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Abstract: In this paper, we present identities of new generalization of Fibonacci and Lucas sequences. The new generalization of Fibonacci and Lucas sequence are defined by recurrence relation $f_k = 2af_{k-1} + (b-a^2)f_{k-2}$ *and* $l_k = 2al_{k-1} + (b-a^2)l_{k-2}$. This was introduced by Goksal Bilgici in 2014. Also we describe and derive sums and connection formulae. We have used their Binet's formula and generating function to derive the identities. The proofs of the main theorems are based on special functions, simple algebra and give several interesting identities involving them.

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1 Introduction

Sequences have been fascinating topic for mathematicians for centuries. The Fibonacci sequence is a source of many nice and interesting identities. It is well known that the Fibonacci numbers and Lucas numbers are closely related. These numbers are of great importance in the study of many subjects such as Algebra, geometry and number theory itself. This sequence in which each number is the sum of the two preceding numbers has proved extremely fruitful and appears in different areas in Mathematics and Science.

The sequence of Fibonacci numbers [11], F_n is defined by

$$F_n = F_{n-1} + F_{n-2},\ n \ge 2 \text{ with } F_0 = 0,\ F_1 = 1 \tag{1.1}$$

The sequence of Lucas numbers [11], L_n is defined by

$$L_n = L_{n-1} + L_{n-2},\ n \ge 2 \text{ with } L_0 = 2,\ L_1 = 1 \tag{1.2}$$

The Fibonacci sequence, Lucas sequence, Pell sequence, Pell-Lucas sequence, Jacobsthal sequence and Jacobsthal-Lucas sequence are most prominent examples of recursive sequences. The second order recurrence sequence has been generalized in two ways mainly, first by preserving the initial conditions and second by preserving the recurrence relation.

Kalman and Mena [10] generalize the Fibonacci sequence by

$$F_n = aF_{n-1} + bF_{n-2}\,,\ n \geq 2 \quad with \quad F_0 = 0\,,\ F_1 = 1 \tag{1.3}$$

Horadam [9] defined generalized Fibonacci sequence $\{H_n\}$ by

$$H_n = H_{n-1} + H_{n-2},\ n \geq 3\ with\ H_1 = p\,,\ H_2 = p + q \tag{1.4}$$

where p and q are arbitrary integers.

The k-Fibonacci numbers defined by Falco'n and Plaza [5], for any positive real number k, the k-Fibonacci sequence is defined recurrently by

$$F_{k,n} = k\,F_{k,n-1} + F_{k,n-2}\,,\ n \geq 2 \quad with \quad F_{k,0} = 0\,,\ F_{k,1} = 1 \tag{1.5}$$

The k-Fibonacci numbers defined by Falco'n [4],

$$L_{k,n} = k\,L_{k,n-1} + L_{k,n-2}\,,\ n \geq 2 \quad with \quad L_{k,0} = 2\,,\ L_{k,1} = k \tag{1.6}$$

Most of the authors introduced Fibonacci pattern based sequences in many ways which are known as Fibonacci-Like sequences and k-Fibonacci-like sequences [7, 8, 13, 17, 22, 25, 26].

Generalized Fibonacci sequence [7], is defined as

$$F_k = pF_{k-1} + qF_{k-2}\,,\ k \geq 2 \quad with \quad F_0 = a\,,\ F_1 = b \tag{1.7}$$

where p, q, a and b are positive integer.

(p,q) - Fibonacci numbers [19], is defined as

$$F_{p,q,n} = pF_{p,q,n} + bF_{p,q,n}\,,\ n \geq 2 \quad with \quad F_{p,q,0} = 0\,,\ F_{p,q,n} = 1 \tag{1.8}$$

(p,q) - Lucas numbers [20], is defined as

$$L_{p,q,n} = pL_{p,q,n} + bL_{p,q,n}\,,\ n \geq 2 \quad with \quad L_{p,q,0} = 2\,,\ L_{p,q,n} = p \tag{1.9}$$

Generalized (p,q) -Fibonacci-Like sequence [21], is defined by recurrence relation

$$S_{p,q,n} = pS_{p,q,n} + qS_{p,q,n}\,,\ n \geq 2 \quad with \quad S_{p,q,0} = 2k\,,\ S_{p,q,n} = 1 + kp \tag{1.10}$$

Goksal Bilgici [2], defined new generalizations of Fibonacci and Lucas sequences

$$f_k = 2af_{k-1} + (b - a^2)\,f_{k-2}\ ,\ k \geq 2\ \ with\ \ f_0 = 0\,,\ f_1 = 1 \tag{1.11}$$

$$l_k = 2al_{k-1} + (b - a^2)l_{k-2}\ ,\ k \geq 2\ \ with\ \ l_0 = 2\,,\ l_1 = 2a \tag{1.12}$$

2 Preliminaries

Before presenting our main theorems, we will need to introduce some known results and notations.

The sequence of new generalization of Fibonacci numbers f_k, [2], is defined by

$$f_k = 2af_{k-1} + (b-a^2)f_{k-2} \ , \ k \geq 2$$

First few generalized Fibonacci numbers are

$$\{f_k\} = \left\{0, 1, 2a, 3a^2+b, 4a^3+4ab, 5a^4+10a^2b+b^2, ...\right\}$$

The sequence of new generalization of Lucas numbers l_k, [2], is defined by

$$l_k = 2al_{k-1} + (b-a^2)l_{k-2} \ , \ k \geq 2$$

First few generalized Lucas numbers are

$$\{l_k\} = \left\{2, 2a, 2a^2+2b, 2a^3+6ab, 2a^4+12a^2b+2b^2, 2a^5+20a^3b+10ab^2, ...\right\}$$

Generating function for new generalization of Fibonacci and Lucas numbers are

$$\sum_{k=0}^{\infty} f_k x^k = \frac{x}{1-2ax-(b-a^2)x^2} \tag{2.1}$$

$$\sum_{k=0}^{\infty} l_k x^k = \frac{2-2ax}{1-2ax-(b-a^2)x^2} \tag{2.2}$$

The Binet's formula for new generalization of Fibonacci and Lucas numbers are

$$f_k = \frac{\Re_1^k - \Re_2^k}{\Re_1 - \Re_2} \ \textit{and} \ l_k = \Re_1^k + \Re_2^k$$

where $\Re_1$ & $\Re_2$ are the roots of the characteristic equation $x^2 - 2ax - (b-a^2) = 0$,
with $\Re_1 = a+\sqrt{b}$, $\Re_2 = a-\sqrt{b}$; $\Re_1 + \Re_2 = 2a$, $\Re_1 - \Re_2 = 2\sqrt{b}$,$\Re_1\Re_2 = a^2 - b$. Also

$$f_{-k} = \frac{-1}{(a^2-b)^k} f_k \ \textit{and} \ l_{-k} = \frac{1}{(a^2-b)^k} l_k$$

3 Sums of New Generalization of Fibonacci and Lucas Numbers

In this section, we study the sums of new generalization of Fibonacci and Lucas numbers. This enables us to give in a straightforward way several formulas for the sums of such numbers.

Theorem 3.1. *Explicit sum Formula for new generalization of Fibonacci numbers*

$$f_k = \sum_{i=0}^{\left[\frac{k-1}{2}\right]} \binom{k-i-1}{i} (2a)^{k-2i-1}(b-a^2)^i \tag{3.1}$$

Proof. By the generating function of new generalization of Fibonacci numbers, the proof is clear. □

Theorem 3.2. *Explicit sum Formula for new generalization of Lucas numbers*

$$l_k = 2\sum_{i=0}^{\left[\frac{k}{2}\right]} \binom{k-i}{i} (2a)^{k-2i}(b-a^2)^i - \sum_{i=0}^{\left[\frac{k-1}{2}\right]} \binom{k-i-1}{i} (2a)^{k-2i}(b-a^2)^i \qquad (3.2)$$

Proof. By the generating function of new generalization of Lucas numbers, the proof is clear. ☐

Lemma 3.3. *For fixed integers* $p,\ q$ *with* $0 \le q \le p-1$, *the following equality holds*

$$f_{p(n+2)+q} = l_p\, f_{p(n+1)+q} - (a^2-b)^p f_{pn+q} \qquad (3.3)$$

Proof. From the Binet's formula of new generalization Fibonacci and Lucas numbers,

$$l_p\, f_{p(n+1)+q} = \left(\Re_1^p + \Re_2^p\right)\left(\frac{\Re_1^{p(n+1)+q} - \Re_2^{p(n+1)+q}}{\Re_1 - \Re_2}\right)$$

$$= \frac{1}{\Re_1 - \Re_2}\left[\Re_1^{p(n+2)+q} + (a^2-b)^p \Re_1^{pn+q} - (a^2-b)^p \Re_2^{pn+q} - \Re_2^{p(n+2)+q}\right]$$

$$= \frac{1}{\Re_1 - \Re_2}\left[\left\{\Re_1^{p(n+2)+q} - \Re_2^{p(n+2)+q}\right\} + (a^2-b)^p \left(\Re_1^{pn+q} - \Re_2^{pn+q}\right)\right]$$

$$= f_{p(n+2)+q} + (a^2-b)^p f_{pn+q}$$

then, the equality becomes,

$$f_{p(n+2)+q} = l_p\, f_{p(n+1)+q} - (a^2-b)^p f_{pn+q}$$ ☐

Theorem 3.4. *For fixed integers* $p,\ q$ *with* $0 \le q \le p-1$, *the following equality holds*

$$\sum_{i=0}^{n} f_{pi+q} = \frac{f_{p(n+1)+q} + (a^2-b)^q l_{p-q} - f_q - (a^2-b)^p f_{pn+q}}{l_p - (a^2-b)^p - 1} \qquad (3.4)$$

Proof. From the Binet's formula of new generalization Fibonacci numbers,

$$\sum_{i=0}^{n} f_{pi+q} = \sum_{i=0}^{n} \frac{\Re_1^{pi+q} - \Re_2^{pi+q}}{\Re_1 - \Re_2}$$

$$= \frac{1}{\Re_1 - \Re_2}\left[\sum_{i=0}^{n} \Re_1^{pi+q} - \sum_{i=0}^{n} \Re_2^{pi+q}\right]$$

$$= \frac{1}{\Re_1 - \Re_2}\left[\frac{\Re_1^{pn+q+p} - \Re_1^q}{\Re_1^p - 1} - \frac{\Re_2^{pn+q+p} - \Re_2^q}{\Re_2^p - 1}\right]$$

$$= \frac{1}{(a^2-b)^p - l_p + 1}\left[(a^2-b)^p f_{pn+q} - f_{p(n+1)+q} + f_q - (a^2-b)^q l_{p-q}\right]$$

$$= \frac{f_{p(n+1)+q} + (a^2-b)^q l_{p-q} - f_q - (a^2-b)^p f_{pn+q}}{l_p - (a^2-b)^p - 1}$$

This completes the proof. ☐

Corollary 3.5. *Sum of odd new generalization of Fibonacci numbers,*

If $p = 2m+1$ *then Eq. (3.4) is*

$$\sum_{i=0}^{n} f_{(2m+1)i+q} = \frac{f_{(2m+1)(n+1)+q} + (a^2-b)^q l_{2m+1-q} - f_q - (a^2-b)^{(2m+1)} f_{(2m+1)n+q}}{l_{(2m+1)} - (a^2-b)^{(2m+1)} - 1} \quad (3.5)$$

For example

(1) If $m=0$ then $p=1$

$$\sum_{i=0}^{n} f_{i+q} = \frac{f_{n+q+1} + (a^2-b)^q l_{1-q} - f_q - (a^2-b) f_{n+q}}{2a-(a^2-b)-1} \quad (3.6)$$

(i) For $q=0$: $\sum_{i=0}^{n} f_i = \frac{f_{n+1} + 2a - (a^2-b) f_n}{2a-(a^2-b)-1}$

(2) If $m=1$ then $p=3$

$$\sum_{i=0}^{n} f_{3i+q} = \frac{f_{3n+q+3} + (a^2-b)^q l_{3-q} - f_q - (a^2-b)^3 f_{3n+q}}{a^3(2-a^3)+3ab(2-ab)+b(3a^4+b^2)-1} \quad (3.7)$$

(i) For $q=0$: $\sum_{i=0}^{n} f_{3i} = \frac{f_{3n+3} + (2a^3+6ab) - (a^2-b)^3 f_{3n}}{a^3(2-a^3)+3ab(2-ab)+b(3a^4+b^2)-1}$

(ii) For $q=1$: $\sum_{i=0}^{n} f_{3i+1} = \frac{f_{3n+4} + 2(a^4-b^2) - 1 - (a^2-b)^3 f_{3n+1}}{a^3(2-a^3)+3ab(2-ab)+b(3a^4+b^2)-1}$

(iii) For $q=2$: $\sum_{i=0}^{n} f_{3i+2} = \frac{f_{3n+5} + 2a(a^4-b)^2 - 2a - (a^2-b)^3 f_{3n+2}}{a^3(2-a^3)+3ab(2-ab)+b(3a^4+b^2)-1}$

(3) If $m=2$ then $p=5$

$$\sum_{i=0}^{n} f_{5i+q} = \frac{f_{5n+q+5} + (a^2-b)^q l_{5-q} - f_q - (a^2-b)^5 f_{5n+q}}{l_5 - (a^2-b)^5 - 1} \quad (3.8)$$

(i) For $q=0$: $\sum_{i=0}^{n} f_{5i} = \frac{f_{5n+5} + l_5 - (a^2-b)^5 f_{5n}}{l_5 - (a^2-b)^5 - 1}$

(ii) For $q=1$: $\sum_{i=0}^{n} f_{5i+1} = \frac{f_{5n+6} + (a^2-b) l_4 - (a^2-b)^5 f_{5n+1}}{l_5 - (a^2-b)^5 - 1}$

(iii) For $q=2$: $\sum_{i=0}^{n} f_{5i+2} = \frac{f_{5n+7} + (a^2-b)^2 l_3 - (a^2-b)^5 f_{5n+2}}{l_5 - (a^2-b)^5 - 1}$

(iv) For $q=3$: $\sum_{i=0}^{n} f_{5i+3} = \frac{f_{5n+8} + (a^2-b)^3 l_2 - (a^2-b)^5 f_{5n+3}}{l_5 - (a^2-b)^5 - 1}$

(v) For $q=4$: $\sum_{i=0}^{n} f_{5i+4} = \frac{f_{5n+9} + 2a(a^2-b)^4 - (a^2-b)^5 f_{5n+4}}{l_5 - (a^2-b)^5 - 1}$

(vi) For $q = 5$: $\sum_{i=0}^{n} f_{5i+5} = \dfrac{f_{5n+10} + \{2 - f_{5n+4}\}(a^2-b)^5}{l_5 - (a^2-b)^5 - 1}$

Corollary 3.6. *Sum of even new generalization of Fibonacci numbers, If $p = 2m$ then Eq. (3.4) is*

$$\sum_{i=0}^{n} f_{2mi+q} = \frac{f_{2m(n+1)+q} + (a^2-b)^q l_{2m-q} - f_q - (a^2-b)^{2m} f_{2mn+q}}{l_{2m} - (a^2-b)^{2m} - 1} \tag{3.9}$$

For example

(1) If $m = 1$ then $p = 2$

$$\sum_{i=0}^{n} f_{2i+q} = \frac{f_{2n+2+q} + (a^2-b)^q l_{2-q} - f_q - (a^2-b)^2 f_{2n+q}}{l_2 - (a^2-b)^2 - 1} \tag{3.10}$$

(i) For $q = 0$: $\sum_{i=0}^{n} f_{2i} = \dfrac{f_{2n+2} + (2a^2+2b) - (a^2-b)^2 f_{2n}}{(2a^2+2b) - (a^2-b)^2 - 1}$

(ii) For $q = 1$: $\sum_{i=0}^{n} f_{2i+1} = \dfrac{f_{2n+3} + 2a(a^2-b) - 1 - (a^2-b)^2 f_{2n+1}}{(2a^2+2b) - (a^2-b)^2 - 1}$

(iii) For $q = 2$: $\sum_{i=0}^{n} f_{2i+2} = \dfrac{f_{2n+4} + 2(a^2-b)^2 - 2a - (a^2-b)^2 f_{2n+2}}{(2a^2+2b) - (a^2-b)^2 - 1}$

(2) If $m = 2$ then $p = 4$

$$\sum_{i=0}^{n} f_{4i+q} = \frac{f_{4n+4+q} + (a^2-b)^q l_{4-q} - f_q - (a^2-b)^4 f_{4n+q}}{l_4 - (a^2-b)^4 - 1} \tag{3.11}$$

(i) For $q = 0$: $\sum_{i=0}^{n} f_{4i} = \dfrac{f_{4n+4} + l_4 - (a^2-b)^4 f_{4n}}{l_4 - (a^2-b)^4 - 1}$

(ii) For $q = 1$: $\sum_{i=0}^{n} f_{4i+1} = \dfrac{f_{4n+5} + (a^2-b) l_3 - 1 - (a^2-b)^4 f_{4n+1}}{l_4 - (a^2-b)^4 - 1}$

(iii) For $q = 2$: $\sum_{i=0}^{n} f_{4i+2} = \dfrac{f_{4n+6} + (a^2-b)^2 l_2 - 2a - (a^2-b)^4 f_{4n+2}}{l_4 - (a^2-b)^4 - 1}$

(iv) For $q = 3$: $\sum_{i=0}^{n} f_{4i+3} = \dfrac{f_{4n+7} + (a^2-b)^3 2a - f_3 - (a^2-b)^4 f_{4n+3}}{l_4 - (a^2-b)^4 - 1}$

(v) For $q = 4$: $\sum_{i=0}^{n} f_{4i+4} = \dfrac{f_{4n+8} + (a^2-b)^4 2 - f_4 - (a^2-b)^4 f_{4n+4}}{l_4 - (a^2-b)^4 - 1}$

(3) If $m = 3$ then $p = 6$

$$\sum_{i=0}^{n} f_{6i+q} = \frac{f_{6n+6+q} + (a^2-b)^q l_{6-q} - f_q - (a^2-b)^6 f_{6n+q}}{l_6 - (a^2-b)^6 - 1} \tag{3.12}$$

(i) For $q=0$: $\sum_{i=0}^{n} f_{6i} = \frac{f_{6n+6} + l_6 - (a^2-b)^6 f_{6n}}{l_6 - (a^2-b)^6 - 1}$

(ii) For $q=1$: $\sum_{i=0}^{n} f_{6i+1} = \frac{f_{6n+7} + (a^2-b)l_5 - 1 - (a^2-b)^6 f_{6n+1}}{l_6 - (a^2-b)^6 - 1}$

(iii) For $q=2$: $\sum_{i=0}^{n} f_{6i+2} = \frac{f_{6n+8} + (a^2-b)^2 l_4 - 2a - (a^2-b)^6 f_{6n+2}}{l_6 - (a^2-b)^6 - 1}$

(iv) For $q=3$: $\sum_{i=0}^{n} f_{6i+3} = \frac{f_{6n+9} + (a^2-b)^2 l_3 - f_3 - (a^2-b)^6 f_{6n+3}}{l_6 - (a^2-b)^6 - 1}$

Theorem 3.7. *For fixed integers p, q with $0 \le q \le p-1$, the following equality holds*

$$\sum_{i=0}^{n} (-1)^i f_{pi+q} = \frac{(-1)^n f_{p(n+1)+q} + (-1)^n (a^2-b)^p f_{pn+q} - (a^2-b)^q f_{p-q} + f_q}{l_p + (a^2-b)^p + 1} \quad (3.13)$$

Proof. Applying Binet's formula of new generalization of Fibonacci and Lucas numbers, the proof is clear. □

For different values of $p \,\&\, q$:

(i) $\sum_{i=0}^{n} (-1)^i f_i = \frac{(-1)^n f_{n+1} + (-1)^n (a^2-b) f_n - 1}{2a + a^2 - b + 1}$

(ii) $\sum_{i=0}^{n} (-1)^i f_{2i} = \frac{(-1)^n f_{2n+2} + (-1)^n (a^2-b)^2 f_{2n} - 2a}{(2a^2+2b) + (a^2-b)^2 + 1}$

(iii) $\sum_{i=0}^{n} (-1)^i f_{2i+1} = \frac{(-1)^n f_{2n+3} + (-1)^n (a^2-b)^2 f_{2n+1} - (a^2-b) + 1}{(2a^2+2b) + (a^2-b)^2 + 1}$

(iv) $\sum_{i=0}^{n} (-1)^i f_{4i} = \frac{(-1)^n f_{4n+4} + (-1)^n (a^2-b)^4 f_{4n} - f_4}{l_4 + (a^2-b)^4 + 1}$

(v) $\sum_{i=0}^{n} (-1)^i f_{4i+1} = \frac{(-1)^n f_{4n+5} + (-1)^n (a^2-b)^4 f_{4n+1} - (a^2-b) f_3 + 1}{l_4 + (a^2-b)^4 + 1}$

(vi) $\sum_{i=0}^{n} (-1)^i f_{4i+2} = \frac{(-1)^n f_{4n+6} + (-1)^n (a^2-b)^4 f_{4n+2} - (a^2-b)^2 2a + 2a}{l_4 + (a^2-b)^4 + 1}$

(vii) $\sum_{i=0}^{n} (-1)^i f_{4i+3} = \frac{(-1)^n f_{4n+7} + (-1)^n (a^2-b)^4 f_{4n+3} - (a^2-b)^3 + f_3}{l_4 + (a^2-b)^4 + 1}$

4 Confluent Hypergeometric Identities of new generalization of Fibonacci and Lucas numbers

K. Dilcher [3], defined Fibonacci numbers in terms of hypergeometric function. C. Berg [1], defined Fibonacci numbers and orthogonal polynomials. In [12], A. Lupas present a guide of

Fibonacci and Lucas Polynomial and defined Fibonacci and Lucas Polynomial in terms of hypergeometric form. In this section, we established some identities of new generalization of Fibonacci and Lucas numbers in terms of confluent hypergeometric function. Proofs of the theorem are based on special function, simple algebra and give several interesting identities involving them.

Theorem 4.1. *If* f_k *and* l_k *are new generalization of Fibonacci and Lucas numbers, then*

$$(i)\ \sum_{k=0}^{\infty} f_k \frac{x^{k-1}}{k!} = e^{2ax}\ {}_1F_0\left[k+1,-,(b-a^2)x^2\right] \tag{4.1}$$

$$(ii)\ \sum_{k=0}^{\infty} l_k \frac{x^{k}}{k!} = (2-2ax)e^{2ax}\ {}_1F_0\left[k+1,-,(b-a^2)x^2\right] \tag{4.2}$$

$$(iii)\ l_k = 2f_{k+1} - 2af_k \tag{4.3}$$

Theorem 4.2. *If* f_k *and* l_k *are new generalization of Fibonacci and Lucas numbers, then*

$$(i)\ \sum_{k=0}^{\infty} f_k \frac{x^{k-1}}{k!} = e^{2ax}\ {}_2F_1\left[k+1,1;1;\ (b-a^2)x^2\right] \tag{4.4}$$

$$(ii)\ \sum_{k=0}^{\infty} l_k \frac{x^{k}}{k!} = (2-2ax)e^{2ax}\ {}_2F_1\left[k+1,1;1;\ (b-a^2)x^2\right] \tag{4.5}$$

Proof (i). By the generating function of new generalization of Fibonacci numbers,

$$\begin{aligned}\sum_{k=0}^{\infty} f_k x^{k-1} &= \frac{1}{1-2ax-(b-a^2)x^2} \\ &= \sum_{k=0}^{\infty}\{2a+(b-a^2)x\}^k x^k \\ &= \sum_{k=0}^{\infty} x^k \sum_{i=0}^{k}\binom{k}{i}(2a)^{k-i}\{(b-a^2)x\}^i \\ &= \sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\binom{k+i}{i}(2a)^k\{(b-a^2)x^2\}^i x^k \\ &= \sum_{k=0}^{\infty}\frac{(2ax)^k}{k!}\sum_{i=0}^{\infty}\frac{(k+i)!}{i!}k!\{(b-a^2)x^2\}^i \end{aligned} \tag{4.6}$$

$$\sum_{k=0}^{\infty} f_k \frac{x^{k-1}}{k!} = e^{2ax}\ {}_1F_0\left[k+1,-,(b-a^2)x^2\right]$$

This is the first part of Theorem 6.

Also from (4.6), $\sum_{k=0}^{\infty} f_k \frac{x^{k-1}}{k!} = \sum_{k=0}^{\infty}\frac{(2ax)^k}{k!}\sum_{i=0}^{\infty}\frac{(k+i)!}{i!}\frac{(1)_i}{(1)_i}\{(b-a^2)x^2\}^i$

$$\sum_{k=0}^{\infty} f_k \frac{x^{k-1}}{k!} = e^{2ax}\ {}_2F_1\left[k+1,1;1;\ (b-a^2)x^2\right]$$

This completes the proof. □

Proof (ii). By the generating function of new generalization of Lucas numbers,

$$\sum_{k=0}^{\infty} l_k x^k = \frac{2-2ax}{1-2ax-(b-a^2)x^2}$$

$$= (2-2ax)\sum_{k=0}^{\infty}\{2a+(b-a^2)x\}^k x^k$$

$$=(2-2ax)\sum_{k=0}^{\infty} x^k \sum_{i=0}^{k}\binom{k}{i}(2a)^{k-i}\{(b-a^2)x\}^i$$

$$=(2-2ax)\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\binom{k+i}{i}(2a)^k\{(b-a^2)x^2\}^i x^k$$

$$=(2-2ax)\sum_{k=0}^{\infty}\frac{(2ax)^k}{k!}\sum_{i=0}^{\infty}\frac{(k+i)!}{i!}k!\{(b-a^2)x^2\}^i \tag{4.7}$$

$$\sum_{k=0}^{\infty} l_k \frac{x^k}{k!} = (2-2ax)e^{2ax}\ {}_1F_0\left[k+1,-,(b-a^2)x^2\right]$$

This is the second part of Theorem 6.

Also from (4.7), $\sum_{k=0}^{\infty} l_k \frac{x^k}{k!} = (2-2ax)\sum_{k=0}^{\infty}\frac{(2ax)^k}{k!}\sum_{i=0}^{\infty}\frac{(k+i)!}{i!}\frac{(1)_i}{(1)_i}\{(b-a^2)x^2\}^i$

$$\sum_{k=0}^{\infty} l_k \frac{x^k}{k!} = (2-2ax)e^{2ax}\ {}_2F_1\left[k+1,1;1;\ (b-a^2)x^2\right]$$

This completes the proof. □

We can easily get the following recurrence relation by using (4.1) and (4.2), also from (4.4) and (4.5),

$$l_k = 2f_{k+1} - 2af_k$$

5 Generalized Identities on the Products of new generalization of Fibonacci and Lucas numbers

Thongmoon [23, 24], defined various identities of Fibonacci and Lucas numbers. Singh, Bhadouria and Sikhwal [13], present some generalized identities involving common factors of Fibonacci and Lucas numbers. Gupta and Panwar [6], present identities involving common factors of generalized Fibonacci, Jacobsthal and jacobsthal-Lucas numbers. Panwar, Singh and Gupta ([14, 15]), present Generalized Identities Involving Common factors of generalized Fibonacci, Jacobsthal and jacobsthal-Lucas numbers. Singh, Sisodiya and Ahmed [18], investigate some products of k-Fibonacci and k-Lucas numbers, also present some generalized identities on the products of k-Fibonacci and k-Lucas numbers to establish connection formulas between them with the help of Binet's formula. In this section, we present identities involving product of new generalization Fibonacci and Lucas numbers and related identities consisting even and odd terms.

Theorem 5.1. *If f_k and l_k are new generalization of Fibonacci and Lucas numbers, then*

$$f_{2k+p}l_{2k+1} = f_{4k+p+1} + (a^2-b)^{2k+1} f_{p-1} \ , \ where\ k \geq 0\ \&\ p \geq 0 \quad (5.1)$$

Proof. Applying Binet's formula of new generalization of Fibonacci and Lucas numbers,

$$f_{2k+p}l_{2k+1} = \left(\frac{\Re_1^{2k+p} - \Re_2^{2k+p}}{\Re_1 - \Re_2}\right)\left(\Re_1^{2k+1} + \Re_2^{2k+1}\right) \quad (5.1)$$

$$= \left(\frac{\Re_1^{4k+p+1} - \Re_2^{4k+p+1}}{\Re_1 - \Re_2}\right) + \frac{\left(\Re_1\Re_2\right)^{2k}}{\left(\Re_1 - \Re_2\right)}\left(\Re_1^p\Re_2 - \Re_2^p\Re_1\right)$$

$$= \left(\frac{\Re_1^{4k+p+1} - \Re_2^{4k+p+1}}{\Re_1 - \Re_2}\right) + \left(\Re_1\Re_2\right)^{2k}(a^2-b)\left(\frac{\Re_1^{p-1} - \Re_2^{p-1}}{\Re_1 - \Re_2}\right)$$

$$= f_{4k+p+1} + (a^2-b)^{2k+1} f_{p-1}$$

This completes the proof. □

Corollary 5.2. *For different values of p, (5.1) can be expressed for even and odd numbers:*

(*i*) *If* $p=0$, *then*: $f_{2k}l_{2k+1} = f_{4k+1} - (a^2-b)^{2k}$ (5.2)

(*ii*) *If* $p=1$, *then*: $f_{2k+1}l_{2k+1} = f_{4k+2}$ (5.3)

(*iii*) *If* $p=2$, *then*: $f_{2k+2}l_{2k+1} = f_{4k+3} + (a^2-b)^{2k+1}$ (5.4)

Following theorems can be solved by Binet's formula of new generalization of Fibonacci and Lucas numbers.

Theorem 5.3. $f_{2k+p}l_{2k+2} = f_{4k+p+2} + (a^2-b)^{2k+2} f_{p-2}$, *where* $k \geq 0$ & $p \geq 0$ (5.5)

Corollary 5.4. *For different values of p, (5.5) can be expressed for even and odd numbers:*

(*i*) *If* $p=0$, *then*: $f_{2k}l_{2k+2} = f_{4k+1} - 2a(a^2-b)^{2k}$ (5.6)

(*ii*) *If* $p=1$, *then*: $f_{2k+1}l_{2k+2} = f_{4k+3} - (a^2-b)^{2k+1}$ (5.7)

(*iii*) *If* $p=2$, *then*: $f_{2k+2}l_{2k+2} = f_{4k+4}$ (5.8)

Theorem 5.5. $f_{2k+p}l_{2k} = f_{4k+p} + (a^2-b)^{2k} f_p$, *where* $k \geq 0$ & $p \geq 0$ (5.9)

Corollary 5.6. *For different values of p, (5.9) can be expressed for even and odd numbers:*

(*i*) *If* $p=0$, *then*: $f_{2k}l_{2k} = f_{4k}$ (5.10)

(*ii*) *If* $p=1$, *then*: $f_{2k+1}l_{2k} = f_{4k+1} + (a^2-b)^{2k}$ (5.11)

(*iii*) *If* $p=2$, *then*: $f_{2k+2}l_{2k} = f_{4k+2} + 2a(a^2-b)^{2k}$ (5.12)

Theorem 5.5. $f_{2k-p}l_{2k+1} = f_{4k-p+1} + (a^2-b)^{2k+1} f_{-p-1}$, *where* $k \geq 0$ & $p \geq 0$ (5.13)

Corollary 5.6. *For different values of p, (5.13) can be expressed for even and odd numbers:*

(*i*) *If* $p=0$, *then*: $f_{2k}l_{2k+1} = f_{4k+1} - (a^2-b)^{2k}$ (5.14)

(*ii*) *If* $p=1$, *then*: $f_{2k-1}l_{2k+1} = f_{4k} - 2a(a^2-b)^{2k-1}$ (5.15)

(*iii*) *If* $p=2$, *then*: $f_{2k-2}l_{2k+1} = f_{4k} - (3a^2+b)(a^2-b)^{2k-2}$ (5.16)

Theorem 5.7. $f_{2k-p}l_{2k-1} = f_{4k-p-1} + (a^2-b)^{2k-1} f_{1-p}$, *where* $k \ge 0$ & $p \ge 0$ (5.17)

Corollary 5.8. *For different values of p, (5.17) can be expressed for even and odd numbers:*

(i) *If* $p=0$, *then*: $f_{2k}l_{2k-1} = f_{4k-1} + (a^2-b)^{2k-1}$ (5.18)

(ii) *If* $p=1$, *then*: $f_{2k-1}l_{2k-1} = f_{4k-2}$ (5.19)

(iii) *If* $p=2$, *then*: $f_{2k-2}l_{2k-1} = f_{4k-3} - (a^2-b)^{2k-2}$ (5.20)

Theorem 5.9. $f_{2k-p}l_{2k} = f_{4k-p} + (a^2-b)^{2k} f_{-p}$, *where* $k \ge 0$ & $p \ge 0$ (5.21)

Corollary 5.10. *For different values of p, (5.21) can be expressed for even and odd numbers:*

(i) *If* $p=0$, *then*: $f_{2k}l_{2k} = f_{4k}$ (5.22)

(ii) *If* $p=1$, *then*: $f_{2k-1}l_{2k} = f_{4k-1} - (a^2-b)^{2k-1}$ (5.23)

(iii) *If* $p=2$, *then*: $f_{2k-2}l_{2k} = f_{4k-2} - 2a(a^2-b)^{2k-2}$ (5.24)

Theorem 5.11. $f_{2k}l_{2k+p} = f_{4k+p} - (a^2-b)^{2k} f_p$, *where* $k \ge 0$ & $p \ge 0$ (5.25)

Corollary 5.12. *For different values of p, (5.25) can be expressed for even and odd numbers:*

(i) *If* $p=0$, *then*: $f_{2k}l_{2k} = f_{4k}$ (5.26)

(ii) *If* $p=1$, *then*: $f_{2k}l_{2k+1} = f_{4k+1} - (a^2-b)^{2k}$ (5.27)

(iii) *If* $p=2$, *then*: $f_{2k}l_{2k+2} = f_{4k+2} - 2a(a^2-b)^{2k}$ (5.28)

Theorem 5.13. $4bf_{2k}f_{2k+p} = l_{4k+p} - (a^2-b)^{2k} l_p$, *where* $k \ge 0$ & $p \ge 0$ (5.29)

Corollary 5.14. *For different values of p, (5.29) can be expressed for even and odd numbers:*

(i) *If* $p=0$, *then*: $4bf_{2k}f_{2k} = l_{4k} - 2(a^2-b)^{2k}$ (5.30)

(ii) *If* $p=1$, *then*: $4bf_{2k}f_{2k+1} = l_{4k+1} - 2a(a^2-b)^{2k}$ (5.31)

(iii) *If* $p=2$, *then*: $4bf_{2k}f_{2k+2} = l_{4k+2} - 2(a^2+b)(a^2-b)^{2k}$ (5.32)

Theorem 5.15. $l_{2k}l_{2k+p} = l_{4k+p} + (a^2-b)^{2k} l_p$, *where* $k \ge 0$ & $p \ge 0$ (5.33)

Corollary 5.16. *For different values of p, (5.33) can be expressed for even and odd numbers:*

(i) *If* $p=0$, *then*: $l_{2k}l_{2k} = l_{4k} + 2(a^2-b)^{2k}$ (5.34)

(ii) *If* $p=1$, *then*: $l_{2k}l_{2k+1} = l_{4k+1} + 2a(a^2-b)^{2k}$ (5.35)

(iii) *If* $p=2$, *then*: $l_{2k}l_{2k+2} = l_{4k+2} + 2(a^2+b)(a^2-b)^{2k}$ (5.36)

6 Sum and difference of squares of new generalization of Fibonacci and Lucas numbers

In this section, sum and difference of new generalization of Fibonacci and Lucas numbers are treated in the following theorem.

Theorem 6.1. $4b\left(f_{n+1}^2 + f_{n-1}^2\right) = l_{2n+2} + l_{2n-2} - 2(a^2 - b)^{n-1}\{(a^2-b)^2 - 1\}$ (6.1)

Theorem 6.2. $4b\left(f_{n+1}^2 - f_{n-1}^2\right) = l_{2n+2} - l_{2n-2} - 2(a^2 - b)^{n-1}\{(a^2-b)^2 - 1\}$ (6.2)

By the Binet's formula of new generalization of Fibonacci and Lucas numbers, the proof is clear.

7 Conclusion

We have derived some fundamental properties in this paper. We describe sums of new generalization of Fibonacci numbers. This enables us to give in a straightforward way several formulas for the sums of such generalized numbers. These identities can be used to develop new identities of numbers and polynomials. We describe some confluent hypergeometric identities and generalized identities involving product of new generalization of Fibonacci and Lucas numbers. Also we present identities related to their sum and difference of squares involving them.

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