# Computing the CD-Number of Strong Product of Graphs

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**Abstract**

The theory of corona domination number was presented by G.Mahadevan et al. [1]. In this work, we carry over the study of corona domination in graphs for the strong product of graphs. We investigate the strong product involving paths, cycles, complete graph and complete bipartite graph.

**Keywords:** Domination number, corona domination number, strong product, pendent vertex, support vertex.

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## 1 Introduction

All the graphs $G = (V(G), E(G)$ consider here are finte, simple, undirected and without isolated vertex. The neighbor of a vertex $v \in V(G)$ is denoted by $N(v)$, where $N(v) = \{u \in V(G) : d(u, v) = 1\}$ and $deg(v) = |N(v)|$ denotes the degree of the vertex $v$. If the degree of a vertex is one then it is called a *pendant vertex* . Let $P_r$ denote a path of length $r$, $C_r$ denotes a cycle of length $r$ and $G^c$ denotes the complement of $G$. Let $G$ be a graph. An *induced subgraph* $H$ is obtained from $G$ by deleting some vertices of $G$. If $G$ is said to be a *bipartite*[4] then its vertex can be separated in to two nonempty subets $Y_1$ and $Y_2$ such that each edge of $G$ has one end in $Y_1$ and other in $Y_2$ and is denoted by $G(Y_1, Y_2)$. A bipartite graph $G(Y_1, Y_2)$ is said to be *complete bipartite* [4] if each vertex of $Y_1$ is adjacent to all the vertices of $Y_2$. If $G(Y_1, Y_2)$ is complete with $|Y_1| = r$ and $|Y_2| = s$ then $G(Y_1, Y_2)$ is denoted by $K_{r,s}$, where $K_{1,s}$ is a *star graph*[4]. A wheel graph $W_{1,r}, r \geq 3$ is obtained by joining a single vertex to all the vertices of a cycle $C_r$. A helm graph $H_r$, is obtained by joining a pendant edge to every vertex of the outer cycle of $W_{1,r}$. $L(G)$ be the *line graph* of $G$ where $V(L(G)) = E(G)$ and two vertices in $L(G)$ are adjacent if and only if the corresponding the edges has an endvertex in common. It is clear that $L(P_r) = P_{r-1}$ and $L(C_r) = C_r$. A *dominating set* $S$ is a set of vertices of $G$ with the condition that every $v \in V - S,\ d(v, S) = 1$. Any dominating set $D$ of $G$ with minimum cardinality, then the domination number $\gamma(G) = |D|$. The *strong product* [3] of graphs $H_1\, and\, H_2$ is the graph $H_1 \boxtimes H_2$ with vertex set $V(H_1) \times V(H_2)$ and any two of its vertices $(v_1, u_1)\ and\ (v_2, u_2)$ are adjacent whenever $v_1v_2 \in E(H_1)\, and\, u_1 = u_2$ or $u_1u_2 \in E(H_2)\, and\, v_1 = v_2$ or $v_1v_2 \in E(H_1)\ \ and\ \ u_1u_2 \in E(H_2)$. The corona domination number(CD number) [1] $\gamma_{CD}$ is a minimum cardinality of the dominating set $S$, with the subgraph induced by $S$ having either pendant or support vertex only. Graph domination and associated concepts have been analyzed for many years; among them, many authors studied the domination number of the various product graphs, especially for the product involving paths, cycles, etc. Motivated by the above, we determined the exact values of $\gamma_{CD}(P_r \boxtimes P_s), \gamma_{CD}(P_r \boxtimes P_s^c), \gamma_{CD}(P_r \boxtimes C_s), \gamma_{CD}(P_r \boxtimes K_s), etc.$ In the future, we will find the exact value of $\gamma_{CD}(P_r \boxtimes G)$ for any given graph G. Also, we see the relationship between the $CD(P_r)$ with $\gamma_{CD}(P_r \boxtimes G)$ and $CD(G)\gamma_{CD}(P_r \boxtimes G)$

Here, we have given the $CD$-number for some graphs. $\gamma_{CD}(P_r^c) = 2,\ r \geq 4$, $\gamma_{CD}(C_4^c) = 4$, $\gamma_{CD}(C_5^c) = 3$, $\gamma_{CD}(C_r^c) = 2,\ r \geq 6$, $\gamma_{CD}(P_2 \boxtimes K_s) = \gamma_{CD}(P_3 \boxtimes K_s) = 2$.

## 2 $CD$-number of strong product for some standard graphs

We begin this section with some observations. The $CD$-number of the strong product of the paths $P_1$, $P_2$, and $P_3$ with $P_r$ is same as the $CD$-number of $P_r$ and the remaining cases are given below.

**Theorem 2.1.** *If $s \geq 4$, then $\gamma(P_4 \boxtimes P_s) = 2\lceil \frac{s}{3} \rceil$.*

*Proof.* Let $P_4 = (v_1, v_2, v_3, v_4)$ and $P_s = (u_1, u_2, ..., u_s)$.
Then $V(P_4 \boxtimes P_s) = \{(v_i, u_j) : 1 \leq i \leq 4, 1 \leq j \leq s\}$ and $E(P_4 \boxtimes P_s) = \{(v_i, u_j)(v_i, u_{j+1}) : 1 \leq i \leq 4, 1 \leq j \leq s-1\} \cup \{(v_i, u_j)(v_{i+1}, u_j) : 1 \leq i \leq 3, 1 \leq j \leq s\} \cup \{(v_i, u_j)(v_{i+1}, u_{j+1}) : 1 \leq i \leq 3, 1 \leq j \leq s-1\} \cup \{(v_i, u_j)(v_{i+1}, u_{j-1}) : 1 \leq i \leq 3, 2 \leq j \leq s\}$. Let $S_1 = \{(v_1, u_j), (v_4, u_j) : j \equiv 2 \, (mod\, 3)\}$. Then

$$S = \begin{cases} S_1 & \text{if } s \equiv 0 \text{ or } 2 \, (mod\, 3), \\ S_1 \cup \{(v_1, u_s), (v_4, u_s)\} & \text{if } s \equiv 1 \, (mod\, 3), \end{cases} \quad \text{is a dominating set of } P_4 \boxtimes P_s.$$

Thus $\gamma(P_4 \boxtimes P_s) \leq |S| = 2\lceil \frac{s}{3} \rceil$. Assume that $D$ is a dominating set of $P_4 \boxtimes P_s$. Since $\gamma(P_s) = \lceil \frac{s}{3} \rceil$ and any dominating set of $i^{th}$ row of $P_4 \boxtimes P_s$ dominating all the vertices in $(i-1)^{th}$ row and $(i+1)^{th}$ row, any dominating set of a row dominates at most three rows. Hence $|D| \geq 2\lceil \frac{s}{3} \rceil$. Then $\gamma(P_4 \boxtimes P_s) \geq 2\lceil \frac{s}{3} \rceil$ and hence the result follows. □

**Theorem 2.2.** *If $s \geq 4$, then $\gamma_{CD}(P_4 \boxtimes P_s) = 2\lceil \frac{s}{3} \rceil$.*

*Proof.* Let $P_4 = (v_1, v_2, v_3, v_4)$ and $P_s = (u_1, u_2, ..., u_s)$.
Then $V(P_4 \boxtimes P_s) = \{(v_i, u_j) : 1 \leq i \leq 4, 1 \leq j \leq s\}$ and $E(P_4 \boxtimes P_s) = \{(v_i, u_j)(v_i, u_{j+1}) : 1 \leq i \leq 4, 1 \leq j \leq s-1\} \cup \{(v_i, u_j)(v_{i+1}, u_j) : 1 \leq i \leq 3, 1 \leq j \leq s\} \cup \{(v_i, u_j)(v_{i+1}, u_{j+1}) : 1 \leq i \leq 3, 1 \leq j \leq s-1\} \cup \{(v_i, u_j)(v_{i+1}, u_{j-1}) : 1 \leq i \leq 3, 2 \leq j \leq s\}$.
Let $S_1 = \{(v_i, u_j) : i \equiv 2 \text{ or } 3 \, (mod\, 4), j \equiv 2 \, (mod\, 3)\}$.

$$Then\ S = \begin{cases} S_1 & \text{if } s \equiv 0 \text{ or } 2 \, (mod\, 3), \\ S_1 \cup \{(v_i, u_s) : i \equiv 2 \text{ or } 3 \, (mod\, 4)\} & \text{if } s \equiv 1 \, (mod\, 3), \end{cases}$$

is a $CD - set$ of $P_4 \boxtimes P_S$. Thus $\gamma_{CD}(P_4 \boxtimes P_s) \leq |S| = 2\lceil \frac{s}{3} \rceil$. Since $\gamma(P_4 \boxtimes P_s) = 2\lceil \frac{s}{3} \rceil$, the result follows. □

**Theorem 2.3.** *If $r \equiv 0$ or $1$ $(mod\ 4)$, $r \geq 5$ and $s \geq r$, then $\gamma_{CD}(P_r \boxtimes P_s) = \lceil \frac{r}{2} \rceil \lceil \frac{s}{3} \rceil$.*

*Proof.* Let $P_r = (v_1, v_2, ..., v_r)$ and $P_s = (u_1, u_2, ..., u_s)$.
Then $V(P_r \boxtimes P_s) = \big\{(v_i, u_j) : 1 \leq i \leq r, 1 \leq j \leq s\big\}$ and $E(P_r \boxtimes P_s) = \{(v_i, u_j)(v_i, u_{j+1}) : 1 \leq i \leq r, 1 \leq j \leq s-1\} \cup \{(v_i, u_j)(v_{i+1}, u_j) : 1 \leq i \leq r-1, 1 \leq j \leq s\} \cup \{(v_i, u_j)(v_{i+1}, u_{j+1}) : 1 \leq i \leq r-1, 1 \leq j \leq s-1\} \cup \{(v_i, u_j)(v_{i+1}, u_{j-1}) : 1 \leq i \leq r-1, 2 \leq j \leq s\}$.

**case 1:** $r \equiv 0\ (mod\ 4)$

Let $S_1 = \big\{(v_i, u_j) :\ i \equiv 2\ or\ 3\ (mod\ 4), j \equiv 2\ (mod\ 3)\big\}$.

$$Then\ S = \begin{cases} S_1 & \text{if } s \equiv 0\ or\ 2\ (mod\ 3), \\ S_1 \cup \{(v_i, u_s) : i \equiv 2\ or\ 3\ (mod\ 4)\} & \text{if } s \equiv 1\ (mod\ 3), \end{cases} \text{ is a } CD-set$$

of $P_r \boxtimes P_s$. Thus $\gamma_{CD}(P_r \boxtimes P_s) \leq |S| = \lceil \frac{r}{2} \rceil \lceil \frac{s}{3} \rceil$. Suppose there exists a dominating set $D_1 \subseteq V$ of cardinality at most $d_1 = \lceil \frac{r}{2} \rceil \lceil \frac{s}{3} \rceil - 1$, then $< D_1 >$ contains an isolated vertex. Hence $|D_1| \geq d_1 + 1 = \lceil \frac{r}{2} \rceil \lceil \frac{s}{3} \rceil$.

**case 2:** $r \equiv 1\ (mod\ 4)$

Let $S_2 = \big\{(v_i, u_j) :\ i \equiv 2\ or\ 3\ (mod\ 4), j \equiv 2\ (mod\ 3)\big\} \cup \big\{(v_{r-1}, u_j) :\ j \equiv 2\ (mod\ 3)\big\}$.

$$Then\ S = \begin{cases} S_2 & \text{if } s \equiv 0\ or\ 2\ (mod\ 3), \\ S_2 \cup \{(v_i, u_s) : i \equiv 2\ or\ 3\ (mod\ 4)\} \cup \{(v_{r-1}, u_s)\} & \text{if } s \equiv 1\ (mod\ 3), \end{cases}$$

is a $CD-set$ of $P_r \boxtimes P_s$. Thus $\gamma_{CD}(P_r \boxtimes P_s) \leq |S| = \lceil \frac{r}{2} \rceil \lceil \frac{s}{3} \rceil$. Suppose there exists a dominating set $D_2 \subseteq V$ of cardinality at most $d_2 = \lceil \frac{r}{2} \rceil \lceil \frac{s}{3} \rceil - 1$, then $< D_2 >$ contains an isolated vertex. Thus $|D_2| \geq d_2 + 1 = \lceil \frac{r}{2} \rceil \lceil \frac{s}{3} \rceil$. Therefore the proof. □

**Theorem 2.4.** *If* $r \equiv 2\ (mod\ 4)$, $r \geq 10$ *and* $s \geq r$, *then*

$$\gamma_{CD}(P_r \boxtimes P_s) = \begin{cases} (\frac{r}{2} + 1)\lceil \frac{s}{3} \rceil & if\ s \equiv 0\ or\ 2\ \ (mod\ 3) \\ (\frac{r}{2} + 1)\lceil \frac{s}{3} \rceil - 4 & if\ s \equiv 1\ (mod\ 3). \end{cases}$$

*Proof.* Let $P_r = (v_1, v_2, ..., v_r)$ and $P_s = (u_1, u_2, ..., u_s)$.
Then $V(P_r \boxtimes P_s) = \big\{(v_i, u_j) : 1 \leq i \leq r, 1 \leq j \leq s\big\}$ and $E(P_r \boxtimes P_s) = \{(v_i, u_j)(v_i, u_{j+1}) : 1 \leq i \leq r, 1 \leq j \leq s-1\} \cup \{(v_i, u_j)(v_{i+1}, u_j) : 1 \leq i \leq r-1, 1 \leq j \leq s\} \cup \{(v_i, u_j)(v_{i+1}, u_{j+1}) : 1 \leq i \leq r-1, 1 \leq j \leq s-1\} \cup \{(v_i, u_j)(v_{i+1}, u_{j-1}) : 1 \leq i \leq r-1, 2 \leq j \leq s\}$.
Let $S_1 = \big\{(v_i, u_j) :\ i \equiv 1\ or\ 2\ (mod\ 4), j \equiv 2\ (mod\ 3)\big\}$, $S_2 = \big\{(v_i, u_j) :\ i \equiv 2\ or\ 3\ (mod\ 4), j \equiv 2\ (mod\ 3)\big\} \cup \{(v_{r-1}, u_j) : j \equiv 2\ (mod\ 3)\} - \big\{(v_r, u_{s-2}), (v_{r-3}, u_{s-2})\big\}$ $and A = (\{(v_i, u_s) : i \equiv 2\ or\ 3\ (mod\ 4)\} \cup \{(v_{r-1}, u_{s-1})(v_{r-4}, u_{s-1})\}) - \{(v_r, u_s), (v_{r-3}, u_s), (v_{r-4}, u_s)\}$.

Then $S = \begin{cases} S_1 & \text{if } s \equiv 0 \text{ or } 2 \ (mod\, 3), \\ S_2 \cup A & \text{if } s \equiv 1 \ (mod\, 3), \end{cases}$ is a $CD-set$ of $P_r \boxtimes P_s$. Thus

$$\gamma_{CD}(P_r \boxtimes P_s) \leq |S| = \begin{cases} (\frac{r}{2}+1)\lceil \frac{s}{3} \rceil & \text{if } s \equiv 0 \text{ or } 2 \ (mod\, 3), \\ (\frac{r}{2}+1)\lceil \frac{s}{3} \rceil - 4 & \text{if } s \equiv 1 \ (mod\, 3). \end{cases}$$

Suppose there exists a dominating set $D \subseteq V$ of cardinality at most

$$d = \begin{cases} (\frac{r}{2}+1)\lceil \frac{s}{3} \rceil - 1 & \text{if } s \equiv 0 \text{ or } 2 \ (mod\, 3), \\ (\frac{r}{2}+1)\lceil \frac{s}{3} \rceil - 5 & \text{if } s \equiv 1 \ (mod\, 3), \end{cases}$$

then $< D >$ contains an isolated vertex.

Thus $|D| \geq d+1 = \begin{cases} (\frac{r}{2}+1)\lceil \frac{s}{3} \rceil & \text{if } s \equiv 0 \text{ or } 2 \ (mod\, 3), \\ (\frac{r}{2}+1)\lceil \frac{s}{3} \rceil - 4 & \text{if } s \equiv 1 \ (mod\, 3). \end{cases}$ Hence the theorem follows.

□

**Theorem 2.5.** *If $s \geq 6$, then* $\gamma_{CD}(P_6 \boxtimes P_s) = \begin{cases} s & \text{if } s \equiv 0 \ (mod\, 4), \\ s+1 & \text{if } s \equiv 1 \text{ or } 3 \ (mod\, 4), \\ s+2 & \text{if } s \equiv 2 \ (mod\, 4). \end{cases}$

*Proof.* Let $P_6 = (v_1, v_2, ..., v_6)$ and $P_s = (u_1, u_2, ..., u_s)$.
Then $V(P_6 \boxtimes P_s) = \{(v_i, u_j) : 1 \leq i \leq 6, 1 \leq j \leq s\}$ and $E(P_6 \boxtimes P_s) = \{(v_i, u_j)(v_i, u_{j+1}) : 1 \leq i \leq 6, 1 \leq j \leq s-1\} \cup \{(v_i, u_j)(v_{i+1}, u_j) : 1 \leq i \leq 5, 1 \leq j \leq s\} \cup \{(v_i, u_j)(v_{i+1}, u_{j+1}) : 1 \leq i \leq 5, 1 \leq j \leq s-1\} \cup \{(v_i, u_j)(v_{i+1}, u_{j-1}) : 1 \leq i \leq 5, 2 \leq j \leq s\}$. Let $S_1 = \{(v_i, u_j) : i \equiv 2 \ (mod\, 3), j \equiv 2 \text{ or } 3 \ (mod\, 4)\}$.

Then $S = \begin{cases} S_1 & \text{if } s \equiv 0 \text{ or } 3 \ (mod\, 4), \\ S_1 \cup \{(v_i, u_{s-1}) : i \equiv 2 \ (mod\, 3)\} & \text{if } s \equiv 1 \text{ or } 2 \ (mod\, 4), \end{cases}$ is a $CD-set$ of $P_6 \boxtimes P_s$. Thus $\gamma_{CD}(P_6 \boxtimes P_s) \leq |S| = \begin{cases} s & \text{if } s \equiv 0 \ (mod\, 4), \\ s+1 & \text{if } s \equiv 1 \text{ or } 3 \ (mod\, 4), \\ s+2 & \text{if } s \equiv 2 \ (mod\, 4). \end{cases}$ Suppose there exists a dominating set $D \subseteq V$ of cardinality at most $d = \begin{cases} s-1 & \text{if } s \equiv 0 \ (mod\, 4), \\ s & \text{if } s \equiv 1 \text{ or } 3 \ (mod\, 4), \\ s+1 & \text{if } s \equiv 2 \ (mod\, 4), \end{cases}$

then $< D >$ has an isolated vertex. Thus $|D| \geq d+1 = \begin{cases} s & \text{if } s \equiv 0 \,(mod\, 4), \\ s+1 & \text{if } s \equiv 1 \text{ or } 3 \,(mod\, 4), \\ s+2 & \text{if } s \equiv 2 \,(mod\, 4). \end{cases}$

Hence the result. □

**Theorem 2.6.** *If $r \equiv 3 \,(mod\, 4)$, $r \geq 7$ and $s \geq r$, then*

$$\gamma_{CD}(P_r \boxtimes P_s) = \begin{cases} \lceil \frac{r}{2} \rceil \lceil \frac{s}{3} \rceil & \text{if } s \equiv 0 \text{ or } 2 \ (mod\, 3), \\ \lceil \frac{r}{2} \rceil \lceil \frac{s}{3} \rceil - 2 & \text{if } s \equiv 1 \,(mod\, 3). \end{cases}$$

*Proof.* Let $P_r = (v_1, v_2, ..., v_r)$ and $P_s = (u_1, u_2, ..., u_s)$.

Then $V(P_r \boxtimes P_s) = \{(v_i, u_j) : 1 \leq i \leq r, 1 \leq j \leq s\}$ and $E(P_r \boxtimes P_s) = \{(v_i, u_j)(v_i, u_{j+1}) : 1 \leq i \leq r, 1 \leq j \leq s-1\} \cup \{(v_i, u_j)(v_{i+1}, u_j) : 1 \leq i \leq r-1, 1 \leq j \leq s\} \cup \{(v_i, u_j)(v_{i+1}, u_{j+1}) : 1 \leq i \leq r-1, 1 \leq j \leq s-1\} \cup \{(v_i, u_j)(v_{i+1}, u_{j-1}) : 1 \leq i \leq r-1, 2 \leq j \leq s\}$.

*Let* $S_1 = \{(v_i, u_j) : i \equiv 2 \text{ or } 3 \,(mod\, 4), j \equiv 2 \,(mod\, 3)\}$ *and*

*let* $A = \big(\{(v_i, u_s) : i \equiv 2 \text{ or } 3 \,(mod\, 4)\} \cup \{(v_{r-1}, u_{s-1})\}\big) - \{(v_r, u_s), (v_{r-1}, u_s)\}$

Then $S = \begin{cases} S_1 & \text{if } s \equiv 0 \text{ or } 2 \,(mod\, 3), \\ (S_1 \cup A) - \{(v_r, u_{s-2})\} & \text{if } s \equiv 1 \,(mod\, 3), \end{cases}$ is a $CD-set$ of $P_r \boxtimes P_s$.

Thus $\gamma_{CD}(P_r \boxtimes P_s) \leq |S| = \begin{cases} \lceil \frac{r}{2} \rceil \lceil \frac{s}{3} \rceil & \text{if } s \equiv 0 \text{ or } 2 \ (mod\, 3), \\ \lceil \frac{r}{2} \rceil \lceil \frac{s}{3} \rceil - 2 & \text{if } s \equiv 1 \,(mod\, 3). \end{cases}$

Suppose there exists a dominating set $D \subseteq V$ of cardinality at most

$d = \begin{cases} \lceil \frac{r}{2} \rceil \lceil \frac{s}{3} \rceil - 1 & \text{if } s \equiv 0 \text{ or } 2 \ (mod\, 3), \\ \lceil \frac{r}{2} \rceil \lceil \frac{s}{3} \rceil - 3 & \text{if } s \equiv 1 \,(mod\, 3), \end{cases}$ then $< D >$ contains an isolated vertex in $\{(v_i, u_j) : i \equiv 2 \text{ or } 3 \,(mod\, 4), j \equiv 2 \,(mod\, 3)\} \cup \{(v_{r-1}, u_j) : j \equiv 1 \,(mod\, 3)\}$.

Thus $|D| \geq d+1 = \begin{cases} \lceil \frac{r}{2} \rceil \lceil \frac{s}{3} \rceil & \text{if } s \equiv 0 \text{ or } 2 \ (mod\, 3), \\ \lceil \frac{r}{2} \rceil \lceil \frac{s}{3} \rceil - 2 & \text{if } s \equiv 1 \,(mod\, 3). \end{cases}$ Hence, the statement of the theorem is established. □

**Theorem 2.7.** *If $r \geq 3$ and $2 \leq s \leq 3$, then* $\gamma_{CD}(P_r \boxtimes P_s^c) = \begin{cases} r+2 & \text{if } r \equiv 2 \ (mod\, 4), \\ 2\lceil \frac{r}{2} \rceil & \text{otherwise.} \end{cases}$

*Proof.* Let $P_r = (v_1, v_2, ..., v_r)$ and $P_s = (u_1, u_2, ..., u_s)$.

Then $V(P_r \boxtimes P_s^c) = \{(v_i, u_j) : 1 \leq i \leq r, 1 \leq j \leq s\}$ and $E(P_r \boxtimes P_s^c) = \{(v_i, u_j)(v_{i+1}, u_j) :$

$1 \leq i \leq r-1, 1 \leq j \leq s\} \cup \{(v_i,u_j)(v_{i+1},u_k) : 1 \leq i \leq r-1, 1 \leq j \leq s-2, j+2 \leq k \leq s\} \cup \{(v_i,u_j)(v_{i+1},u_l) : 1 \leq i \leq r-1, 3 \leq j \leq s, 1 \leq l \leq j-2\} \cup \{(v_i,u_j)(v_i,u_k) : 1 \leq i \leq r, 1 \leq j \leq s-2, j+2 \leq k \leq s\}$. Let $S_1 = \{(v_i,u_j) : i \equiv 2 \; or \; 3 \; (mod\; 4), j = 1,2\}$. Then

$S = \begin{cases} S_1 \cup \{(v_{r-1},u_j) : j = 1,2\} & \text{if } r \equiv 1 \; or \; 2 \; (mod\; 4), \\ S_1 & \text{otherwise,} \end{cases}$ is a $CD$- set of $(P_r \boxtimes P_s^c)$.

Thus $\gamma_{CD}(P_r \boxtimes P_s^c) \leq |S| = \begin{cases} r+2 & \text{if } r \equiv 2 \; (mod\; 4), \\ 2\lceil \frac{r}{2} \rceil & \text{otherwise.} \end{cases}$

Suppose there exists a dominating $D$ of cardinality at most

$d = \begin{cases} r+1 & \text{if } r \equiv 1 \; or \; 2 \; (mod\; 4), \\ 2\lceil \frac{r}{2} \rceil - 1 & \text{otherwise,} \end{cases}$ then $< D >$ has an isolated vertex. Thus

$|D| \geq d+1 = \begin{cases} r+2 & \text{if } r \equiv 1 \; or \; 2 \; (mod\; 4), \\ 2\lceil \frac{r}{2} \rceil & \text{otherwise.} \end{cases}$ Hence the theorem follows. □

**Theorem 2.8.** *If $r \geq 4, s \geq 4$ and $s \geq r$, then $\gamma_{CD}(P_r \boxtimes P_s^c) = 2\lceil \frac{r}{3} \rceil$.*

*Proof.* Let $P_r = (v_1, v_2, ..., v_r)$ and $P_s = (u_1, u_2, ..., u_s)$.

Then $V(P_r \boxtimes P_s^c) = \{(v_i,u_j) : 1 \leq i \leq r, 1 \leq j \leq s\}$ and $E(P_r \boxtimes P_s^c) = \{(v_i,u_j)(v_{i+1},u_j) : 1 \leq i \leq r-1, 1 \leq j \leq s\} \cup \{(v_i,u_j)(v_{i+1},u_k) : 1 \leq i \leq r-1, 1 \leq j \leq s-2, j+2 \leq k \leq s\} \cup \{(v_i,u_j)(v_{i+1},u_l) : 1 \leq i \leq r-1, 3 \leq j \leq s, 1 \leq lj-2\} \cup \{(v_i,u_j)(v_i,u_k) : 1 \leq i \leq r, 1 \leq j \leq s-2, j+2 \leq k \leq s\}$. Let $S_1 = \{(v_i,u_1),(v_i,u_{s-1}) : i \equiv 2 \; (mod\; 3)\}$. Then

$S = \begin{cases} S_1 & \text{if } r \equiv 0 \; or \; 2 \; (mod\; 3). \\ S_1 \cup \{(v_r,u_1),(v_r,u_{s-1})\} & \text{if } r \equiv 1 \; (mod\; 3), \end{cases}$ is a $CD-set$ of $P_r \boxtimes P_s^c$.

Thus $\gamma_{CD}(P_r \boxtimes P_s^c) \leq |S| = 2\lceil \frac{r}{3} \rceil$.

Suppose there exists a dominating set $D \subseteq V$ of cardinality at most $d = 2\lceil \frac{r}{3} \rceil - 1$, then $< D >$ contains an isolated vertex. Thus $|D| \geq d+1 = 2\lceil \frac{r}{3} \rceil$. Hence the theorem follows. □

**Observation 2.1.**

1. *If $2 \leq r \leq 3$ and $s \geq r$, then $\gamma_{CD}(P_r \boxtimes P_s^c) = 4$.*

2. *If $s \geq 3$, then $\gamma_{CD}(C_3 \boxtimes P_s^c) = 4$.*

3. *If $r, s \geq 4$, then $\gamma_{CD}(C_r \boxtimes P_s^c) = 2\lceil \frac{r}{3} \rceil$.*

4. *If* $2 \le r \le 3$ *and* $s \ge 4$*, then* $\gamma_{CD}(P_r \boxtimes C_s^c) = 4$.

5. *If* $r, s \ge 4$*, then* $\gamma_{CD}(P_r \boxtimes C_s^c) = 2\lceil \frac{r}{3} \rceil$.

6. *If* $s \ge 3$*, then* $\gamma_{CD}(C_3 \boxtimes C_s^c) = 4$.

7. *If* $r, s \ge 4$*, then* $\gamma_{CD}(C_r \boxtimes C_s^c) = 2\lceil \frac{r}{3} \rceil$.

**Theorem 2.9.** *If* $r \ge 4$ *and* $s \ge r$*, then* $\gamma_{CD}(P_r \boxtimes K_s) = \begin{cases} \frac{r}{2}+1 & \textit{if } r \equiv 2 \ (mod\, 4), \\ \lceil \frac{r}{2} \rceil & \textit{otherwise.} \end{cases}$

*Proof.* Let $P_r = (v_1, v_2, ..., v_r)$ and $V(K_s) = \{u_1, u_2, ..., u_s\}$.
Then $V(P_r \boxtimes K_s) = \big\{(v_i, u_j) : 1 \le i \le r, 1 \le j \le s\big\}$ and $E(P_r \boxtimes K_s) = \{(v_i, u_j)(v_i, u_{j+1}) : 1 \le i \le r, 1 \le j \le j \le s-1\} \cup \{(v_i, u_j)(v_i, u_k) : 1 \le i \le r, 1 \le j \le s-2, j+2 \le k \le s\} \cup \{(v_i, u_j)(v_{i+1}, u_k) : 1 \le i \le r-1, j+1 \le k \le s, 1 \le j \le s-1\} \cup \{(v_i, u_j)(v_{i-1}, u_k) : 2 \le i \le r, 1 \le j \le s-1, j+1 \le k \le s\} \cup \{(v_i, u_j)(v_{i+1}, u_j) : 1 \le i \le r-1, 1 \le j \le s\}$.

Let $S_1 = \big\{(v_i, u_1) : i \equiv 2 \text{ or } 3 \ (mod\, 4)\big\}$. Then
$S = \begin{cases} S_1 \cup \{(v_{r-1}, u_1)\} & \text{if } r \equiv 1 \text{ or } 2 \ (mod\, 4), \\ S_1 & \text{otherwise,} \end{cases}$ is a $CD-$set of $P_r \boxtimes K_s$.

Thus $\gamma_{CD}(P_r \boxtimes K_s) \le |S| = \begin{cases} \frac{r}{2}+1 & \text{if } r \equiv 2 \ (mod\, 4), \\ \lceil \frac{r}{2} \rceil & \text{otherwise.} \end{cases}$

Suppose there exists a dominating set $D \subseteq V$ of cardinality at most
$d = \begin{cases} \frac{r}{2} & \text{if } r \equiv 2 \ (mod\, 4), \\ \lceil \frac{r}{2} \rceil - 1 & \text{otherwise.} \end{cases}$,

then $< D >$ contains an isolated vertex. Thus $|D| \ge d+1 = \begin{cases} \frac{r}{2}+1 & \text{if } r \equiv 2 \ (mod\, 4), \\ \lceil \frac{r}{2} \rceil & \text{otherwise.} \end{cases}$

Therefore the proof. □

**Theorem 2.10.** $\gamma_{CD}(C_r \boxtimes C_s) = \gamma_{CD}(P_r \boxtimes P_s)$, $s \ge r$.

*Proof.* Let $C_r = (v_1, v_2, ..., v_r, v_1)$ and $C_s = (u_1, u_2, ..., u_s, u_1)$.
Then $V(C_r \boxtimes C_s) = \big\{(v_i, u_j) : 1 \le i \le r, 1 \le j \le s\big\}$ and $E(C_r \boxtimes C_s) = \{(v_i, u_j)(v_i, u_{j+1}) : 1 \le i \le r, 1 \le j \le s-1\} \cup \{(v_i, u_j)(v_{i+1}, u_j) : 1 \le i \le r-1, 1 \le j \le s\} \cup \{(v_i, u_j)(v_{i+1}, u_{j+1}) : 1 \le i \le r-1, 1 \le j \le s-1\} \cup \{(v_i, u_j)(v_{i+1}, u_{j-1}) : 1 \le i \le r-1, 2 \le j \le s\} \cup \{(v_i, u_1)(v_{i+1}, u_s) :$

$1 \leq i \leq r-1\} \cup \{(v_i, u_1)(v_{i-1}, u_s) : 2 \leq i \leq r\} \cup \{(v_i, u_1)(v_i, u_s) : 1 \leq i \leq r\} \cup \{(v_1, u_i)(v_r, u_i) : 1 \leq i \leq s\} \cup \{(v_1, u_1)(v_r, u_s)\} \cup \{(v_r, u_1)(v_1, u_s)\}$.

The proof of this theorem had subdivided into many cases and each case will be explained in a identical way as in Theorem 2.3, 2.4, 2.5, 2.6. □

**Observation 2.2.**

1. *If $r \geq 3$ and $s \geq r$, then $\gamma_{CD}(C_r \boxtimes K_s) = \gamma_{CD}(P_r \boxtimes K_s)$.*
2. *$\gamma_{CD}(P_1 \boxtimes K_s) = \gamma_{CD}(P_1^c \boxtimes K_s) = 2$.*
3. *If $2 \leq r \leq 3$, then $\gamma_{CD}(P_r^c \boxtimes K_s) = 4$.*
4. *$\gamma_{CD}(P_r^c \boxtimes P_s^c)$ does not exists if $r \leq 2$.*
5. *$\gamma_{CD}(P_3^c \boxtimes P_3^c)$ does not exists.*
6. *If $r \geq 3$ and $s \geq 4$, then $\gamma_{CD}(P_r^c \boxtimes P_s^c) = 4$.*

**Theorem 2.11.** *If $r \leq 3$ and $s \geq 3$, then* $\gamma_{CD}(P_r \boxtimes C_s) = \begin{cases} \frac{s}{2}+1 & \text{if } s \equiv 2 \ (mod\, 4), \\ \lceil \frac{s}{2} \rceil & \text{otherwise.} \end{cases}$

*Proof.* Let $P_r = (v_1, v_2, ..., v_r)$ and $C_s = (u_1, u_2, ..., u_s, u_1)$.

Then $V(P_r \boxtimes P_s) = \{(v_i, u_j) : 1 \leq i \leq r, 1 \leq j \leq s\}$ and $E(P_r \boxtimes P_s) = \{(v_i, u_j)(v_i, u_{j+1}) : 1 \leq i \leq r, 1 \leq j \leq s-1\} \cup \{(v_i, u_j)(v_{i+1}, u_j) : 1 \leq i \leq r-1, 1 \leq j \leq s\} \cup \{(v_i, u_j)(v_{i+1}, u_{j+1}) : 1 \leq i \leq r-1, 1 \leq j \leq s-1\} \cup \{(v_i, u_j)(v_{i+1}, u_{j-1}) : 1 \leq i \leq r-1, 2 \leq j \leq s\} \cup \{(v_i, u_1)(v_{i+1}, u_s) : 1 \leq i \leq r-1\} \cup \{(v_i, u_1)(v_{i-1}, u_s) : 2 \leq i \leq r\} \cup \{(v_i, u_1)(v_i, u_s) : 1 \leq i \leq r\} \cup \{(v_1, u_i)(v_r, u_i) : 1 \leq i \leq s\} \cup \{(v_i, u_1)(v_{i+1}, u_s) : 1 \leq i \leq r-1\} \cup \{(v_i, u_1)(v_{i-1}, u_s) : 2 \leq i \leq r\} \cup \{(v_i, u_1)(v_i, u_s) : 1 \leq i \leq r\}$.

Let $S_1 = (v_2, u_i) : i \equiv 2 \text{ or } 3 \, (mod\, 4)$. Then $S = \begin{cases} S_1 \cup \{(v_2, u_{s-1})\} & \text{if } s \equiv 2 \, (mod\, 4), \\ S_1 & \text{otherwise,} \end{cases}$

is a $CD-$set of $P_r \boxtimes C_s$.

Thus $\gamma_{CD}(P_r \boxtimes C_s) \leq |S| = \begin{cases} \frac{s}{2}+1 & \text{if } s \equiv 2 \ (mod\, 4), \\ \lceil \frac{s}{2} \rceil & \text{otherwise.} \end{cases}$ Suppose there exists a dominating set $D \subseteq V$ of cardinality at most $d = \begin{cases} \frac{s}{2} & \text{if } s \equiv 2 \ (mod\, 4), \\ \lceil \frac{s}{2} \rceil - 1 & \text{otherwise.} \end{cases}$,

then $< D >$ contains an isolated vertex. Thus $|D| \geq d+1 = \begin{cases} \frac{s}{2}+1 & \text{if } s \equiv 2 \ (mod\, 4), \\ \lceil \frac{s}{2} \rceil & \text{otherwise.} \end{cases}$

Therefore the proof. □

**Theorem 2.12.** *If $r, s \geq 4$, then $\gamma_{CD}(P_r \boxtimes C_s) = \gamma_{CD}(P_r \boxtimes P_s)$.*

*Proof.* Let $P_r = (v_1, v_2, ..., v_r)$ and $C_s = (u_1, u_2, ..., u_s, u_1)$.
Then $V(P_r \boxtimes P_s) = \{(v_i, u_j) : 1 \leq i \leq r, 1 \leq j \leq s\}$ and $E(P_r \boxtimes P_s) = \{(v_i, u_j)(v_i, u_{j+1}) : 1 \leq i \leq r, 1 \leq j \leq s-1\} \cup \{(v_i, u_j)(v_{i+1}, u_j) : 1 \leq i \leq r-1, 1 \leq j \leq s\} \cup \{(v_i, u_j)(v_{i+1}, u_{j+1}) : 1 \leq i \leq r-1, 1 \leq j \leq s-1\} \cup \{(v_i, u_j)(v_{i+1}, u_{j-1}) : 1 \leq i \leq r-1, 2 \leq j \leq s\} \cup \{(v_i, u_1)(v_{i+1}, u_s) : 1 \leq i \leq r-1\} \cup \{(v_i, u_1)(v_{i-1}, u_s) : 2 \leq i \leq r\} \cup \{(v_i, u_1)(v_i, u_s) : 1 \leq i \leq r\} \cup \{(v_1, u_i)(v_r, u_i) : 1 \leq i \leq s\} \cup \{(v_i, u_1)(v_{i+1}, u_s) : 1 \leq i \leq r-1\} \cup \{(v_i, u_1)(v_{i-1}, u_s) : 2 \leq i \leq r\} \cup \{(v_i, u_1)(v_i, u_s) : 1 \leq i \leq r\}$. The proof of this theorem had subdivided into many cases and each case will be explained in a identical way as in Theorem 2.3, 2.4, 2.5, 2.6. □

**Theorem 2.13.** *For any non trivial path $P_r$ and $K_{t,s}, t \geq 1$, $\gamma_{CD}(P_r \boxtimes K_{t,s}) = 2\lceil \frac{r}{3} \rceil$.*

*Proof.* Let $P_r = (v_1, v_2, v_3, ..., v_r)$ and $V(K_{t,s}) = (V_1, V_2), where\ V_1 = (x_1, x_2, x_3, \dots, x_t)\ and$ $V_2 = (y_1, y_2, y_3, \dots, y_s)$. Then $V(P_r \boxtimes K_{t,s}) = \{(v_i, x_j), (v_i, y_k) : 1 \leq i \leq r, 1 \leq j \leq t, 1 \leq k \leq s\}$ and $E(P_r \boxtimes K_{t,s}) = \{(v_i, x_j)(v_{i+1}, x_j) : 1 \leq i \leq r-1, 1 \leq j \leq t\} \cup \{(v_i, y_j)(v_{i+1}, y_j) : 1 \leq i \leq r-1, 1 \leq j \leq t\} \cup \{(v_i, x_j)(v_i, y_k) : 1 \leq i \leq r, 1 \leq j \leq t, 1 \leq k \leq s\} \cup \{(v_i, x_j)(v_{i+1}, y_k) : 1 \leq i \leq r-1, 1 \leq j \leq t, 1 \leq k \leq s\} \cup \{(v_i, x_j)(v_{i-1}, y_k) : 2 \leq i \leq r, 1 \leq j \leq t, 1 \leq k \leq s\}$.
Let $S_1 = \{(v_i, u_1), (v_i, u_{s-1}) : i \equiv 2\, (mod\, 3)\}$. Then
$S = \begin{cases} S_1 & \text{if } r \equiv 0 \text{ or } 2\, (mod\, 3). \\ S_2 \cup \{(v_r, u_1), (v_r, u_{s-1})\} & \text{if } r \equiv 1\, (mod\, 3), \end{cases}$ is a $CD-set$ of $P_r \boxtimes K_{t,s}$.
Thus $\gamma_{CD}(P_r \boxtimes P_s^c) \leq |S| = 2\lceil \frac{r}{3} \rceil$. Suppose there exists a dominating set $D \subseteq V$ of cardinality at most $d = 2\lceil \frac{r}{3} \rceil - 1$, then $< D >$ contains an isolated vertex. Thus $|D| \geq d+1 = 2\lceil \frac{r}{3} \rceil$.
Hence the theorem follows. □

**Theorem 2.14.** *If $r \geq 2$ and $s \geq 3$, then $\gamma_{CD}(P_r \boxtimes H_s) = \begin{cases} \frac{(s+1)(r+2)}{2} & if\ s \equiv 2\ (mod\, 4), \\ (s+1)\lceil \frac{r}{2} \rceil & otherwise. \end{cases}$*

*Proof.* Let $P_r = (v_1, v_2, v_3, \ldots, v_r)$ and $V(H_s) = \{u_1, u_2, u_3, \ldots, u_s, u'_1, u'_2, u'_3, \ldots, u'_s, u\}$, where $\Delta(H_s) = deg(u), deg(u'_i) = 1$ and the vertices $u_1, u_2, u_3, \ldots, u_s$ form a cycle. Then $V(P_r \boxtimes H_s) = \{(v_i, u_j)(v_i, u'_j)(v_i, u) : 1 \leq i \leq r, 1 \leq j \leq s\}$ and $E(P_r \boxtimes H_s) = \{(v_i, u_j)(v_{i+1}, u_j), (v_i, u'_j)(v_{i+1}, u'_j), (v_i, u)(v_{i+1}, u) : 1 \leq i \leq r-1, 1 \leq j \leq s\} \cup \{(v_i, u_j)(v_{i+1}, u_j), (v_i, u_1)(v_i, u_s) : 1 \leq i \leq r, 1 \leq j \leq s-1\} \cup \{(v_i, u_j)(v_i, u'_j), (v_i, u_j)(v_i, u) : 1 \leq i \leq r, 1 \leq j \leq s\} \cup \{(v_i, u_j)(v_{i+1}, u_{j+1}) : 1 \leq i \leq r-1, 1 \leq j \leq s-1\} \cup \{(v_i, u_j)(v_{i+1}, u'_j) : 1 \leq i \leq r-1, 1 \leq j \leq s\} \cup \{(v_i, u_j)(v_{i+1}, u) : 1 \leq i \leq r-1, 1 \leq j \leq s\} \cup \{(v_{i+1}, u_j)(v_i, u_{j+1}) : 1 \leq i \leq r-1, 2 \leq j \leq s-1\} \cup \{(v_{i+1}, u_j)(v_i, u'_j) : 1 \leq i \leq r-1, 1 \leq j \leq s\} \cup \{(v_{i+1}, u_j)(v_i, u) : 1 \leq i \leq r-1, 1 \leq j \leq s\}$. Let $S_1 = \{(v_i, u'_j), (v_i, u) : i \equiv 2 \; or \; 3 \; (mod\, 4), 1 \leq j \leq s\}$. Then $S = \begin{cases} S_1 \cup \{(v_{r-1}, u'_j), (v_{r-1}, u) : 1 \leq j \leq s\} & \text{if } r \equiv 1 \; or \; 2 \; (mod\, 4), \\ S_1 & \text{otherwise,} \end{cases}$ is a $CD$- set of $P_r \boxtimes H_s$.

Thus $\gamma_{CD}(P_r \boxtimes H_s) \leq |S| = \gamma_{CD}(P_r \boxtimes H_s) = \begin{cases} \frac{(s+1)(r+2)}{2} & \text{if } s \equiv 2 \; (mod\, 4), \\ (s+1)\lceil \frac{r}{2} \rceil & \text{otherwise.} \end{cases}$ Suppose there exists a $D$ dominating of cardinality at most $d = \begin{cases} \frac{(s+1)(r+2)}{2} - 1 & \text{if } s \equiv 2 \; (mod\, 4), \\ (s+1)\lceil \frac{r}{2} \rceil - 1 & \text{otherwise,} \end{cases}$ then $< D >$ has an isolated vertex. Thus $|D \geq d + 1 = \begin{cases} \frac{(s+1)(r+2)}{2} & \text{if } s \equiv 2 \; (mod\, 4), \\ (s+1)\lceil \frac{r}{2} \rceil & \text{otherwise.} \end{cases}$ Hence the theorem. □

**Observation 2.3.**

1. *Let $G$ be a totally disconnected graph of order $s$, then*
$$\gamma_{CD}(P_r \boxtimes G) = \begin{cases} \frac{s(r+2)}{2} & \textit{if } s \equiv 2 \; (mod\, 4), \\ s\lceil \frac{r}{2} \rceil & \textit{otherwise.} \end{cases}$$

2. $\gamma_{CD}(P_r \boxtimes W_{1,s}) = \gamma_{CD}(P_r) = \begin{cases} \frac{r}{2} + 1 & \textit{if } s \equiv 2 \; (mod\, 4), \\ \lceil \frac{r}{2} \rceil & \textit{otherwise.} \end{cases}$

3. $\gamma_{CD}(L(P_r) \boxtimes L(P_s)) = \gamma_{CD}(P_{r-1} \boxtimes P_{s-1})$.

4. $\gamma_{CD}(L(P_r) \boxtimes P_s) = \gamma_{CD}(P_{r-1} \boxtimes P_s)$.

5. $\gamma_{CD}(L(C_r) \boxtimes L(C_s)) = \gamma_{CD}(C_r \boxtimes C_s)$.

**Conclusion**

Finding the $CD$-number for a general graph ia an NP-complete problem. In this paper we find out the exact value of $\gamma_{CD}(P_r \boxtimes P_s), \gamma_{CD}(P_r \boxtimes P_s^c), \gamma_{CD}(P_r \boxtimes C_s), \gamma_{CD}(P_r \boxtimes K_s), \gamma_{CD}(P_r \boxtimes W_{1,s}), \gamma_{CD}(L(P_r) \boxtimes P_s), etc.$ Also we have given the minimum $CD$-set for the above noticed graph and the comparsion of this parameter with other dominating parameter will be described in the successive paper.

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