.1

Be *M* and *N* two convex subsets of a vector subspace *L* such that the kernel[2] of, at least, one of them, for instance the one of *M*, is non-empty and do not intersect the other set; So, there is a non-null linear functional on *L* that separates *M* and *N*.

[2] The kernel of a set $E \subset L$, designated *J(E)* , is the set of points $x \in E$ such that, given any $y \in L$, it is determined $\varepsilon = \epsilon(y) > 0$ such that $x + ty \in E\ since\ |t| < \varepsilon$.

A convex set with non-empty kernel is a convex body.

Dem.:

Less than one translation it is possible to suppose that the point 0 belongs to kernel of M, which will be designated$\breve{M}$. So, given y_0 natural, $-y_0$ belongs to the kernel of M-N and to the kernel of M-N+y_0. As $\breve{M} \cap N = \emptyset$ (by hypothesis), 0 does not belong to the kernel of M-N and y_0 does not belong to the one of M-N+y_0 .

Put K=M-N+y_0 and be p the Minkowsky functional[3] of $\breve{K}$. So $p(y_0) \geq 1$ since y_0 does not belong to $\breve{K}$. Define, now, the linear functional

$$f_0(\alpha y_0) = \alpha p(y_0).$$

Note that f_0 is defined in a space with dimension 1, constituted by elements αy_0, and is such that $f_0(\alpha y_0) \leq p(\alpha y_0)$. In fact, $p(\alpha y_0)$= $\alpha p(y_0)$, when $\alpha < 0$ and $f_0(\alpha y_0) = \alpha f_0(y_0) < 0 < p(\alpha y_0)$, when $\alpha < 0$.

With these conditions, by the Hahn-Banach Theorem, it is possible to state the existence of a linear functional f , defined in L, that extends f_0, fulfilling $f(y) \leq p(y), \underset{y \in L}{\forall}$.

Then $f(y) \leq 1, \underset{y \in K}{\forall}$ and $f(y_0) \geq 1$. So,

-f separates the sets K and $\{y_0\}$, that is

-f separates the sets M-N and $\{0\}$,that is

-f separates the sets M and N.□

4. CONCLUSIONS

After a detailed study of the Hahn-Banach Theorem, in both its forms: real and complex, an important result, a separation theorem, consequence of it is presented.

In fact, it is determinant for important results in Optimization and Convex Programming, see for instance (4,5), that are among the most important mathematical tools used in Management and Economics, see (1) and (5-10).

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[3] Be L any vector space and A a convex body in L which kernel contains 0. The Minkowsky functional of the convex body A , designated $p_A(x)$, is the functional $p_A(x) = inf\left\{r : \frac{x}{r} \in A, r > 0\right\}$.

A Minkowsky functional is convex, positively homogeneous and assumes only positive values. Reciprocally, if *p(x)* is a positively homogeneous functional, assuming only positive values, if k is a positive number, the set $A = \{x : p(x) \leq k\}$ is a convex body with kernel $\{x : p(x) < k\}$, which contains the point 0. If in $A = \{x : p(x) \leq k\}$ it is assumed k=1, the initial functional *p(x)* is the Minkowsky functional of A.

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