



## Morgan-Stone Lattices

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# MORGAN-STONE LATTICES

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ABSTRACT. *Morgan-Stone (MS) lattices* are axiomatized by the constant-free identities of those axiomatizing *Morgan-Stone (MS) algebras*. Applying the technique of characteristic functions of prime filters as homomorphisms from lattices onto the two-element chain one and their products, we prove that the variety of MS lattices is the abstract hereditary multiplicative class generated by a six-element one with an equational disjunctive system expanding the direct product of the three- and two-element chain distributive lattices, in which case subdirectly-irreducible MS lattices are exactly isomorphic copies of nine non-one-element subalgebras of the six-element generating MS lattice, and so we get a 29-element non-chain distributive lattice of varieties of MS lattices subsuming the four-/three-element chain one of “De Morgan”/Stone lattices/algebras (viz., constant-free versions of De Morgan algebras)/(more precisely, their term-wise definitionally equivalent constant-free versions, called *Stone lattices*). Among other things, we provide an REDPC scheme for MS lattices. Laying a special emphasis onto the [quasi-]equational join (viz., the [quasi-]variety generated by the union) of De Morgan and Stone lattices, we find a fifteen-element non-chain distributive lattice of its sub-quasi-varieties subsuming the eight-element one of those of the variety of De Morgan lattices found earlier, each of the rest being the quasi-equational join of its intersection with the variety of De Morgan lattices and the variety of Stone lattices. In this connection, we also prove that any relatively simple relatively subdirectly-representable abstract hereditary multiplicative subclass of the equational join of the varieties of De Morgan lattices/algebras and Stone ones is a sub-variety of the former.

## 1. INTRODUCTION

The notion of *De Morgan lattice*, being originally due to [15], has been independently explored in [10] under the term *distributive  $i$ -lattice* w.r.t. their subdirectly-irreducibles and the lattice of varieties. They satisfy so-called *De Morgan identities*. On the other hand, these are equally satisfied in *Stone algebras* (cf., e.g., [7]). This has inevitably raised the issue of unifying such varieties. Perhaps, a first way of doing it within the framework of De Morgan algebras (viz., bounded De Morgan lattices; cf., e.g., [1]) has been due to [2] (cf. [23]) under the term *Morgan-Stone (MS) algebra* providing a description of their subdirectly-irreducibles, among which there are those being neither De Morgan nor Stone algebras. Here, we study unbounded MS algebras naturally called *Morgan-Stone (MS) lattices*. Demonstrating the usefulness of the technique of the characteristic functions of prime filters and functional products of former ones as well as disjunctive systems, we briefly discuss the issues of subdirectly-irreducible Morgan-Stone lattices and their varieties. Likewise, summarizing construction of REDPC schemes (cf. [6]) for distributive lattice[ expansion]s originally being due to [8] [and [12, 21]], we provide that for Morgan-Stone lattices and an enhanced one for the {quasi-}equational join of De Morgan and Stone lattices. Nevertheless, the main purpose of this study is to find the lattice of sub-quasi-varieties of the latter upon the basis of that of the variety

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of De Morgan lattices found in [17]. In this connection, we also prove that any relatively simple relatively subdirectly-representable abstract hereditary multiplicative subclass of the equational join of the varieties of De Morgan lattices/algebras and Stone ones is a sub-variety of the former.

The rest of the work is as follows. Section 2 is a concise summary of basic set-theoretical and algebraic issues underlying the work. Then, in Section 3 we briefly summarize general issues concerning REDPC in the sense of [6] as well as equational implicative/disjunctive systems in the sense of [20]/[19] in connection with simplicity/“subdirect irreducibility”. Next, Section 4 is devoted to preliminary study of Morgan-Stone lattices. Further, Section 5 is a thorough collection of culminating results on sub-quasi-varieties of the [quasi-]equational join of De Morgan and Stone lattices. Finally, Section 6 is a concise collection of open issues and necessary statements.

## 2. GENERAL BACKGROUND

**2.1. Set-theoretical background.** Non-negative integers are identified with the sets/ordinals of lesser ones, “their set/ordinal”|“the ordinal||set class” being denoted by  $\omega|\infty||\Upsilon$ . Unless any confusion is possible, one-element sets are identified with their elements.

For any sets  $A, B$  and  $D$  as well as  $\theta \subseteq A^2$ ,  $h : A \rightarrow B$  and  $g : A^2 \rightarrow A$ , let  $\wp_{[K]}((B, )A)$  be the set of all subsets of  $A$  (including  $B$ ) [of cardinality in  $K \subseteq \infty$ ,  $D \subseteq_K A$  standing for  $D \in \wp_K(A)$ ],  $((\Delta_A|\nu_\theta)|(A/\theta)|\chi_A^B) \triangleq (\{\langle a, a|\theta[\{a\}] \rangle \mid a \in A\}|\nu_\theta[A]|(((A \cap B) \times \{1\}) \cup ((A \setminus B) \times \{0\})))$ ,  $A^{*|+} \triangleq (\bigcup_{m \in (\omega \setminus \{0|1\})} A^m)$ ,  $h_* : A^* \rightarrow B^* : a \mapsto (a \circ h)$ ,  $g_+ : A^+ \rightarrow A$ ,  $\langle [a, b], ]c \rangle \mapsto [g][g_+(\langle a, b \rangle), ]c$  and  $\varepsilon_B : (\Upsilon^B)^2 \rightarrow \wp(B)$ ,  $\langle d, e \rangle \mapsto \{b \in B \mid \pi_b(d) = \pi_b(e)\}$ ,  $A$ -tuples {viz., functions with domain  $A$ } being written in the sequence form  $\bar{t}$  with  $t_a$ , where  $a \in A$ , standing for  $\pi_a(\bar{t})$ . Then, for any  $(\bar{a}|C) \in (A^*|\wp(A))$ , by induction on the length (viz., domain) of any  $\bar{b} = \langle [\bar{c}, d] \rangle \in A^*$ , put  $((\bar{a} * \bar{b})|(\bar{b}(\cap/\wedge)C)) \triangleq ((([\bar{a} * \bar{c}, d])|([\bar{c}(\cap/\wedge)C, d])])|[(\text{provided } d \in / \notin C)])$ . Likewise, given any  $\bar{S} \in \wp(D)^B$  and  $\bar{f} \in \prod_{b \in B} S_b^A$ , let  $(\prod \bar{f}) : A \rightarrow (\prod_{b \in B} S_b)$ ,  $a \mapsto \langle f_b(a) \rangle_{b \in B}$ , in which case

$$(2.1) \quad \ker(\prod \bar{f}) = (A^2 \cap (\bigcap_{b \in B} (\ker f_b))),$$

$$(2.2) \quad \forall b \in B : f_b = ((\prod \bar{f}) \circ \pi_b),$$

$f_0 \times f_1$  standing for  $(\prod \bar{f})$ , whenever  $B = 2$ .

A *lower/upper cone* of a poset  $\mathcal{P} = \langle P, \leq \rangle$  is any  $C \subseteq P$  such that, for all  $a \in C$  and  $b \in P$ ,  $(a \geq / \leq b) \Rightarrow (b \in C)$ . Then, an  $a \in S \subseteq P$  is said to be *minimal/maximal in S*, if  $\{a\}$  is a lower/upper cone of  $S$ , their set being denoted by  $(\min / \max)_{\mathcal{P}|\leq}(S)$ , in case of the equality of which to  $S$ , this being called an *anti-chain* of  $\mathcal{P}$ .

An  $X \in Y \subseteq \wp(A)$  is said to be  $[K\text{-}]$ meet-irreducible in  $Y$ , [where  $K \subseteq \infty$ ], if  $\forall Z \in \wp_{[K]}(Y) : ((A \cap (\bigcap Z)) = X) \Rightarrow (X \in Z)$ , their set being denoted by  $\text{MI}^{[K]}(Y)$ , “finitely-” standing for “ $\omega$ -” within any related context. Next, a  $\mathcal{U} \subseteq \wp(A)$  is said to be *upward-directed*, if  $\forall \mathcal{S} \in \wp_\omega(\mathcal{U}) : \exists T \in (\mathcal{U} \cap \wp(\bigcup \mathcal{S}, A))$ , subsets of  $\wp(A)$  closed under unions of upward directed subsets being called *inductive*. Further, a [finitary] *closure operator* over  $A$  is any unary operation on  $\wp(A)$  such that  $\forall X \in \wp(A), \forall Y \in \wp(X) : (X \cup C(C(X)) \cup C(Y)) \subseteq C(X) [= (\bigcup C[\wp_\omega(X)])]$ . Finally, a *closure system* over  $A$  is any  $\mathcal{C} \subseteq \wp(A)$  containing  $A$  and closed under intersections of subsets containing  $A$ , any  $\mathcal{B} \subseteq \mathcal{C}$  with  $\mathcal{C} = \{A \cap (\bigcap \mathcal{S}) \mid \mathcal{S} \subseteq \mathcal{B}\}$  being called a (closure) *basis* of  $\mathcal{C}$  and determining the closure operator  $C_{\mathcal{B}} \triangleq \{\langle Z, A \cap (\bigcap (X \cap \wp(Z, A))) \rangle \mid Z \in \wp(A)\}$  over  $A$  with  $(\text{img } C_{\mathcal{B}}) = \mathcal{C}$ . Conversely,  $\text{img } C$  is a closure system over  $A$  with

$C_{\text{img } C} = C$ , being inductive iff  $C$  is finitary, and forming a complete lattice under the partial ordering by inclusion with meet/join  $(\Delta_{\wp(A)}/C)(A \cap ((\bigcap/\bigcup)\mathcal{S}))$  of any  $\mathcal{S} \subseteq (\text{img } C)$ ,  $C$  and  $\text{img } C$  being called *dual to one another*. Then,  $C(X) \in (\text{img } C)$  is said to be *generated by* an  $X \subseteq A$ , elements of  $C_{[\wp_\omega/\{n\}]}(A)$  / “with  $n \in (\omega|\{1\})$ ” being said to be *finitely/n-generated/principal*.

*Remark 2.1.* Due to Zorn Lemma, according to which any non-empty inductive set has a maximal element,  $MI^{[K]}(\mathcal{C})$  is a basis of any inductive closure system  $\mathcal{C}$ .  $\square$

A *filter/ideal on A* is any  $\mathcal{F} \subseteq \wp(A)$  such that, for all  $\mathcal{S} \in \wp_\omega(\wp(A))$ ,  $(\mathcal{S} \subseteq \mathcal{F}) \Leftrightarrow ((A \cap ((\bigcap/\bigcup)\mathcal{S})) \in \mathcal{F})$  “the set  $\text{Fi}(A)$  of them being an inductive closure system over  $\wp(A)$  with dual finitary closure operator (of filter generation)  $\text{Fg}_A$  such that

$$(2.3) \quad \text{Fg}_A(\mathcal{F}) = \wp(A \cap (\bigcap \mathcal{F}), A),$$

for all  $\mathcal{F} \in \wp_\omega(\wp(A))$ ”. Then, an *ultra-filter on A* is any filter  $\mathcal{U}$  on  $A$  such that  $\wp(A) \setminus \mathcal{U}$  is an ideal on  $A$ .

**2.2. Algebraic background.** Unless otherwise specified, we deal with a fixed but arbitrary finitary functional signature  $\Sigma$ ,  $\Sigma$ -algebras/“their carriers” being denoted by same capital Fraktur/Italic letters (with same indices, if any) “with denoting the class of all [one-element] ones by  $\mathbf{A}_\Sigma^{[=1]}$ ”. Given any  $\alpha \in (\infty \setminus 1)$ , let  $\text{Tm}_\Sigma^\alpha$  be the carrier of the absolutely-free  $\Sigma$ -algebra  $\mathfrak{Tm}_\Sigma^\alpha$ , freely-generated by the set  $V_\alpha \triangleq \{x_\beta\}_{\beta \in \alpha}$  of (*first*  $\alpha$ ) *variables*, and  $\text{Eq}_\Sigma^\alpha \triangleq (\text{Tm}_\Sigma^\alpha)^2$ ,  $\phi \approx /(\lesssim | \gtrsim)\psi$ , where  $\phi, \psi \in \text{Tm}_\Sigma^\alpha$  / “and  $\wedge \in \Sigma$ ”, meaning  $\langle \phi/(\phi \wedge \psi), \psi/(\phi \psi) \rangle$  “and being called a  $\Sigma$ -equation of rank  $\alpha$ ”. / “Likewise, for any  $\Sigma$ -algebra  $\mathfrak{A}$  and  $a, b \in A$ ,  $(a \leq | \geq)^\mathfrak{A} b$ ”  $\| [a, b]_\mathfrak{A}$  stands for  $((a|b) = (a \wedge^\mathfrak{A} b)) \| \{c \in A \mid a \leq^\mathfrak{A} c \leq^\mathfrak{A} b\}$ .” Then, any  $\langle \Gamma, \Phi \rangle \in (\wp_{\infty/(\{1\} \cup \omega)}(\text{Eq}_\Sigma^\alpha) \times \text{Eq}_\Sigma^\alpha)$  / “with  $\alpha \in \omega$ ” is called a  $\Sigma$ -implication/-[quasi-]identity of rank  $\alpha$ , written as  $\Gamma \rightarrow \Phi$  and identified with  $\Phi$ , if  $\Gamma = \emptyset$ , as well as treated as the universal infinitary/first-order strict Horn sentence  $\forall_{\beta \in \alpha} x_\beta ((\bigwedge \Gamma) \rightarrow \Phi)$ , the class/set of those of any /finite rank true in a  $\mathbf{K} \subseteq \mathbf{A}_\Sigma$  being called the *implicational/[quasi-]equational theory of K* and denoted by  $(\mathcal{J}/[\mathcal{Q}]\mathcal{E})(\mathbf{K})$ .

Subclasses of  $\mathbf{A}_\Sigma$  “closed under  $\mathbf{I}|\mathbf{S}_{(>1)}|\mathbf{P}^{[\text{SD}|\text{U}]}$ ” / “containing each  $\Sigma$ -algebra with finitely-generated subalgebras in them” / “containing no infinite finitely-generated member” are referred to as “*abstract|(non-trivially-)hereditary|[ultra-|sub-]multiplicative*”/local/locally-finite (cf. [14]). Then, a *skeleton* {of a(n abstract)  $\mathbf{K} \subseteq \mathbf{A}_\Sigma$ } is any  $\mathbf{S} \subseteq \mathbf{A}_\Sigma$  without pair-wise distinct isomorphic members {such that  $\mathbf{S} \subseteq \mathbf{K} \subseteq \mathbf{IS}$  (i.e.,  $\mathbf{K} = \mathbf{IS}$ )}. Given a  $\mathbf{K} \subseteq \mathbf{A}_\Sigma \ni \mathfrak{A}$ , set  $\text{hom}^{[\text{S}]}(\mathfrak{A}, \mathbf{K}) \triangleq \{h \in \text{hom}(\mathfrak{A}, \mathfrak{B}) \mid \mathfrak{B} \in \mathbf{K}, (\text{img } h) = B\}$  and  $\text{Co}_\mathbf{K}(\mathfrak{A}) \triangleq \{\theta \in \text{Co}(\mathfrak{A}) \mid (\mathfrak{A}/\theta) \in \mathbf{K}\}$ , whose elements are called  $\mathbf{K}$ -(*relative*) *congruences of A*,  $\mathfrak{A} \preceq \mathbf{K}$  standing for  $\mathfrak{A} \in \mathbf{ISK}$  and thus providing a quasi-ordering on  $\mathbf{A}_\Sigma$ , in which case, by the Homomorphism Theorem, we have

$$(2.4) \quad (\ker[\text{hom}^{[\text{S}]}(\mathfrak{A}, \mathbf{K})] \setminus \{(\emptyset|\{A^2\})\}) = \text{Co}_{(\mathbf{I}|\mathbf{IS}_{(>1)})\mathbf{K}}(\mathfrak{A}),$$

and so “by the Homomorphism Theorem”, for all  $\mathfrak{B} \in \mathbf{A}_\Sigma$  and  $h \in \text{hom}^{[\text{S}]}(\mathfrak{B}|\mathfrak{A}, \mathfrak{A}|\mathfrak{B})$ :

$$(2.5) \quad \forall \theta \in (\text{Co}_{(\mathbf{I}|\mathbf{IS}_{(>1)})\mathbf{K}}(\mathfrak{B}) \cap \wp((\ker h)|\Delta_B, B^2)) :$$

$$h_*^{[(-1)]}[\theta] \in (\text{Co}_{(\mathbf{I}|\mathbf{IS}_{(>1)})\mathbf{K}}(\mathfrak{A}) \cap \wp(\Delta_A|(\ker h), A^2)),$$

$$h_*^{[(-1)]}[h_*^{[(-1)]}[\theta]] = (\theta \cap (B|h[A])^2)$$

“yielding an isomorphism between the posets  $\text{Co}_{[\mathbf{K}]}(\mathfrak{B}) \cap \wp(\ker h, B^2)$  and  $\text{Co}_{[\mathbf{K}]}(\mathfrak{A})$  ordered by inclusion as well as”  $\|$  “implying:

$$(2.6) \quad h_*^{-1}[\text{Cg}_{(\mathbf{I}|\mathbf{IS}_{(>1)})\mathbf{K}}^\mathfrak{B}(h_*[X])] = \| \supseteq \text{Cg}_{(\mathbf{I}|\mathbf{IS}_{(>1)})\mathbf{K}}^\mathfrak{A}(X \cup (\ker h)),$$

for all  $X \subseteq A^{2^n}$ , while, as, for any set  $I$ ,  $\overline{\mathfrak{B}} \in \mathbf{A}_\Sigma^I$  and  $\bar{f} \in (\prod_{i \in I} \text{hom}(\mathfrak{A}, \mathfrak{B}_i))$ :

$$(2.7) \quad \left( \prod \bar{f} \right) \in \text{hom}(\mathfrak{A}, \prod_{i \in I} \mathfrak{B}_i),$$

by (2.1) and (2.2) with  $I \triangleq \text{Co}_{(\mathbf{I}|\mathbf{S})\mathbf{K}}(\mathfrak{A})$  for  $B$ ,  $\overline{\mathfrak{B}} \triangleq \langle \mathfrak{B}/i \rangle_{i \in I}$ ,  $D \triangleq (\bigcup_{i \in I} B_i)$  and  $\bar{f} \triangleq \langle \nu_i \rangle_{i \in I}$ , we get:

$$(2.8) \quad (\mathfrak{A} \in \mathbf{IP}^{\text{SD}}(\mathbf{I}|\mathbf{S})\mathbf{K}) \Leftrightarrow ((A^2 \cap (\bigcap \ker[\text{hom}^{\text{Sll}}(\mathfrak{A}, \mathbf{K})])) = \Delta_A),$$

whereas, since, for any  $I \triangleq \Theta \subseteq \text{Co}_{\mathbf{K}}(\mathfrak{A})$ ,  $\theta \triangleq (A^2 \cap (\bigcap \Theta)) \in \text{Co}(\mathfrak{A})$ ,  $\overline{\mathfrak{B}} \triangleq \langle \mathfrak{A}/i \rangle_{i \in I} \in (\mathbf{A}_\Sigma \langle \cap \mathbf{K} \rangle)^I$  as well as, by the Homomorphism Theorem,  $\bar{f} \triangleq \langle \nu_\theta^{-1} \circ \nu_i \rangle_{i \in I} \in (\prod_{i \in I} \text{hom}(\mathfrak{A}/\theta, \mathfrak{B}_i))$ , taking (2.1), (2.2) and (2.7) into account, we see that  $e \triangleq (\prod \bar{f})$  is an embedding of  $\mathfrak{A}/\theta$  into  $\mathfrak{C} \triangleq (\prod_{i \in I} \mathfrak{B}_i)$  such that  $\mathfrak{C} \upharpoonright (\text{img } e)$ , being isomorphic to  $\mathfrak{A}/\theta$ , is a subdirect product of  $\overline{\mathfrak{B}}$  (in which case  $(\mathfrak{A}/\theta) \in \mathbf{IP}^{\text{SD}}\mathbf{K}$ , and so, providing  $\mathbf{K}$  is both abstract and sub-multiplicative,  $\theta \in \text{Co}_{\mathbf{K}}(\mathfrak{A})$ ). In particular, [providing  $\mathbf{K}$  is both abstract and sub-multiplicative],  $\text{Co}_{[\mathbf{K}]}(\mathfrak{A})$  is a closure system over  $A^2$ , the dual closure operator being denoted by  $\text{Cg}_{[\mathbf{K}]}^{\mathfrak{A}}$ .

*Remark 2.2.* By (2.4), the  $\downarrow$ -right alternative of (2.5) with  $h = \nu_\theta$ , where  $\theta \in \text{Co}_{\mathbf{IP}^{\text{SD}}(\mathbf{I}|\mathbf{S})\mathbf{K}}(\mathfrak{A})$ ,  $\mathfrak{B} = (\mathfrak{A}/\theta)$  and  $\theta = \Delta_B$  as well as (2.8), since  $\vartheta = h_*^{-1}[\theta]$ , while  $h_*^{-1}$  preserves intersections,  $\text{Co}_{(\mathbf{I}|\mathbf{S})\mathbf{K}}(\mathfrak{A})$  is a basis of the closure system  $\text{Co}_{\mathbf{IP}^{\text{SD}}(\mathbf{I}|\mathbf{S})\mathbf{K}}(\mathfrak{A})$  over  $A^2$ .  $\square$

According to [22], *pre-varieties* are abstract hereditary multiplicative subclasses of  $\mathbf{A}_\Sigma$  (these are exactly model classes of theories constituted by  $\Sigma$ -implications of unlimited rank, and so are also called *implicative/implicational*; cf., e.g., [3]/[17]),  $\mathbf{ISP}\mathbf{K} = \mathbf{IP}^{\text{SD}}(\mathbf{I}\mathbf{S}_{\geq 1})\mathbf{K} = \text{Mod}(\mathcal{J}(\mathbf{K}))$  being the least one including and so called *generated by a  $\mathbf{K} \subseteq \mathbf{A}_\Sigma$* . Likewise, [*quasi-*]*varieties* are [ultra-multiplicative] pre-varieties closed under  $\mathbf{H}^{[1]} \triangleq \mathbf{I}$  (these are exactly model classes of sets of  $\Sigma$ -[quasi-]identities of unlimited finite rank, and so are local and also called [*quasi-*]*equational*; cf., e.g., [14]),  $\mathbf{H}^{[1]}\mathbf{SP}[\mathbf{P}^{\text{U}}]\mathbf{K} = \text{Mod}([\mathcal{Q}]\mathcal{J}(\mathbf{K}))$  being the least one including and so called *generated by a  $\mathbf{K} \subseteq \mathbf{A}_\Sigma$* . Then, intersections of a  $\mathbf{K} \subseteq \mathbf{A}_\Sigma$  with [quasi-]varieties are called its *relative sub-[quasi-]varieties*, in which case, for any  $\mathcal{E} \subseteq \text{Eq}_\Sigma^\omega$ ,

$$(2.9) \quad (\mathbf{IP}^{\text{SD}}(\mathbf{K}) \cap \text{Mod}(\mathcal{E})) = \mathbf{IP}^{\text{SD}}(\mathbf{K} \cap \text{Mod}(\mathcal{E})),$$

and so  $\mathbf{S} \mapsto (\mathbf{S} \cap \mathbf{K})$  and  $\mathbf{R} \mapsto \mathbf{IP}^{\text{SD}}\mathbf{R}$  are inverse to one another isomorphisms between the lattices of relative sub-varieties of  $\mathbf{IP}^{\text{SD}}\mathbf{K}$  and those of  $\mathbf{K}$ .

Then, a [pre-]variety  $\mathbf{P} \subseteq \mathbf{A}_\Sigma$  is said to be [(relatively)] *congruence-distributive*, if, for each  $\mathfrak{A} \in \mathbf{P}$ ,  $\text{Co}_{(\mathbf{P})}(\mathfrak{A})$  is distributive.

*Remark 2.3.* Given a [quasi-equational] pre-variety  $\mathbf{P} \subseteq \mathbf{A}_\Sigma$  and  $\alpha \in (\infty \setminus 1)$ , by the  $\downarrow$ -right alternative of (2.4) with  $\mathbf{K} = \mathbf{P}$  and  $\mathfrak{A} = \text{Tm}_\Sigma^\alpha$ , any  $\Sigma$ -implication  $\Gamma \rightarrow \Phi$  of rank  $\alpha$  is true in  $\mathbf{P}$  iff  $\Phi \in \text{Cg}_{\mathbf{P}}^\alpha(\Gamma)$  [in which case, by the Compactness Theorem for ultra-multiplicative classes of algebras (cf., e.g., [14]),  $\text{Cg}_{\mathbf{P}}^\alpha$  is finitary, and so is  $\text{Cg}_{\mathbf{P}}^\beta$ , for any  $\mathfrak{B} \in \mathbf{A}_\Sigma$ , in view of the left  $\|\text{-}$ alternative of (2.6), when taking  $\alpha = |B|$  and  $h$  to extend any bijection from  $V_\alpha$  onto  $B$ ].  $\square$

Furthermore, [given an abstract  $\mathbf{K} \subseteq \mathbf{A}_\Sigma$ ] an  $\mathfrak{A} \in (\mathbf{A}_\Sigma[\cap \mathbf{K}])$  is said to be [ $\mathbf{K}$ -*relatively*] *simple* / [ $\mathbf{K}$ -*subdirectly-irreducible*] (where  $\mathbf{K} \subseteq \infty$ ), if  $\Delta_A \in (\max_{\subseteq} / \mathbf{M}\mathbf{I}^{(\mathbf{K})})(\text{Co}_{[\mathbf{K}]}(\mathfrak{A}) \setminus (\{A^2\}/\emptyset))$ , in which case  $|A| \neq 1$ , the class of (those of) them (which are in a  $\mathbf{K}' \subseteq (\mathbf{A}_\Sigma[\cap \mathbf{K}])$ ) being denoted by  $(\text{Si} / \text{SI}^{(\mathbf{K})})_{[\mathbf{K}]}(\mathbf{K}')$ ,<sup>1</sup> and so, by

<sup>1</sup>This is abstract (whenever  $\mathbf{K}'$  is so), in view of (2.5).

(2.4) and (2.8),

$$(2.10) \quad (\text{Si} \mid \text{SI})_{[\mathbf{IP}^{\text{SD}}(\mathbf{S})K'']}(\mathbf{IP}^{\text{SD}}(\mathbf{S})K'') \subseteq \mathbf{I}(\mathbf{S}_{>1})K'',$$

for any  $K'' \subseteq \mathbf{A}_\Sigma$ . Then, a [pre-]variety  $\mathbf{P}$  is said to be  $[\{\text{relatively}\}]$  (finitely) *semi-simple/subdirectly-representable*, if

$$(\text{SI}_{[\{\mathbf{P}\}]}^{(\omega)}(\mathbf{P})/\mathbf{P}) \subseteq \mid = (\text{Si}_{[\{\mathbf{P}\}]}(\mathbf{P})/\mathbf{IP}^{\text{SD}}(\text{Si} \mid \text{SI}_{[\{\mathbf{P}\}]}^{(\omega)}(\mathbf{P}))),$$

any variety  $\mathbf{V} \subseteq \mathbf{A}_\Sigma$  being well-known, due to Birkhoff's Theorem, to be subdirectly-representable. More generally, we have:

*Remark 2.4.* Given any [quasi-]variety  $\mathbf{Q} \subseteq \mathbf{A}_\Sigma$  and  $\mathfrak{A} \in (\{\mathbf{Q} \cap \} \mathbf{A}_\Sigma)$ , by Remarks 2.1, 2.2, 2.3 and the right  $\parallel$ -alternative of (2.5),  $\text{MI}^{(\omega)}(\text{Co}_{\mathbf{Q}}(\mathfrak{A})) = \text{Co}_{\text{SI}_{\mathbf{Q}}^{(\omega)}(\mathbf{Q})}(\mathfrak{A})$  is a basis of both  $\text{Co}_{\mathbf{Q}}(\mathfrak{A})$  and  $\text{Co}_{\mathbf{IP}^{\text{SD}} \text{SI}_{\mathbf{Q}}^{(\omega)}(\mathbf{Q})}(\mathfrak{A})$ , in which case these are equal {and so, since  $\nu_{\Delta_A} \in \text{hom}^{\text{S}}(\mathfrak{A}, \mathfrak{A}/\Delta_A)$  is injective,  $\mathfrak{A} \in \mathbf{IP}^{\text{SD}} \text{SI}_{\mathbf{Q}}^{(\omega)}(\mathbf{Q})$ . In particular,  $\mathbf{Q}$  is [relatively] (finitely) subdirectly-representable.  $\square$

Recall that, according to [13], a [n implicational]  $\mathbf{K} \subseteq \mathbf{A}_\Sigma$  is *congruence-permutable*, i.e., for each  $\mathfrak{A} \in \mathbf{K}$  and all  $\theta, \vartheta \in \text{Co}(\mathfrak{A})$ ,  $(\theta \circ \vartheta) \subseteq (\vartheta \circ \theta)$ , if [f] it has a *congruence-permutation term*, viz., a  $\pi \in \text{Tm}_\Sigma^3$  such that  $\mathbf{K}$  satisfies the  $\Sigma$ -identities in  $\{x_1 \approx (\sigma_i(\pi)) \mid i \in \{0, 2\}\}$ , where, for every  $j \in 3$ ,  $\sigma_j \triangleq [x_j/x_1; x_k/x_0]_{k \in (3 \setminus \{j\})}$ . Likewise, a *minority|majority term* for  $\mathbf{K}$  {with  $\Sigma_+ \triangleq \{\wedge, \vee\} \subseteq \Sigma$  and the  $\Sigma_+$ -reducts of members of  $\mathbf{K}$  being lattices} is any  $\mu \in \text{Tm}_\Sigma^3$  such that  $\mathbf{K}$  satisfies the  $\Sigma$ -identities in  $\{x_{(1-\min(2-i,i))0} \approx (\sigma_i(\mu)) \mid i \in 3\} \mid \{\mu_+ \triangleq (\wedge_+ \langle x_i \vee (x_{\max(1-i,0)} \wedge x_{2+\min(i,1-i)}) \rangle)_{i \in 3}\}$  being so}, in which case it is so “as well as a congruence-permutation term” for the variety generated by  $\mathbf{K}$ , and so this is congruence-distributive [16], while, for any congruence-permutation term  $\pi$  for  $\mathbf{K}$ ,  $\pi[x_1/\mu]$  is a majority|minority term for  $\mathbf{K}$  “and so  $\mu[x_1/\mu]$  is a majority term for  $\mathbf{K}$ ”. Finally, a (*ternary*) |*dual discriminator (term)* for  $\mathbf{K}$  is any  $\delta \in \text{Tm}_\Sigma^3$  such that, for each  $\mathfrak{A} \in \mathbf{K}$ ,  $\delta^{\mathfrak{A}} = ((\pi_{2|0} \upharpoonright (\Delta_A \times A)) \cup (\pi_{0|2} \upharpoonright ((A^2 \setminus \Delta_A) \times A)))$ , in which case  $\mathfrak{A}$  is simple, because, for every  $\theta \in (\text{Co}(\mathfrak{A}) \setminus \{\Delta_A\})$ , any  $\langle a, b \rangle \in (\theta \setminus \Delta_A) \neq \emptyset$  and all  $c \in A$ , we have  $\langle a|c \rangle = \delta^{\mathfrak{A}}(a, b, c)$   $\theta^{\mathfrak{A}}(a, a, c) = \langle c|a \rangle$ , so getting  $\theta = A^2$ , while  $\delta$  is a |*dual discriminator* for  $\mathbf{IP}^{\text{U}} \mathbf{K}$  as well as a minority|majority term for  $\mathbf{K}$ , whereas, for any congruence-permutation term  $\pi$  for  $\mathbf{K}$ ,  $\pi[x_1/\delta]$  is a dual| discriminator for  $\mathbf{K}$  “and so  $\delta[x_1/\delta]$  is a dual discriminator for  $\mathbf{K}$ ”, {(quasi-/pre-)varieties generated by classes of}  $\Sigma$ -algebras with [dual] discriminator  $\delta$  being called [*dual*]  $\delta$ -*discriminator*, with denoting the class of [dual]  $\delta$ -discriminator members of a  $\mathbf{C} \subseteq \mathbf{A}_\Sigma$  by  $\mathbf{C}_\delta^{[\delta]}$ . Then, [dual]  $\delta$ -discriminator quasi-varieties are exactly quasi-equational [dual]  $\delta$ -discriminator pre-varieties.

2.2.1. *Filtral congruences.* Let  $I$  be a set,  $\mathcal{F}$  a {n ultra-}filter on  $I$  [ $\mathbf{P} \subseteq \mathbf{A}_\Sigma$  a (quasi-equational) pre-variety],  $\overline{\mathfrak{A}} \in (\mathbf{A}_\Sigma[\cap \mathbf{P}])^I$  and  $\mathfrak{B}$  a subalgebra of its direct product. Then, by (2.5), for each  $i \in I$ ,  $(B^2 \cap (\ker \pi_i)) = ((\pi_i \upharpoonright B)_2^{-1}[\Delta_{A_i}] \in \text{Co}_{[\mathbf{P}]}(\mathfrak{B}))$ , as  $(\pi_i \upharpoonright B) \in \text{hom}(\mathfrak{B}, \mathfrak{A}_i)$  and  $\Delta_{A_i} \in \text{Co}_{[\mathbf{P}]}(\mathfrak{A}_i)$ , in which case, for all  $K \subseteq J \subseteq I$ , the closure system  $\text{Co}_{[\mathbf{P}]}(\mathfrak{B})$  on  $B^2$  contains  $\theta_J^B \triangleq (B^2 \cap \varepsilon_I^{-1}[\emptyset(J, I)]) = (B^2 \cap (\bigcap_{j \in J} \ker \pi_j)) \subseteq \theta_K^B$ ,  $\Theta_{\mathcal{F}}^B \triangleq \{\theta_L^B \mid L \in \mathcal{F}\}$  being thus upward-directed (and so  $\text{Co}_{[\mathbf{P}]}(\mathfrak{B})$ , being inductive, in view of Remark 2.3, contains  $\theta_{\mathcal{F}}^B \triangleq (\bigcup \Theta_{\mathcal{F}}^B) = (B^2 \cap \varepsilon_I^{-1}[\mathcal{F}])$ , called  $\langle \mathcal{F} \rangle$ -{ultra-}filtral). Clearly, for any  $\mathcal{X} \subseteq \text{Fi}(I)$  | “with  $(\bigcup \mathcal{X}) \in \text{Fi}(I)$ ”,

$$(2.11) \quad \theta_{\emptyset(I) \cap ((\bigcap \mid \bigcup) \mathcal{X})}^B = (B^2 \cap ((\bigcap \mid \bigcup) \{\theta_{\mathcal{F}}^B \mid \mathcal{F} \in \mathcal{X}\})).$$

A [pre-]variety  $\mathbf{P} \subseteq \mathbf{A}_\Sigma$  is said to be [*relatively*] (*subdirectly*) (*finitely/principally*) *filtral*, if every (finitely-generated/principal) [ $\mathbf{P}$ -]congruence of each member of  $\mathbf{SP} \text{SI}_{[\mathbf{P}]}(\mathbf{P}) (\cap \mathbf{P}^{\text{SD}} \text{SI}_{[\mathbf{P}]}(\mathbf{P}))$  is filtral (cf. [6] for the equational case).

## 2.2.1.1. Filtrality versus semi-simplicity.

**Lemma 2.5.** *Any [relatively] subdirectly principally filtral [pre-]variety  $\mathbf{P} \subseteq \mathbf{A}_\Sigma$  is [relatively] semi-simple.*

*Proof.* Consider any  $\mathfrak{A} \in \mathbf{SI}_{[\mathbf{P}]}(\mathbf{P})$ , in which case  $|A| > 1$ , and any  $\theta \in (\text{Co}_{[\mathbf{P}]}(\mathfrak{A}) \setminus \{\Delta_A\})$  as well as any  $\bar{a} \in (\theta \setminus \Delta_A) \neq \emptyset$ , in which case  $\mathfrak{B} \triangleq \mathfrak{A}^1 \in \mathbf{P}^{\text{SD}} \mathbf{SI}_{[\mathbf{P}]}(\mathbf{P})$ , while  $h \triangleq (\pi_0|_B) \in \text{hom}^S(\mathfrak{B}, \mathfrak{A})$  is injective, whereas  $B^2 \ni \bar{b} \triangleq (\bar{a} \circ h^{-1}) \in \vartheta \triangleq \text{Cg}_{[\mathbf{P}]}^{\mathfrak{B}}(\bar{b}) = \theta_{\mathcal{F}}^B$ , for some  $\mathcal{F} \in \text{Fi}(1)$ , and so, by (2.5),  $\eta \triangleq h_*^{-1}[\theta] \in (\text{Co}_{[\mathbf{P}]}(\mathfrak{B}) \cap \varphi(\vartheta, B^2))$ , while  $\theta = h_*[\eta]$ , whereas  $\varnothing = \varepsilon_1(\bar{b}) \in \mathcal{F}$ . Then,  $\mathcal{F} = \varphi(1)$ , in which case  $\eta \supseteq \vartheta = B^2$ , and so  $\theta \supseteq h_*[B^2] = A^2$ . Thus,  $\mathfrak{A} \in \mathbf{Si}_{[\mathbf{P}]}(\mathbf{P})$ , as required.  $\square$

## 2.2.1.2. Filtrality versus congruence-distributivity.

**Lemma 2.6** (cf. [9] for the  $\square$ (-)-non-optional case). *Let  $\mathbf{Q} \subseteq \mathbf{A}_\Sigma$  be a [quasi-]variety,  $I$  a set,  $\overline{\mathfrak{A}} \in \mathbf{Q}^I$ ,  $\mathfrak{B} \in \mathbf{S}(\prod \overline{\mathfrak{A}})$  and  $\theta \in \text{MI}^{(\omega)}(\text{Co}_{[\mathbf{Q}]}(\mathfrak{B}))$ . Suppose  $\text{Co}_{[\mathbf{Q}]}(\mathfrak{B})$  is distributive. Then, there is an ultra-filter  $\mathcal{U}$  on  $I$  such that  $\theta_{\mathcal{U}}^B \subseteq \theta$ .*

*Proof.* By (2.11),  $S \triangleq \{\mathcal{F} \in \text{Fi}(I) \mid \theta_{\mathcal{F}}^B \subseteq \theta\} \ni \{I\}$  is inductive, for  $\text{Fi}(I)$  is so, in which case, by Zorn Lemma, it, being non-empty, has a maximal element  $\mathcal{U}$ , and so, for any  $\mathcal{X} \in \wp_\omega(\varphi(I))$  such that  $Y \triangleq (\bigcup \mathcal{X}) \in \mathcal{U}$ ,  $(\mathcal{X} \cap \mathcal{U}) \neq \emptyset$ , as, for each  $Z \in \mathcal{X}$ ,  $\theta_{\mathcal{F}_Z}^B \in \text{Co}_{[\mathbf{Q}]}(\mathfrak{B})$  with  $\mathcal{U} \subseteq \mathcal{F}_Z \triangleq \text{Fg}_I(\mathcal{U} \cup \{Z\}) \in \text{Fi}(I)$ , while  $\mathcal{U} = \text{Fg}_I(\mathcal{U}) = \text{Fg}_I(\mathcal{U} \cup \{Y\}) = (\varphi(I) \cap (\bigcap \{\mathcal{F}_Z \mid Z \in \mathcal{X}\}))$ , in view of (2.3), since  $\text{Fg}_I$  is finitary, whereas, by (2.11),  $\theta = \text{Cg}_{[\mathbf{Q}]}^{\mathfrak{B}}(\theta \cup \theta_{\mathcal{U}}^B) = \text{Cg}_{[\mathbf{Q}]}^{\mathfrak{B}}(\theta \cup (B^2 \cap (\bigcap \{\theta_{\mathcal{F}_Z}^B \mid Z \in \mathcal{X}\}))) = (B^2 \cap (\bigcap \{\text{Cg}_{[\mathbf{Q}]}^{\mathfrak{B}}(\theta \cup \theta_{\mathcal{F}_Z}^B) \mid Z \in \mathcal{X}\}))$ , that is, for some  $Z \in \mathcal{X}$ ,  $\theta = \text{Cg}_{[\mathbf{Q}]}^{\mathfrak{B}}(\theta \cup \theta_{\mathcal{F}_Z}^B) \supseteq \theta_{\mathcal{F}_Z}^B$ , i.e.,  $\mathcal{U} \subseteq \mathcal{F}_Z \in S$ , viz.,  $Z \in \mathcal{F}_Z = \mathcal{U}$ , as required.  $\square$

This, by (2.5), Birkhoff's and the Homomorphism Theorems [as well as [5, Corollary 2.3]/[20, Lemma 2.1]], immediately yields:

**Corollary 2.7.** *Let  $\mathbf{K}$  be a [finite/] class of [finite/]  $\Sigma$ -algebras (with {dual} discriminator  $\delta$ ) and  $\mathbf{P} \triangleq \mathbf{H}^{(1)}\mathbf{SPK}$ . Suppose  $\mathbf{P}$  is a [relatively] congruence-distributive [/locally-finite] [quasi-]variety. Then,*

$$(\mathbf{P}_\delta^{\{\theta\}} \subseteq \mathbf{Si}_{[\mathbf{P}]}(\mathbf{P}) \subseteq) \mathbf{SI}_{[\mathbf{P}]}^{\omega}(\mathbf{P}) \subseteq \mathbf{H}^{(\|\langle 1 \rangle\|)} \mathbf{SP}^{\mathbf{U}}\mathbf{K}[\subseteq \mathbf{H}^{(\|\langle 1 \rangle\|)} \mathbf{SK}][\subseteq \mathbf{P}_\delta^{\{\theta\}}]$$

*[in which case its members are finite, and so  $\mathbf{SI}_{[\mathbf{P}]}^{\omega}(\mathbf{P}) = \mathbf{SI}_{[\mathbf{P}]}(\mathbf{P})$ ]/. In particular, {dual} ( $\delta$ -)discriminator quasi-varieties are exactly [semi-simple] {dual} ( $\delta$ -)discriminator varieties.*

**Corollary 2.8.** *Let  $\mathbf{Q} \subseteq \mathbf{A}_\Sigma$  be a ([relatively] semi-simple) [quasi-]variety,  $I \in \Upsilon$ ,  $\overline{\mathfrak{A}} \in \mathbf{Si}_{[\mathbf{Q}]}(\mathbf{Q})^I$ ,  $\mathfrak{D} \triangleq (\prod \overline{\mathfrak{A}})$ ,  $\mathfrak{B} \in \mathbf{S}\{\mathfrak{D}\}$  and  $\theta \in (\text{Co}_{[\mathbf{Q}]}(\mathfrak{B}) \setminus \{B^2\})$ . Suppose  $\mathbf{Si}_{[\mathbf{Q}]}(\mathbf{Q})^I$  is both ultra-multiplicative and non-trivially-hereditary {while  $\text{Co}_{[\mathbf{Q}]}(\mathfrak{B})$  is distributive}. Then,  $\theta$  is maximal in  $\text{Co}_{[\mathbf{Q}]}(\mathfrak{B}) \setminus \{B^2\}$  if {f} it is ultra-filtral. {In particular, all elements of  $\text{Co}_{[\mathbf{Q}]}(\mathfrak{B})$  are filtral.}*

*Proof.* First, assume  $\theta = \theta_{\mathcal{U}}^B$ , for some ultra-filter  $\mathcal{U}$  on  $I$ , in which case  $\mathfrak{C} \triangleq (\mathfrak{D}/\theta_{\mathcal{U}}^D) \in \mathbf{P}^{\mathbf{U}} \mathbf{Si}_{[\mathbf{Q}]}(\mathbf{Q}) \subseteq \mathbf{Si}_{[\mathbf{Q}]}(\mathbf{Q})$ , while  $h \triangleq (\Delta_B \circ \nu_{\theta_{\mathcal{U}}^D}) \in \text{hom}(\mathfrak{B}, \mathfrak{C})$ , whereas  $(\ker h) = (\Delta_B)_*^{-1}[\theta_{\mathcal{U}}^D] = \theta$ , and so by (2.4) and Footnote 1, as  $\theta \neq B^2$ ,  $(\mathfrak{B}/\theta) \in \mathbf{IS}_{>1} \mathbf{Si}_{[\mathbf{Q}]}(\mathbf{Q}) \subseteq \mathbf{Si}_{[\mathbf{Q}]}(\mathbf{Q})$ . Then, by (2.5),  $\theta \in \max(\text{Co}_{[\mathbf{Q}]}(\mathfrak{B}) \setminus \{B^2\})$ . {Conversely, assume  $\theta \in \max(\text{Co}_{[\mathbf{Q}]}(\mathfrak{B}) \setminus \{B^2\}) \subseteq \text{MI}(\text{Co}_{[\mathbf{Q}]}(\mathfrak{B}))$ , in which case, by Lemma 2.6, there is some ultra-filter  $\mathcal{U}$  on  $I$  such that, as  $\theta \neq B^2$ ,  $(\text{Co}_{[\mathbf{Q}]}(\mathfrak{B}) \setminus \{B^2\}) \ni \theta_{\mathcal{U}}^B \subseteq \theta$ , and so, by the "if" part,  $\theta = \theta_{\mathcal{U}}^B$ . (Then, Remarks 2.1, 2.3, 2.4, (2.5) and (2.11) complete the argument.)}  $\square$

### 2.2.2. Subdirect products versus subalgebras.

**Lemma 2.9** (cf. [11]). *Let  $\mathfrak{A} \in \mathbf{A}_\Sigma$  and  $\mathfrak{B}$  a subalgebra of  $\mathfrak{A}$ . Then,  $h_A^B \triangleq \{(\bar{a}, b) \in (A^\omega \times B) \mid |\omega \setminus \varepsilon_\omega(\bar{a}, \omega \times \{b\})| \in \omega\} \supseteq (\bigcup\{\{\langle \omega \times \{b\}, b \rangle\} \cup \{(\langle (\omega \setminus \{i\}) \times \{b\}) \cup \{(i, a)\}, b) \mid i \in \omega, a \in A\} \mid b \in B\})$  is a function forming a subalgebra of  $\mathfrak{A}^\omega \times \mathfrak{B}$ , in which case it is a surjective homomorphism from  $\mathfrak{C}_A^B \triangleq (\mathfrak{A}^\omega \upharpoonright (\text{dom } h_A^B))$  onto  $\mathfrak{B}$ , and so  $\mathfrak{C}_A^B$  is a subdirect product of  $\omega \times \{\mathfrak{A}\}$ .*

#### 2.2.2.1. Filtrality versus non-trivial heredity of simplicity.

**Corollary 2.10.** *Let  $\mathbf{P} \subseteq \mathbf{A}_\Sigma$  be a [relatively] subdirectly principally filtral [pre-]variety. Then,  $(\text{SI}_{[\mathbf{P}]}(\mathbf{P}) \cup \mathbf{A}_\Sigma^{-1})(\setminus \mathbf{A}_\Sigma^{-1})$  is (non-trivially-)hereditary.*

*Proof.* Let  $\mathfrak{A} \in (\text{SI}_{[\mathbf{P}]}(\mathbf{P}) \cup \mathbf{A}_\Sigma^{-1})$  and  $\mathfrak{B}$  a non-one-element subalgebra of  $\mathfrak{A}$ , in which case  $|A| \neq 1$ , and so, by Lemma 2.9,  $h \triangleq h_A^B$  is a surjective homomorphism from the subdirect product  $\mathfrak{C} \triangleq \mathfrak{C}_A^B$  of  $(\omega \times \{\mathfrak{A}\}) \in \text{SI}_{[\mathbf{P}]}(\mathbf{P})^\omega$  onto  $\mathfrak{B}$ . Consider any  $\theta \in (\text{Co}_{[\mathbf{P}]}(\mathfrak{B}) \setminus \{\Delta_B\})$  and take any  $\langle a, b \rangle \in (\theta \setminus \Delta_B) \neq \emptyset$ , in which case, by (2.5),  $\text{Co}_{[\mathbf{P}]}(\mathfrak{C}) \ni \vartheta \triangleq h_*^{-1}[\theta] \ni \langle \bar{c}, \bar{d} \rangle \triangleq \langle \omega \times \{a\}, \omega \times \{b\} \rangle$ , while  $h_*[\vartheta] = \theta$ , and so  $\vartheta \supseteq \eta \triangleq \text{Cg}_{[\mathbf{P}]}^{\mathfrak{C}}(\langle \bar{c}, \bar{d} \rangle) = \theta_{\mathcal{F}}^{\mathfrak{C}}$ , for some  $\mathcal{F} \in \text{Fi}(\omega)$ . Then,  $\emptyset = \varepsilon_\omega(\bar{c}, \bar{d}) \in \mathcal{F}$ , in which case  $\mathcal{F} = \wp(\omega)$ , and so  $\vartheta \supseteq \eta = C^2$ . Thus,  $\theta \supseteq h_*[C^2] = B^2$ , in which case  $\theta = B^2$ , and so  $\mathfrak{B} \in \text{Si}_{[\mathbf{P}]}(\mathbf{P})$ , as required.  $\square$

**2.2.3. Locality versus local finiteness.** As an immediate consequence of [20, Lemma 2.1], in its turn, being that of [5, Corollary 2.3], we, first, have the following useful universal observation:

**Corollary 2.11.** *Any abstract hereditary local subclass of a locally-finite quasi-variety is ultra-multiplicative.*

Aside from quasi-varieties as such, certain representative subclasses of them are local as well.

#### 2.2.3.1. Local subclasses of local pre-varieties.

**Lemma 2.12.** *Let  $\mathbf{P} \subseteq \mathbf{A}_\Sigma$  be a [local (more specifically, quasi-equational) pre-]variety. Then,  $(\text{SI}^\omega \upharpoonright \text{Si})_{[\mathbf{P}]}(\mathbf{P}) \cup \mathbf{A}_\Sigma^{-1}$  is local.*

*Proof.* Consider any  $\mathfrak{B} \in (\mathbf{P} \setminus ((\text{SI}^\omega \upharpoonright \text{Si})_{[\mathbf{P}]}(\mathbf{P}) \cup \mathbf{A}_\Sigma^{-1}))$ , in which case there are some  $\bar{a} \in (B^2 \setminus \Delta_B) \neq \emptyset$ ,  $n \in (\omega \setminus \{1\})$  and  $\bar{\theta} \in (\text{Co}_{[\mathbf{P}]}(\mathfrak{B}) \setminus (\text{img } \bar{\vartheta}^B))^n$ , where, for any  $C \subseteq B$ ,  $\bar{\vartheta}^C \triangleq (\langle \Delta_C \rangle \upharpoonright \langle \Delta_C, C^2 \rangle)$ , “such that  $(B^2 \cap (\bigcap (\text{img } \bar{\theta}))) = \Delta_B$ ”, and so some  $\langle \bar{b}^{i,j} \rangle_{i \in n}^{j \in (1|2)} \in (\prod_{i \in n}^{j \in (1|2)} ((\theta_i \setminus \vartheta_j^B) \cup (\vartheta_j^B \setminus \theta_i))) \neq \emptyset$ . Let  $\mathfrak{A}$  be the finitely-generated subalgebra of  $\mathfrak{B}$  generated by  $\{a_0, a_1\} \cup \{b_k^{i,j} \mid i \in n, j \in (1|2), k \in 2\}$ , in which case, by (2.5) with  $h = \Delta_A$ ,  $\bar{\eta} \triangleq \langle \theta_i \cap A^2 \rangle_{i \in n} \in (\text{Co}_{[\mathbf{P}]}(\mathfrak{A}) \setminus (\text{img } \bar{\vartheta}^A))^n$ , as  $\langle \bar{b}^{i,j} \rangle_{i \in n}^{j \in (1|2)} \in (\prod_{i \in n}^{j \in (1|2)} ((\eta_i \setminus \vartheta_j^A) \cup (\vartheta_j^A \setminus \eta_i)))$ , so  $\mathfrak{A} \in (\mathbf{P} \setminus ((\text{SI}^\omega \upharpoonright \text{Si})_{[\mathbf{P}]}(\mathbf{P}) \cup \mathbf{A}_\Sigma^{-1}))$ , for  $\bar{a} \in (A^2 \setminus \Delta_A)$  “and  $(A^2 \cap (\bigcap (\text{img } \bar{\eta}))) = (A^2 \cap (\bigcap (\text{img } \bar{\theta}))) = (A^2 \cap \Delta_B) = \Delta_A$ ”.  $\square$

**2.2.3.1.1. Finite semi-simplicity versus semi-simplicity and local finiteness.** Lemma 2.12 immediately yields:

**Corollary 2.13.** *Any locally-finite [relatively] semi-simple [local (more specifically, quasi-equational) pre-]variety  $\mathbf{P} \subseteq \mathbf{A}_\Sigma$  with hereditary  $\text{SI}_{[\mathbf{P}]}^\omega(\mathbf{P}) \cup \mathbf{A}_\Sigma^{-1}$  is [relatively] finitely semi-simple.*

## 3. PRELIMINARIES: QUATERNARY EQUATIONAL SCHEMES

A quaternary  $\Sigma$ -(equational) scheme is any  $\mathcal{U} \subseteq \text{Eq}_\Sigma^4$ . This is called an *implication scheme* for a  $\mathbf{K} \subseteq \mathbf{A}_\Sigma$ , if this satisfies the  $\Sigma$ -implication:

$$(3.1) \quad (\{x_0 \approx x_1\} \cup \mathcal{U}) \rightarrow (x_2 \approx x_3).$$



Likewise, it is called an *identity|reflexive|symmetric|transitive* one, if  $\mathbf{K}$  satisfies the  $\Sigma$ -implications of the form  $(\emptyset|\emptyset|\mathcal{U}|(\mathcal{U} \cup (\mathcal{U}[x_{2+i}/x_{3+i}]_{i \in 2}))) \rightarrow \Psi$ , where  $\Psi \in (\mathcal{U}([x_3/x_2][x_{2+i}/x_i]_{i \in 2}[x_3/x_2, x_2/x_3][x_3/x_4]))$ , reflexive symmetric transitive ones being also called *equivalence* ones. Then,  $\mathcal{U}$  is called a *congruence* one, if it is an equivalence one, while, for each  $\varsigma \in \Sigma$  of arity  $n \in (\omega \setminus 1)$ ,  $\mathbf{K}$  satisfies the  $\Sigma$ -implications of the form  $(\bigcup_{j \in n} (\mathcal{U}[x_{2+i}/x_{2+i+(2 \cdot j)}]_{i \in 2})) \rightarrow \Psi$ , where  $\Psi \in (\mathcal{U}[x_{2+i}/\varsigma((x_{2+i+(2 \cdot j)})_{j \in n})]_{i \in 2})$ . Finally,  $\mathcal{U}$  [being finite] is called a “*restricted equationally definable principal {relative} congruence (REDP{R}C)*”/“(equational) *implicative|disjunctive scheme/system* for a “{pre-}variety”/  $\mathbf{K} \subseteq \mathbf{A}_\Sigma$ , if, for each  $\mathfrak{A} \in \mathbf{K}$  and all  $\bar{a} \in A^4$ ,  $(\forall \theta \in (\text{Co}_{\{\mathbf{K}\}}(\mathfrak{A})/\{\Delta_A\}) : (\langle a_0, a_1 \rangle \in | \notin \theta) \Rightarrow (\langle a_3, a_3 \rangle \in \theta)) \Leftrightarrow (\mathfrak{A} \models (\bigwedge \mathcal{U})[x_i/a_i]_{i \in I}$  [cf. [6]/[20][19]] / “and so for  $\mathbf{IS}[\mathbf{P}^{\mathcal{U}}]\mathbf{K}$ , (pre-varieties generated by classes of)  $\Sigma$ -algebras with [finite] implicative|disjunctive system  $\mathcal{U}$  being called  $\langle$ [finitely]  $\mathcal{U}$ -implicative|-disjunctive with the class of  $\mathcal{U}$ -implicative|-disjunctive members of any  $\mathbf{K}' \subseteq \mathbf{A}_\Sigma$  denoted by  $\mathbf{K}'_{\mathcal{U}}$  “in which case  $\mathcal{U}$ , being an implication scheme for (the pre-variety generated by)  $\mathbf{K}$ , providing this is quasi-equational, includes a finite one, by the Compactness Theorem for ultramultiplicative classes of algebras [14]”, and so implicative quasi-varieties, being thus finitely so, are exactly those in the original sense of [20]. Then, by Remark 2.4 therein, quasi-equational/finitely implicative pre-varieties are finitely disjunctive.

Given any  $\tau \in \text{Tm}_\Sigma^3$ , put

$$\begin{aligned} \mathcal{U}_\tau^\supset &\triangleq \{\tau \approx (\tau[x_2/x_3])\}, \\ \mathcal{U}_\tau^{\partial\supset} &\triangleq \{(\tau[x_0/x_{2+k}, x_1/x_{3-k}, x_2/(\tau[x_2/x_{2+k}])]) \approx x_{2+k} \mid k \in 2\}, \\ \mathcal{U}_\tau^\vee &\triangleq \{(\tau[x_0/\tau, x_1/(\tau[x_2/x_3])]) \approx (\tau[x_0/\tau, x_1/(\tau[x_2/x_3]), x_2/x_3])\}, \end{aligned}$$

in which case  $\mathcal{U}_\tau^\vee$  is defined by  $\mathcal{U}_\tau^\supset$  according to [20, Remark 2.4].

*Remark 3.1.* Given any [dual] discriminator  $\tau \in \text{Tm}_\Sigma^3$  for a  $\mathbf{K} \subseteq \mathbf{A}_\Sigma$ ,  $\mathcal{U}_\tau^{([\partial\supset])^\vee}$  is a finite implicative/disjunctive system for  $\mathbf{K}$ . In particular, any [dual] discriminator pre-variety is finitely both implicative and disjunctive.  $\square$

This enables us to build easily an example of a non-quasi-equational finitely both implicative and disjunctive pre-variety well-justifying the generic framework of pre-varieties we follow here:

**Example 3.2.** Let  $\Sigma = \{\neg, \nabla, \tau\}$ , where  $\neg$  and  $\nabla$  are unary, while  $\tau$  is ternary,  $\mathfrak{A}$  the  $\Sigma$ -algebra such that  $A \triangleq \omega$ ,  $\tau(x_0, x_1, x_2)$  is a (dual) discriminator for  $\mathfrak{A}$  and, for all  $a \in A$ ,  $\nabla^{\mathfrak{A}}(a) \triangleq \min(a, 1)$ , whereas  $\neg^{\mathfrak{A}}(a) \triangleq \max(0, a - 1)$ . Then, by Remark 3.1, the pre-variety  $\mathbf{P}$  generated by  $\mathfrak{A}$ , being (dual)  $\tau$ -discriminator, is finitely both implicative and disjunctive. Let us show, by contradiction, that it is not a quasi-variety. For suppose it is a quasi-variety. By induction on any  $n \in \omega$ , put  $\neg^{0[+n+1]}x_i \triangleq [\neg^{\neg^n}]x_i [= \neg^n \neg x_i]$ , where  $i \in 2$ , and set  $\varepsilon^n \triangleq (\nabla(\neg^n x_0) \approx \nabla(\neg^n x_1))$ . Then, given any  $N \subseteq \omega$ , set  $\varepsilon_N \triangleq \{\varepsilon^n \mid n \in N\}$ . Note that the  $\Sigma$ -implication  $\varepsilon_\omega \rightarrow (x_0 \approx x_1)$  is true in  $\mathfrak{A}$ , and so in  $\mathbf{P}$ . Hence, by Remark 2.3, there is some  $N \in \wp_\omega(\omega)$  such that the  $\Sigma$ -quasi-identity  $\varepsilon_N \rightarrow (x_0 \approx x_1)$  is true in  $\mathbf{P} \ni \mathfrak{A}$ . However,  $\mathfrak{A} \models \varepsilon_N[x_i/(i + m + 1)]_{i \in 2}$ , where  $m \triangleq (\bigcup N) \in \omega$ , though  $(m + 1) \neq (m + 2)$ . This contradiction means that  $\mathbf{P}$  is not a quasi-variety.  $\square$

### 3.1. Implicativity versus REDPRC and relative semi-simplicity.

**Lemma 3.3.** Let  $\mathcal{U} \subseteq \text{Eq}_\Sigma^4$  be an implication scheme for a [pre-]variety  $\mathbf{P} \subseteq \mathbf{A}_\Sigma$ ,  $\mathfrak{A} \in \mathbf{P}$ ,  $\bar{a}, \bar{b} \in A^2$  and  $\theta \triangleq \text{Cg}_{\mathbf{P}}^{\mathfrak{A}}(\bar{a})$ . Suppose  $\mathfrak{A} \models (\bigwedge \mathcal{U})[x_i/a_i, x_{2+i}/b_i]_{i \in 2}$ . Then,  $\bar{b} \in \theta$ .

*Proof.* As (3.1) is true in  $\mathbf{P} \ni (\mathfrak{A}/\theta) \models (\bigwedge \mathcal{U})[x_i/\nu_\theta(a_i), x_{2+i}/\nu_\theta(b_i)]_{i \in 2}$ , while  $\bar{a} \in \theta = (\ker \nu_\theta)$ , we get  $\bar{b} \in \theta$ .  $\square$

**Corollary 3.4.** *Let  $\mathcal{U} \subseteq \text{Eq}_{\Sigma}^4$  be an implication/REDPC scheme for a [pre-]variety  $\mathbf{P} \subseteq \mathbf{A}_{\Sigma}$ . Then,  $\mathbf{P}_{\mathcal{U}} \subseteq / = (\text{Si}_{[\mathbf{P}]}(\mathbf{P}) \cup \mathbf{A}_{\Sigma}^{-1})$ . In particular, any implicative [pre-]variety is [relatively] both semi-simple and subdirectly representable.*

*Proof.* Consider any non-one-element  $\mathfrak{A} \in \mathbf{P}_{\mathcal{U}}$  and  $\vartheta \in (\text{Co}_{[\mathbf{P}]}(\mathfrak{A}) \setminus \{\Delta_A\})$ , in which case there is some  $\bar{a} \in (\vartheta \setminus \Delta_A) \neq \emptyset$ , and so, for any  $\bar{b} \in A^2$ ,  $\mathfrak{A} \models (\bigwedge \mathcal{U})[x_i/a_i, x_{2+i}/b_i]_{i \in 2}$ . Then, “by Lemma 3.3”/  $\bar{b} \in \vartheta$ , in which case  $\vartheta = A^2$ , and so  $\mathfrak{A}$  is [P-]simple. Conversely, for any  $\mathbf{A} \in \text{Si}_{[\mathbf{P}]}(\mathbf{P})$ ,  $\text{Co}_{[\mathbf{P}]}(\mathfrak{A}) = \{\Delta_A, A^2\}$ , in which case, for all  $\bar{a} \in A^4$ , as  $\langle a_2, a_3 \rangle \in A^2$ , we have  $(\forall \theta \in \text{Co}_{[\mathbf{P}]}(\mathfrak{A}) : (a_0 \theta a_1) \Rightarrow (a_2 \theta a_3)) \Leftrightarrow ((a_0 = a_1) \Rightarrow (a_2 = a_3))$ , and so  $\mathfrak{A}$  is  $\mathcal{U}$ -implicative, whenever  $\mathcal{U}$  is an REDP[R]C scheme for  $\mathbf{P} \ni \mathfrak{A}$ .  $\square$

**Theorem 3.5.** *Any  $\mathcal{U} \subseteq \text{Eq}_{\Sigma}^4$  is an identity congruence implication scheme for a [n equational] pre-variety  $\mathbf{K} \subseteq \mathbf{A}_{\Sigma}$  iff [f] it is an REDPC one.*

*Proof.* The “if” part is immediate. [Conversely, if  $\mathcal{U}$  is an identity congruence implication scheme for  $\mathbf{K}$ , then, by induction on construction of any  $\varphi \in \text{Tm}_{\Sigma}^{\omega}$ , we conclude that  $\mathbf{K}$  satisfies the  $\Sigma$ -identities in  $\mathcal{U}[x_{2+i}/(\varphi[x_0/x_i])]_{i \in 2}$ , in which case, by Mal’cev Lemma [13] (cf. [6, Lemma 2.1]), for any  $\mathfrak{A} \in \mathbf{A}$ ,  $\bar{a} \in A^2$  and  $\bar{b} \in \text{Cg}^{\mathfrak{A}}(\bar{a})$ , we have  $\mathfrak{A} \models (\bigwedge \mathcal{U})[x_i/a_i, x_{2+i}/b_i]_{i \in 2}$ , and so Lemma 3.3 completes the argument].  $\square$

This, by Lemma 3.3 and the Compactness Theorem for ultra-multiplicative classes of algebras (cf., e.g., [14]), immediately yields:

**Corollary 3.6.** *Any quasi-variety with REDPRC scheme  $\mathcal{U}$  has a finite one  $\subseteq \mathcal{U}$ .*

**Theorem 3.7.** *Let  $\mathcal{U} \subseteq \text{Eq}_{\Sigma}^4$ . Then, any [(not necessarily) quasi-equational pre-]variety  $\mathbf{P} \subseteq \mathbf{A}_{\Sigma}$  is  $\mathcal{U}$ -implicative iff it is [relatively (both subdirectly-representable and)] semi-simple with REDP[R]C scheme  $\mathcal{U}$ , in which case  $((\text{SI} | \text{Si})_{[\mathbf{P}]}(\mathbf{P}) \cup \mathbf{A}_{\Sigma}^{-1}) = \mathbf{P}_{\mathcal{U}}$ .*

*Proof.* If  $\mathbf{P}$  is  $\mathcal{U}$ -implicative, that is, is the pre-variety generated by  $\mathbf{P}_{\mathcal{U}}$ , then, for any  $\mathfrak{A} \in \mathbf{P}$  and  $\bar{a} \in A^4$  such that  $\mathfrak{A} \not\models (\bigwedge \mathcal{U})[x_i/a_i]_{i \in 4}$ , by (2.8), there are some  $\mathfrak{B} \in \mathbf{P}_{\mathcal{U}}$  and  $h \in \text{hom}(\mathfrak{A}, \mathfrak{B})$  such that  $\mathfrak{B} \not\models (\bigwedge \mathcal{U})[x_i/h(a_i)]_{i \in 4}$ , that is,  $h(a_{0|2}) = | \neq h(a_{1|3})$ , in which case, by (2.4),  $\langle a_{0|2}, a_{1|3} \rangle \in | \notin (\ker h) \in \text{Co}_{[\mathbf{P}]}(\mathfrak{A})$ , and so Remark 2.4, Lemma 3.3 and Corollary 3.4 complete the argument.  $\square$

### 3.1.1. Implicativity versus filtrality.

**Definition 3.8.** Given any  $n \in \omega$ , a  $\mathcal{U} \subseteq \text{Eq}_{\Sigma}^{2 \cdot (n+1)}$  is called a(n)/an “restricted equationally definable  $n$ -generated [relative] congruence ( $n$ -REDG[R]C) scheme”/ “(equational)  $n$ -multiple{-premise} implicative system” for a “[pre-]variety”/  $\mathbf{K} \subseteq \mathbf{P}_{\Sigma}$ , if for each  $\mathfrak{A} \in \mathbf{K}$  and every  $\bar{a} \in (A^2)^{n+1}$ ,  $(\forall \theta \in (\text{Co}_{[\mathbf{K}]}(\mathfrak{A})/\{\Delta_A\}) : ((\bar{a}|n) \in \theta^n) \Rightarrow (a_n \in \theta)) \Leftrightarrow (\mathfrak{A} \models (\bigwedge \mathcal{U})[x_{i+j}/\pi_j(a_i)]_{i \in (n+1), j \in 2})$ .  $\square$

**Theorem 3.9.** *Any [quasi-]equational/ [pre-]variety  $\mathbf{P} \subseteq \mathbf{A}_{\Sigma}$  is implicative iff it is [relatively] / “both subdirectly-representable and” (subdirectly) / “finitely|principally” filtral.*

*Proof.* First, assume  $\mathbf{P}$  is {both quasi-equational and}  $\mathcal{U}$ -implicative, for some  $\mathcal{U} \subseteq \text{Eq}_{\Sigma}^4$ , in which case, by Corollary 3.4,  $\mathbf{P}$  is [relatively] both subdirectly-representable and semi-simple, while  $\mathbf{K} \triangleq \mathbf{P}_{\mathcal{U}} \supseteq \text{Si}_{[\mathbf{P}]}(\mathbf{P})$  is both abstract and hereditary, whereas  $\mathbf{P} = \text{ISP}\mathbf{K} = \text{IP}^{\text{SD}}\text{SK}$ . Consider any set  $I$ , any  $\bar{\mathfrak{A}} \in (\text{Si} || \text{SI})_{[\mathbf{P}]}(\mathbf{P})^I$ , any subalgebra  $\mathfrak{B}$  of  $\prod \bar{\mathfrak{A}}$ , any  $\{Y \subseteq\} X \subseteq_{\omega} \{Z \subseteq\} B^2$  and any  $\bar{b} \in B^2$ . Let  $I_X \triangleq (I \cap (\bigcap \varepsilon_I^{-1}[X])) \{ \subseteq I_Y \} \subseteq I$ , in which case  $\text{Fi}(I) \ni \mathcal{F}_X \triangleq \wp(I_X, I) \{ \supseteq \mathcal{F}_Y$ , and so  $\mathcal{G}_Z \triangleq \{ \mathcal{F}_W \mid W \in \wp_{\omega}(Z) \} \subseteq \text{Fi}(I)$  is upward-directed. Then,  $\text{Fi}(I)$ , being inductive, contains  $\mathcal{H}_Z \triangleq (\bigcup \mathcal{G}_Z)$ . Take any bijection  $\bar{a}$  from  $m \triangleq |X| \in \omega$

onto  $X$ . By induction on any  $n \in \omega$ , define  $\mathcal{U}_n \subseteq \text{Eq}_\Sigma^{2 \cdot (n+1)}$  by  $\mathcal{U}_0 \triangleq \{x_0 \approx x_1\}$  and  $\mathcal{U}_{n+1} \triangleq (\bigcup \{\mathcal{U}[x_{2+i}/\varphi_i]_{i \in 2} \mid \bar{\varphi} \in (\mathcal{U}_n[x_j/x_{j+2}]_{j \in (2 \cdot (n+1))}\})$ , in which case  $\mathcal{U}_n$  is an  $n$ -multiple implicative system for  $\mathbf{K}$ , and so, by (2.4), (2.8) and Remark 2.2, is an  $n$ -REDG[R]C scheme for  $\mathbf{P}$ . Then, as  $(\text{img } \prod \mathfrak{A}) \subseteq \text{Si}_{|\mathbf{P}|}(\mathbf{P}) \subseteq \mathbf{K}$ ,  $(\bar{b} \in \text{Cg}_{|\mathbf{P}|}^{\mathfrak{B}}(X)) \Leftrightarrow (\mathfrak{B} \models (\bigwedge \mathcal{U}_m)[x_{j+k}/\pi_k(a_j); x_{(2 \cdot m)+l}/b_l]_{j \in m; k, l \in 2}) \Leftrightarrow (\forall i \in I : \mathfrak{A}_i \models (\bigwedge \mathcal{U}_m)[x_{j+k}/\pi_i(\pi_k(a_j)); x_{(2 \cdot m)+l}/\pi_i(b_l)]_{j \in m; k, l \in 2}) \Leftrightarrow (\forall i \in I : (i \in I_X) \Rightarrow (i \in \varepsilon_I(\bar{b}))) \Leftrightarrow (I_X \subseteq \varepsilon_I(\bar{b})) \Leftrightarrow (\varepsilon_I(\bar{b}) \in \mathcal{F}_X) \Leftrightarrow (\bar{b} \in \theta_{\mathcal{F}_X}^{\mathfrak{B}})$ , in which case  $\text{Cg}_{|\mathbf{P}|}^{\mathfrak{B}}(X) = \theta_{\mathcal{F}_X}^{\mathfrak{B}}$  {and so, by Remark 2.3 and (2.11),  $\text{Cg}_{|\mathbf{P}|}^{\mathfrak{B}}(Z) = \theta_{\mathcal{F}_Z}^{\mathfrak{B}}$ }.

Conversely, assume  $\mathbf{P}$  is [relatively] /“both subdirectly-representable and” subdirectly principally filtral, in which case, by “Remark 2.4 as well as”/ Footnote 1, Lemma 2.5 and Corollary 2.10,  $\mathbf{P}$  is [relatively] both subdirectly-representable and semi-simple with abstract and non-trivially-hereditary  $\mathbf{K} \triangleq (\text{Si} \parallel \text{SI})_{|\mathbf{P}|}(\mathbf{P})$ . Let  $I \triangleq \{\theta \in \text{Co}_{\mathbf{K}}(\mathfrak{Tm}_\Sigma^4) \mid (x_0 \theta x_1) \Rightarrow (x_2 \theta x_3)\}$ ,  $\mathfrak{A} \triangleq \langle \mathfrak{A}/i \rangle_{i \in I} \in \mathbf{K}^I$ ,  $\mathfrak{D} \triangleq (\prod \mathfrak{A})$ ,  $h \triangleq (\prod_{i \in I} \nu_i)$  and  $\bar{a} \triangleq \langle h(v_j) \rangle_{j \in 4}$ , in which case, by (2.2) and (2.7),  $h \in \text{hom}(\mathfrak{Tm}_\Sigma^4, \mathfrak{D})$ , while  $\mathfrak{B} \triangleq (\mathfrak{D} \upharpoonright (\text{img } h))$  is a subdirect product of  $\mathfrak{A}$ , whereas  $h \in \text{hom}^S(\mathfrak{Tm}_\Sigma^4, \mathfrak{B})$ , and so  $\vartheta \triangleq \text{Cg}_{|\mathbf{P}|}^{\mathfrak{B}}(\langle a_0, a_1 \rangle) = \theta_{\mathcal{F}}^{\mathfrak{B}}$ , for some  $\mathcal{F} \in \text{Fi}(I)$ . Then,  $\langle a_0, a_1 \rangle \in \vartheta$ , in which case  $\varepsilon_I(\langle a_2, a_3 \rangle) \supseteq \varepsilon_I(\langle a_0, a_1 \rangle) \in \mathcal{F}$ , and so  $\varepsilon_I(\langle a_2, a_3 \rangle) \in \mathcal{F}$ , i.e.,  $\langle a_2, a_3 \rangle \in \vartheta$ . Let  $\mathcal{U} \triangleq (\ker h) \subseteq \text{Eq}_\Sigma^4$ . Consider any  $\mathfrak{C} \in \mathbf{K}$  and  $g \in \text{hom}(\mathfrak{Tm}_\Sigma^4, \mathfrak{C})$ . Then, providing  $\mathcal{U} \subseteq \eta \triangleq (\ker g) \ni \langle x_0, x_1 \rangle$ , by the Homomorphism Theorem,  $f \triangleq (h^{-1} \circ g) \in \text{hom}(\mathfrak{B}, \mathfrak{C})$ , in which case, by (2.5),  $\langle a_0, a_1 \rangle \in \zeta \triangleq (\ker f) = f_*^{-1}[\Delta_C] \in \text{Co}_{|\mathbf{P}|}(\mathfrak{B})$ , and so  $\langle a_2, a_3 \rangle \in \vartheta \subseteq \zeta$ . In that case,  $\langle x_2, x_3 \rangle \in \eta$ . Now, assume  $(\langle x_0, x_1 \rangle \in \eta) \Rightarrow (\langle x_2, x_3 \rangle \in \eta)$ , in which case  $\mathcal{U} \subseteq \eta$ , i.e.,  $\mathfrak{C} \models (\bigwedge \mathcal{U})[g]$ , whenever  $\eta = \text{Eq}_\Sigma^4$ . Otherwise, by the  $\{\}$ -optional version of the right alternative of (2.4),  $\eta \in I$ , in which case, by (2.1),  $\mathcal{U} \subseteq \eta$ , i.e.,  $\mathfrak{C} \models (\bigwedge \mathcal{U})[g]$ , and so  $\mathcal{U}$  is an implicative system for  $\mathbf{K}$ . Thus,  $\mathbf{P}$ , being [relatively] subdirectly-representable, is  $\mathcal{U}$ -implicative.  $\square$

**Corollary 3.10.** *Any finitely implicative pre-variety is relatively both subdirectly-representable and filtral.*

*Proof.* Any implicative system  $\mathcal{U} \subseteq_\omega \text{Eq}_\Sigma^4$  for any  $\mathbf{K} \subseteq \mathbf{A}_\Sigma$  is so for  $\mathbf{P}^{\mathbf{U}}\mathbf{K}$ , in which case  $\mathbf{ISPP}^{\mathbf{U}}\mathbf{K} \supseteq \mathbf{ISPK}$  is  $\mathcal{U}$ -implicative, and so Theorem 3.9 ends the proof.  $\square$

Whether the converse holds remains an open problem.

**3.1.2. Generic identity equivalence implication schemes for distributive lattice expansions.** Here, it is supposed that  $\Sigma_+ \subseteq \Sigma$ . Given any  $\mathfrak{A} \in \mathbf{A}_\Sigma$ ,  $X \subseteq A$  and  $\Omega \subseteq \text{Tm}_\Sigma^1$ , we have  $\Omega_X^{\mathfrak{A}} : A \rightarrow \wp(\Omega)$ ,  $a \mapsto \{\varphi \in \Omega \mid \varphi^{\mathfrak{A}}(a) \in X\}$ .

Given any  $\bar{\varphi} \in (\text{Tm}_\Sigma^1)^*$  with  $x_0 \in \Xi \triangleq (\text{img } \bar{\varphi})$ ,  $\iota \in \Omega \in \wp(V_1, \Xi)$ ,  $i \in 2$  and  $\Delta \in \wp(\Xi)$ , let  $\varepsilon_{\bar{\varphi}, \Delta}^{i, \iota} \triangleq ((\wedge_+ \langle (\bar{\varphi} \cap \Delta) * ((\bar{\varphi} \cap \Delta) \circ [x_0/x_1]), \iota(x_{2+i}) \rangle) \lesssim (\vee_+ \langle (\bar{\varphi} \setminus \Delta) * ((\bar{\varphi} \setminus \Delta) \circ [x_0/x_1]), \iota(x_{3-i}) \rangle)) \in \text{Eq}_\Sigma^4$  and  $\mathcal{U}_\Omega^{\bar{\varphi}} \triangleq \{\varepsilon_{\bar{\varphi}, \Delta}^{i, \iota} \mid i \in 2, \iota \in \Omega, \Delta \in \wp(\Xi)\} \in \wp_\omega(\text{Eq}_\Sigma^4)$ .

**Lemma 3.11.** *Let  $\mathfrak{A}$  be a  $\Sigma$ -algebra with (distributive) lattice  $\Sigma_+$ -reduct,  $\bar{\varphi} \in (\text{Tm}_\Sigma^1)^*$  with  $x_0 \in \Xi \triangleq (\text{img } \bar{\varphi})$  and  $\Omega \in \wp(V_1, \Xi)$ . Then,  $\mathcal{U}_\Omega^{\bar{\varphi}}$  is an identity reflexive symmetric (transitive implication) scheme for  $\mathfrak{A}$ .*

*Proof.* Clearly, for all  $j \in 2$ ,  $\iota \in \Xi$  and  $\Delta \in \wp(\Xi)$ , there are some  $\phi, \psi, \xi \in \text{Tm}_\Sigma^3$  such that  $(\varepsilon_{\bar{\varphi}, \Delta}^{j, \iota}[x_3/x_2]) = ((\phi \wedge \xi) \lesssim (\psi \vee \xi))$ , in which case this is satisfied in lattice  $\Sigma$ -expansions, and so in  $\mathfrak{A}$ . Likewise, there are then some  $\bar{\eta}, \bar{\zeta} \in (\text{Tm}_\Sigma^2)^+$  with  $((\text{img } \bar{\eta}) \cap (\text{img } \bar{\zeta})) \neq \emptyset$  such that  $(\varepsilon_{\bar{\varphi}, \Delta}^{j, \iota}[x_{2+i}/x_i]_{i \in 2}) = ((\wedge_+ \bar{\eta}) \lesssim (\vee_+ \bar{\zeta}))$ , in which case this is satisfied in lattice  $\Sigma$ -expansions, and so in  $\mathfrak{A}$ . Furthermore,  $(\mathcal{U}_\Omega^{\bar{\varphi}}[x_2/x_3, x_3/x_2]) = \mathcal{U}_\Omega^{\bar{\varphi}}$ . (Next, since the  $\Sigma_+$ -quasi-identity  $\{(x_0 \wedge x_1) \lesssim (x_2 \vee$

$x_3), (x_0 \wedge x_3) \lesssim (x_2 \vee x_4)\} \rightarrow ((x_0 \wedge x_1) \lesssim (x_2 \vee x_4))$ , being satisfied in distributive lattices, is so in  $\mathfrak{A}$ , so are logical consequences of its substitutional  $\Sigma$ -instances  $(\mathcal{U}_\Omega^\varphi \cup (\mathcal{U}_\Omega^\varphi[x_{2+i}/x_{3+i}]_{i \in 2})) \rightarrow \Psi$ , where  $\Psi \in (\mathcal{U}_\Omega^\varphi[x_3/x_4])$ . Finally, consider any  $a \in A$  and  $\bar{b} \in (A^2 \setminus \Delta_A)$ , in which case, by the Prime Ideal Theorem, there are some  $k \in 2$  and some prime filter  $F$  of  $\mathfrak{A}$  such that  $b_k \in F \not\approx b_{1-k}$ , and so, as  $\Delta \triangleq \Xi_F^\mathfrak{A}(a) \in \wp(\Xi)$  and  $x_0 \in \Omega$ ,  $\mathfrak{A} \not\models (\bigwedge \mathcal{U}_\Omega^\varphi)[x_i/a, x_{2+i}/b_i]_{i \in 2}$ , for  $\mathfrak{A} \not\models \varepsilon_{\bar{\varphi}, \Delta}^{k, x_0}[x_i/a, x_{2+i}/b_i]_{i \in 2}$ .  $\square$

This, by Corollary 3.4, immediately yields:

**Corollary 3.12.** *Let  $\mathfrak{A}$  be a non-one-element  $\Sigma$ -algebra with distributive lattice  $\Sigma_+$ -reduct,  $\bar{\varphi} \in (\text{Tm}_\Sigma^1)^*$  with  $x_0 \in \Xi \triangleq (\text{img } \bar{\varphi})$  and  $\Omega \in \wp(V_1, \Xi)$ . Suppose  $\mathcal{U}_\Omega^\varphi$  is an implicative system for  $\mathfrak{A}$ . Then,  $\mathfrak{A}$  is simple.*

3.1.2.1. Equality determinants versus implicativity. Recall that a (logical)  $\Sigma$ -matrix is any pair  $\mathcal{A} = \langle \mathfrak{A}, D \rangle$  with a  $\Sigma$ -algebra  $\mathfrak{A}$  and a  $D \subseteq A$ , in which case an  $\Omega \subseteq \text{Tm}_\Sigma^1$  is called an equality/identity determinant for  $\mathcal{A}$ , if  $\Omega_D^\mathfrak{A}$  is injective (cf. [19]), and so one for a class  $\mathbf{M}$  of  $\Sigma$ -matrices, if it is so for each member of  $\mathbf{M}$ .

**Theorem 3.13.** *Let  $\mathbf{M}$  be a class of  $\Sigma$ -matrices and  $\bar{\varphi} \in (\text{Tm}_\Sigma^1)^*$  with  $x_0 \in \Xi \triangleq (\text{img } \bar{\varphi})$ . Suppose, for all  $\mathcal{A} \in \mathbf{M}$ ,  $\pi_0(\mathcal{A})|_{\Sigma_+}$  is a distributive lattice with set of its prime filters  $\pi_1[\mathbf{M} \cap \pi_0^{-1}[\{\pi_0(\mathcal{A})\}]]$ . Then,  $\Xi$  is an equality determinant for  $\mathbf{M}$  iff  $\mathcal{U}_{V_1}^\varphi$  is an implicative system for  $(\mathbf{IS}_{>1}\{\mathbf{P}^U\})\pi_0[\mathbf{M}]$  (in which case its members are simple).*

*Proof.* Let  $\mathcal{A} = \langle \mathfrak{A}, D \rangle \in \mathbf{M}$ ,  $\bar{a} \in A^2$  and, for any  $\bar{b} \in A^2$ ,  $h_{\bar{b}} \triangleq [x_i/a_i, x_{2+i}/b_i]_{i \in 2}$ . First, assume  $\Xi$  is an equality determinant for  $\mathbf{M}$ . Consider any  $\bar{b} \in A^2$ . Assume  $\mathfrak{A} \not\models \varepsilon_{\bar{\varphi}, \Delta}^{j, x_0}[h_{\bar{b}}]$ , for some  $j \in 2$  and  $\Delta \subseteq \Xi$ , in which case, by the Prime Ideal Theorem,  $\exists \mathcal{B} = \langle \mathfrak{A}, D' \rangle \in \mathbf{M} : \forall k \in 2 : \Delta = \Xi_{D'}^\mathfrak{A}(a_k)$ , and so  $a_0 = a_1$ . Then, by Lemma 3.11 with  $\Omega = \Xi$ ,  $\mathcal{U}_{V_1}^\varphi$  is an implicative system for  $\mathfrak{A}$ . Conversely, assume  $\mathcal{U}_{V_1}^\varphi$  is an implicative system for  $\mathfrak{A}$  and  $\Delta \triangleq \Xi_D^\mathfrak{A}(a_0) = \Xi_D^\mathfrak{A}(a_1)$ . Take any  $\bar{b} \in (D \times (A \setminus D)) \neq \emptyset$ , in which case, as  $\Delta \subseteq \Xi \ni x_0$ ,  $\mathfrak{A} \not\models \varepsilon_{\bar{\varphi}, \Delta}^{0, x_0}[h_{\bar{b}}]$ , for  $D$  is a prime filter of  $\mathfrak{A}|_{\Sigma_+}$ , and so  $a_0 = a_1$ . (Finally, Corollary 3.12 completes the argument.)  $\square$

3.2. **Disjunctivity.** Unless otherwise specified, fix any  $\mathcal{U} \subseteq \text{Eq}_\Sigma^4$ .

3.2.1. *Disjunctivity versus finite subdirect irreducibility and congruence-distributivity.*

**Lemma 3.14.** *Any  $\mathcal{U}$ -disjunctive /finite non-one-element  $\mathfrak{A} \in \mathbf{A}_\Sigma$  is finitely/ subdirectly-irreducible. In particular, any disjunctive pre-variety is (relatively) finitely subdirectly-representable.*

*Proof.* Consider any  $\theta, \vartheta \in (\text{Co}(\mathfrak{A}) \setminus \{\Delta_A\})$  and take any  $(\bar{a}|\bar{b}) \in ((\theta|\vartheta) \setminus \{\Delta_A\}) \neq \emptyset$ , in which case the  $\Sigma$ -identities in  $\mathcal{U}[x_{1|3}/x_{0|2}]$ , being true in  $\mathfrak{A}$ , are so in  $\mathfrak{A}/(\theta|\vartheta)$  (in particular, under  $[x_{0|2}/\nu_{\theta|\vartheta}((a|b)_0), x_{(2|0)+i}/\nu_{\theta|\vartheta}((b|a)_i)]_{i \in 2}$ ), and so  $\Delta_A \not\subseteq \{\langle \phi^\mathfrak{A}[x_i/a_i, x_{2+i}/b_i]_{i \in 2}, \phi^\mathfrak{A}[x_i/a_i, x_{2+i}/b_i]_{i \in 2} \mid (\phi \approx \psi) \in \mathcal{U}\} \subseteq (\theta \cap \vartheta)$ . Then,  $(\theta \cap \vartheta) \neq \Delta_A$ . Thus, induction on the cardinality of finite subsets of  $\text{Co}(\mathfrak{A})$  ends the proof.  $\square$

Given any  $\mathfrak{A} \in \mathbf{A}_\Sigma$ , let  $\mathcal{U}^\mathfrak{A} : \wp(A)^2 \rightarrow \wp(A), \langle X, Y \rangle \mapsto \{\langle \phi^\mathfrak{A}[x_i/a_i, x_{2+i}/b_i]_{i \in 2}, \phi^\mathfrak{A}[x_i/a_i, x_{2+i}/b_i]_{i \in 2} \mid \bar{\phi} \in \mathcal{U}, \bar{a} \in X, \bar{b} \in Y \}$ .

**Lemma 3.15.** *Let  $\mathbf{P} \subseteq \mathbf{A}_\Sigma \ni \mathfrak{A}$  be a  $\mathcal{U}$ -disjunctive pre-variety and  $X, Y, Z \subseteq A^2$ . Then,  $\text{Cg}_\mathbf{P}^\mathfrak{A}(\mathcal{U}^\mathfrak{A}(X, Y) \cup Z) = (\text{Cg}_\mathbf{P}^\mathfrak{A}(X \cup Z) \cap \text{Cg}_\mathbf{P}^\mathfrak{A}(Y \cup Z))$ .*

*Proof.* In that case,  $\mathbf{P}$  is generated by  $\mathbf{K} \triangleq \mathbf{P}_{\mathcal{U}} = \mathbf{ISK}$ , so, by Remark 2.2 and (2.8),  $\text{Co}_{\mathbf{K}}(\mathfrak{A})$  is a basis of  $\text{Co}_{\mathbf{P}}(\mathfrak{A})$ . Then, for any  $\theta \in \text{Co}_{\mathbf{K}}(\mathfrak{A})$ ,  $\mathfrak{A}/\theta$  is  $\mathcal{U}$ -disjunctive, in which case  $(\mathcal{U}^{\mathfrak{A}}(X, Y) \cup Z) \subseteq \theta$  iff either  $(X \cup Z) \subseteq \theta$  or  $(Y \cup Z) \subseteq \theta$ , and so, for any  $\bar{a} \in A^2$ ,  $(\bar{a} \in \text{Cg}_{\mathbf{P}}^{\mathfrak{A}}(\mathcal{U}^{\mathfrak{A}}(X, Y) \cup Z)) \Leftrightarrow (\forall \theta \in \text{Co}_{\mathbf{K}}(\mathfrak{A}) : ((\mathcal{U}^{\mathfrak{A}}(X, Y) \cup Z) \subseteq \theta) \Rightarrow (\bar{a} \in \theta)) \Leftrightarrow ((\forall \theta \in \text{Co}_{\mathbf{K}}(\mathfrak{A}) : (X \cup Z) \subseteq \theta) \Rightarrow (\bar{a} \in \theta)) \& (\forall \theta \in \text{Co}_{\mathbf{K}}(\mathfrak{A}) : ((Y \cup Z) \subseteq \theta) \Rightarrow (\bar{a} \in \theta)) \Leftrightarrow (\bar{a} \in (\text{Cg}_{\mathbf{P}}^{\mathfrak{A}}(X \cup Z) \cap \text{Cg}_{\mathbf{P}}^{\mathfrak{A}}(Y \cup Z)))$ , as required.  $\square$

**Corollary 3.16.** *Any  $\mathcal{U}$ -disjunctive [pre-]variety  $\mathbf{P} \subseteq \mathbf{A}_{\Sigma}$  is [relatively] congruence-distributive, and so is any [quasi-equational/finitely] implicative one.*

*Proof.* Then, by Lemma 3.15, for any  $\mathfrak{A} \in \mathbf{P}$  and  $\theta, \vartheta, \eta \in \text{Co}_{\mathbf{P}}(\mathfrak{A})$ , we have  $(\text{Cg}_{\mathbf{P}}^{\mathfrak{A}}(\theta \cup \eta) \cap \text{Cg}_{\mathbf{P}}^{\mathfrak{A}}(\vartheta \cup \eta)) = \text{Cg}_{\mathbf{P}}^{\mathfrak{A}}(\mathcal{U}^{\mathfrak{A}}(\theta, \vartheta) \cup \eta) = \text{Cg}_{\mathbf{P}}^{\mathfrak{A}}(\text{Cg}_{\mathbf{P}}^{\mathfrak{A}}(\mathcal{U}^{\mathfrak{A}}(\theta, \vartheta)) \cup \eta) = \text{Cg}_{\mathbf{P}}^{\mathfrak{A}}((\text{Cg}_{\mathbf{P}}^{\mathfrak{A}}(\theta) \cap \text{Cg}_{\mathbf{P}}^{\mathfrak{A}}(\vartheta)) \cup \eta) = \text{Cg}_{\mathbf{P}}^{\mathfrak{A}}((\theta \cap \vartheta) \cup \eta)$ , as required.  $\square$

**Lemma 3.17.** *Let  $\mathbf{P} \subseteq \mathbf{A}_{\Sigma}$  be a  $\mathcal{U}$ -implicative pre-variety and  $\mathcal{U}'$  a disjunctive system for  $\mathbf{P}_{\mathcal{U}}$ . Then, every  $\mathcal{U}'$ -disjunctive member of  $\mathbf{P}$  is  $\mathcal{U}$ -implicative.*

*Proof.* In that case,  $\mathcal{U}$ , being is an identity implication scheme for  $\mathbf{P}_{\mathcal{U}}$ , is so for  $\mathbf{P} = \mathbf{ISPP}_{\mathcal{U}}$ , while the  $\Sigma$ -identities in  $\bigcup\{\mathcal{U}'[x_{2+i}/\varphi_i]_{i \in 2} \mid \bar{\varphi} \in \mathcal{U}\}$ , being true in  $\mathbf{P}_{\mathcal{U}}$ , are so in  $\mathbf{P}$ , and so  $\mathcal{U}'$ -disjunctive members of  $\mathbf{P}$  are  $\mathcal{U}$ -implicative, as required.  $\square$

**Corollary 3.18.** *For any  $\mathcal{U}$ -disjunctive [pre-]variety  $\mathbf{P} \subseteq \mathbf{A}_{\Sigma}$ ,  $\mathbf{P}_{\mathcal{U}} = (\text{SI}_{[\mathbf{P}]}^{\omega}(\mathbf{P}) \cup \mathbf{A}_{\Sigma}^{-1})$ . In particular, any [quasi-equational/finitely] implicative [pre-]variety is [relatively] finitely semi-simple.*

*Proof.* Then, any one-element  $\Sigma$ -algebra is  $\mathcal{U}$ -disjunctive, while, for any  $\mathfrak{A} \in \text{SI}_{[\mathbf{P}]}^{\omega}(\mathbf{P})$  and  $\bar{a}, \bar{b} \in (A^2 \setminus \Delta_A)$ , since  $\text{Cg}_{[\mathbf{P}]}^{\mathfrak{A}}(\bar{a}|\bar{b}) \in (\text{Co}_{[\mathbf{P}]}(\mathfrak{A}) \setminus \{\Delta_A\})$ , whereas, by Lemma 3.15,  $(\text{Cg}_{[\mathbf{P}]}^{\mathfrak{A}}(\bar{a}) \cap \text{Cg}_{[\mathbf{P}]}^{\mathfrak{A}}(\bar{b})) = \text{Cg}_{[\mathbf{P}]}^{\mathfrak{A}}(\mathcal{U}^{\mathfrak{A}}(\bar{a}|\bar{b}))$ , we have  $\mathcal{U}^{\mathfrak{A}}(\bar{a}|\bar{b}) \neq \Delta_A = \text{Cg}_{[\mathbf{P}]}^{\mathfrak{A}}(\Delta_A)$ , i.e.,  $\mathfrak{A} \not\models (\bigwedge \mathcal{U})[x_i/a_i, x_{2+i}/b_i]_{i \in 2}$ , in which case  $\mathfrak{A}$  is  $\mathcal{U}$ -disjunctive, because the  $\Sigma$ -identities in  $\bigcup_{j \in 2} \mathcal{U}[x_{(2 \cdot j)}/x_{(2 \cdot j)+1}]$ , being true in  $\mathbf{P}_{\mathcal{U}}$ , are so in  $\mathbf{ISPP}_{\mathcal{U}} = \mathbf{P} \ni \mathfrak{A}$ , and so Lemmas 3.4, 3.14, 3.17 and [20, Remark 2.4] complete the argument.  $\square$

**Theorem 3.19.** *Any [pre-]variety  $\mathbf{P} \subseteq \mathbf{A}_{\Sigma}$  is disjunctive iff it is [relatively both] congruence-distributive [and finitely-subdirectly-representable] with  $\text{SI}_{[\mathbf{P}]}^{\omega}(\mathbf{P}) \cup \mathbf{A}_{\Sigma}^{-1}$  being “a universal (infinitary) model class”/hereditary.*

*Proof.* The “only if” part is by Lemma 3.2.1 and Corollary 3.18. Conversely, assume  $\mathbf{P}$  is [relatively both] congruence-distributive [and finitely-subdirectly-representable] with hereditary  $\text{SI}_{[\mathbf{P}]}^{\omega}(\mathbf{P}) \cup \mathbf{A}_{\Sigma}^{-1}$ , in which case, by Remark 2.4, it is [relatively] finitely-subdirectly-representable, while, by (2.5),  $\text{Co}_{[\mathbf{P}]}(\mathfrak{I}\mathfrak{m}_{\Sigma}^4) \cap \wp(\theta, \text{Eq}_{\Sigma}^4)$ , where  $\theta \triangleq (\text{Eq}_{\Sigma}^4 \cap (\bigcap \text{Co}_{\text{SI}_{[\mathbf{P}]}^{\omega}(\mathbf{P})}(\mathfrak{I}\mathfrak{m}_{\Sigma}^4))) \in \text{Co}_{[\mathbf{P}]}(\mathfrak{I}\mathfrak{m}_{\Sigma}^4)$ , is distributive, for  $\text{Co}_{[\mathbf{P}]}(\mathfrak{I}\mathfrak{m}_{\Sigma}^4/\theta)$  is so. Let  $\forall j \in 2 : \vartheta_j \triangleq \text{Cg}_{[\mathbf{P}]}^{\mathfrak{I}\mathfrak{m}_{\Sigma}^4}(\theta \cup \{(x_{2 \cdot j}, x_{(2 \cdot j)+1})\}) \in (\text{Co}_{[\mathbf{P}]}(\mathfrak{I}\mathfrak{m}_{\Sigma}^4) \cap \wp(\theta, \text{Eq}_{\Sigma}^4)) \ni \mathcal{U} \triangleq (\vartheta_0 \cap \vartheta_1) \subseteq \text{Eq}_{\Sigma}^4$ . Consider any  $\mathfrak{A} \in \text{SI}_{[\mathbf{P}]}^{\omega}(\mathbf{P})$  and any  $\bar{a} \in A^4$ . Let  $h \in \text{hom}(\mathfrak{I}\mathfrak{m}_{\Sigma}^4, \mathfrak{A})$  extend  $\{(x_i, a_i) \mid i \in 4\}$ , in which case  $\mathfrak{B} \triangleq (\mathfrak{A} \upharpoonright (\text{img } h)) \in (\text{SI}_{[\mathbf{P}]}^{\omega}(\mathbf{P}) \cup \mathbf{A}_{\Sigma}^{-1})$ , and so  $(\{(a_0, a_1), (a_2, a_3)\} \cap \Delta_A) \neq \emptyset$  &  $\Leftrightarrow (\mathfrak{A} \models \Phi_{\mathcal{U}}^4[h \upharpoonright V_4])$ , unless  $\mathfrak{B} \in \text{SI}_{[\mathbf{P}]}^{\omega}(\mathbf{P})$ . Otherwise, by (2.5) and the Homomorphism Theorem,  $\theta \subseteq \eta \triangleq (\ker h) \in \text{MI}^{\omega}(\text{Co}_{[\mathbf{P}]}(\mathfrak{I}\mathfrak{m}_{\Sigma}^4))$ , in which case we have:

$$\begin{aligned} (\mathfrak{A} \models \Phi_{\mathcal{U}}^4[h \upharpoonright V_4]) &\Leftrightarrow ((\vartheta_0 \cap \vartheta_1) = \mathcal{U} \subseteq \eta) \Leftrightarrow (\eta = \text{Cg}_{[\mathbf{P}]}^{\mathfrak{I}\mathfrak{m}_{\Sigma}^4}(\eta \cup (\vartheta_0 \cap \vartheta_1))) = \\ &(\text{Cg}_{[\mathbf{P}]}^{\mathfrak{I}\mathfrak{m}_{\Sigma}^4}(\eta \cup \vartheta_0) \cap \text{Cg}_{[\mathbf{P}]}^{\mathfrak{I}\mathfrak{m}_{\Sigma}^4}(\eta \cup \vartheta_1)) \Leftrightarrow (\exists j \in 2 : \eta = \text{Cg}_{[\mathbf{P}]}^{\mathfrak{I}\mathfrak{m}_{\Sigma}^4}(\eta \cup \vartheta_j)) \Leftrightarrow \\ &(\exists j \in 2 : \vartheta_j \subseteq \eta) \Leftrightarrow (\exists j \in 2 : (x_{2 \cdot j}, x_{(2 \cdot j)+1}) \in \eta) \Leftrightarrow (\exists j \in 2 : a_{2 \cdot j} = a_{(2 \cdot j)+1}), \end{aligned}$$

and so  $\mathcal{U}$  is a disjunctive system for  $\text{SI}_{[\mathbf{P}]}^{\omega}(\mathbf{P})$ . Thus,  $\mathbf{P}$ , being [relatively] finitely-subdirectly-representable, is  $\mathcal{U}$ -disjunctive, as required.  $\square$

This, by Remark 2.4 and Corollary 3.18 (as well as the Compactness Theorem for ultra-multiplicative classes; cf., e.g., [14]), immediately yields:

**Corollary 3.20.** *Any [quasi-]variety  $\mathbf{Q} \subseteq \mathbf{A}_\Sigma$  is (finitely) disjunctive iff it is [relatively] congruence-distributive with  $\text{SI}_{[\mathbf{Q}]}^\omega(\mathbf{Q}) \cup \mathbf{A}_\Sigma^{-1}$  being “a universal (first-order) model class”/“hereditary (and ultra-multiplicative)”.*

This, in its turn, by Footnote 1, Corollary 2.11 and Lemma 2.12, immediately yields:

**Corollary 3.21.** *Any locally-finite [quasi-]variety  $\mathbf{Q} \subseteq \mathbf{A}_\Sigma$  is (finitely) disjunctive iff it is [relatively] congruence-distributive with  $\text{SI}_{[\mathbf{Q}]}^\omega(\mathbf{Q}) \cup \mathbf{A}_\Sigma^{-1}$  being “a universal {infinitary} model class”/hereditary.*

Finally, this, by the congruence-distributivity of lattice expansions (cf., e.g., [16]) and Corollary 2.7, immediately yields:

**Corollary 3.22.** *Suppose  $\Sigma_+ \subseteq \Sigma$ . Then, any finitely-generated variety  $\mathbf{V} \subseteq \mathbf{A}_\Sigma$  of lattice expansions with non-trivially-hereditary  $\text{SI}^{(\omega)}(\mathbf{V})$  is finitely disjunctive.*

This provides an immediate (though far from being constructive) insight into the finite disjunctivity of the finitely-generated variety of distributive/Stone|“De Morgan” lattices/algebras|lattices, a constructive one being given by [18, Example 1/2] and [19, Lemma 11].

3.2.1.1. Implicativity versus finite semi-simplicity and disjunctivity. By Footnote 1, Theorem 3.9, Corollaries 2.8, 2.11, 2.13, 3.18, 3.20, 3.21, Lemma 2.12 and [20, Remark 2.4], we eventually get:

**Theorem 3.23.** *Any locally-finite/ [quasi-]variety  $\mathbf{Q} \subseteq \mathbf{A}_\Sigma$  is implicative iff it is /finitely both disjunctive and [relatively] semi-simple iff it is [relatively] both congruence-distributive and semi-simple with  $\text{Si}_{[\mathbf{Q}]}(\mathbf{Q}) \cup \mathbf{A}_\Sigma^{-1}$  being “a universal /first-order model class”|“hereditary /“and ultra-multiplicative”.*

This, by the congruence-distributivity of lattice expansions (cf., e.g., [16]), Corollaries 2.7, 3.4 and Footnote 1, immediately yields:

**Corollary 3.24.** *Suppose  $\Sigma_+ \subseteq \Sigma$ . Then, any locally-finite variety  $\mathbf{V} \subseteq \mathbf{A}_\Sigma$  of lattice expansions is implicative iff it is semi-simple “and (finitely) disjunctive”|“with non-trivially-hereditary  $(\text{Si} | \text{SI})(\mathbf{V})$ ”.*

**Corollary 3.25.** *Suppose  $\Sigma_+ \subseteq \Sigma$ . Let  $\mathbf{K} \subseteq \mathbf{A}_\Sigma$  be a finite set of finite lattice expansions without non-simple non-one-element subalgebras and  $\mathbf{V}$  the variety generated by  $\mathbf{K}$ . Then,  $\mathbf{V}$  is implicative with  $(\text{Si} | \text{SI})(\mathbf{V}) = \mathbf{IS}_{>1}\mathbf{K}$ .*

These provide an immediate /{though far from being constructive} insight into the not/ implicativity of (and so not/ REDPC for; cf. Theorem 3.7) the not/ semi-simple finitely-generated variety of Stone/distributive|“De Morgan” algebras/lattices|algebras|lattices / (cf. [8][21]) / “a constructive one being given by Theorem 3.13 and [18, Example 1] “Remark 4.3”.

Whether the /-alternative stipulations are necessary in Theorem 3.23 remains an open issue. On the other hand, the necessity of the “[relative] congruence-distributivity”//“lattice expansion” stipulation therein// as well as in Corollaries 3.20, 3.21, 3.22, 3.24, 3.25 and Theorem 3.19 is demonstrated by:

**Example 3.26.** Let  $\Sigma = \{\wedge\}$  and  $\text{SL}$  the variety of semi-lattices, in which case, for any filter  $F \neq A$  of any  $\mathfrak{A} \in \text{SL}$ ,  $\chi_A^F$  is a surjective homomorphism from  $\mathfrak{A}$  onto  $\mathfrak{S}_2 \in \text{SL}$  with  $S_2 \triangleq 2$  and  $\wedge^{\mathfrak{A}} \triangleq (\cap | 2^2)$ , and so, by (2.8),  $\text{SL} = \mathbf{IP}^{\text{SD}}\mathfrak{S}_2$ . Now, assume  $|A| > 2$ , in which case, providing  $\mathfrak{A}$  is a chain, for any  $\bar{a} \in A^3$  with  $|\text{img } \bar{a}| = 3$  such that  $a_0 \leq^{\mathfrak{A}} a_1 \leq^{\mathfrak{A}} a_2$  and  $i \in 2$ ,  $\Delta_A \neq \theta_i \triangleq ([a_i, a_{i+1}]_{\mathfrak{A}}^2 \cup$

$\Delta_A) = \text{Cg}^{\mathfrak{A}}(\{\langle a_i, a_{i+1} \rangle\}) \in \text{Co}(\mathfrak{A})$ , while  $(\theta_0 \cap \theta_1) = \Delta_A$ , and so  $\mathfrak{A}$  is not finitely-sibdirectly-irreducible. Otherwise, take any  $\bar{b} \in A^2$  such that  $c \triangleq (b_0 \wedge^{\mathfrak{A}} b_1) \notin (\text{img } \bar{b})$ , in which case, for each  $j \in 2$ ,  $\vartheta_j \triangleq ((\bigcup\{[c \wedge^{\mathfrak{A}} d, b_j \wedge^{\mathfrak{A}} d]_{\mathfrak{A}}^2 \mid d \in A\}) \cup \Delta_A) \supseteq \Delta_A$  is symmetric and forms a subalgebra of  $\mathfrak{A}^2$ , and so the transitive closure  $\eta_j = \text{Cg}^{\mathfrak{A}}(\{\langle c, b_j \rangle\}) \supseteq \vartheta_j$  of  $\vartheta_j$  is a congruence of  $\mathfrak{A}$  distinct from  $\Delta_A$ . By contradiction, prove that  $(\eta_0 \cap \eta_1) \subseteq \Delta_A$ . For suppose  $(\eta_0 \cap \eta_1) \not\subseteq \Delta_A$ . Take any  $\bar{e} \in ((\eta_0 \cap \eta_1) \setminus \Delta_A) \neq \emptyset$ , in which case, for all  $k, l \in 2$ ,  $\langle e_k, e_{1-k} \rangle \in (\theta_l \setminus \Delta_A)$ , that is, there are some  $m_l \in \omega$ ,  $\bar{f}^l \in A^{m_l+2}$  and  $\bar{g}^l \in A^{m_l+1}$  such that  $f_0^l = e_k$ ,  $f_{m_l+1}^l = e_{1-k}$  and, for every  $n \in (m_l + 1)$ ,  $f_{n+1}^l \in [c \wedge^{\mathfrak{A}} g_n^l, b_l \wedge^{\mathfrak{A}} g_n^l]_{\mathfrak{A}}$ , and so  $e_k \leq^{\mathfrak{A}} c$ , when taking  $n = 0$ , because  $\{l, 1-l\} = 2$ , while  $e_k = f_0^{l(1-l)} \leq^{\mathfrak{A}} (b_{l(1-l)} \wedge^{\mathfrak{A}} g_0^{l(1-l)}) \leq^{\mathfrak{A}} b_{l(1-l)}$ . By induction on any  $\ell \in (m_l + 2)$ , show that  $e_k \leq^{\mathfrak{A}} f_{\ell}^l$ . The case  $\ell = 0$  is by the equality  $e_k = f_0^l$ . Otherwise,  $(m_l + 2) \ni (\ell - 1) < \ell$ , in which case, by induction hypothesis, we have  $c \geq^{\mathfrak{A}} e_k \leq^{\mathfrak{A}} f_{\ell-1}^l \leq^{\mathfrak{A}} (b_l \wedge^{\mathfrak{A}} g_{\ell-1}^l) \leq^{\mathfrak{A}} g_{\ell-1}^l$ , and so we get  $e_k \leq^{\mathfrak{A}} (c \wedge^{\mathfrak{A}} g_{\ell-1}^l) \leq^{\mathfrak{A}} f_{\ell}^l$ . In particular,  $e_k \leq^{\mathfrak{A}} e_{1-k}$ , when taking  $\ell = (m_l + 1)$ , since  $f_{m_l+1}^l = e_{1-k}$ . Then,  $e_0 = e_1$ , in which case this contradiction shows that  $(\eta_0 \cap \eta_1) = \Delta_A$ , and so  $\mathfrak{A}$  is not finitely-sibdirectly-irreducible. Thus, by (2.10) as well as the simplicity of two-element algebras and absence of their proper non-one-element subalgebras,  $((\text{SI}^{(\omega)} \mid \text{Si})(\text{SL})\{\cup \mathbf{A}_{\Sigma}^{\bar{1}}\}) = (\mathbf{IG}_2\{\cup \mathbf{A}_{\Sigma}^{\bar{1}}\})$  is the class of {no-more-than-}two-element semi-lattices {that is, the universal first-order model subclass of SL relatively axiomatized by the single universal first-order sentence  $\forall_{i \in 3} x_i ((x_2 \approx x_1) \vee (x_2 \approx x_0) \vee (x_1 \approx x_0))$ }, while SL, being finitely-semi-simple and finitely-generated, is semi-simple and locally-finite. On the other hand, since  $\text{Fi}(2) = \{\wp(N, 2) \mid N \subseteq 2\}$ , the set  $\{\Delta_{2^2}, (2^2)^2\} \cup \{\ker(\pi_j \upharpoonright 2^2) \mid j \in 2\}$  of filtral congruences of  $\mathfrak{S}_2^2$  does not contain its congruence  $\Delta_{2^2} \cup \{\langle (0, \mathbb{k}), \langle 0, 1 - \mathbb{k} \rangle \mid \mathbb{k} \in 2\}$ , in which case, by Theorem 3.9, SL, not being filtral, is not implicative, and so, by Theorem 3.23, is neither congruence-distributive nor disjunctive.  $\square$

### 3.2.2. Disjunctivity versus distributivity of lattices of sub-varieties.

**Lemma 3.27.** *Let  $\mathbf{K}$  be a class of  $\Sigma$ -algebras with a disjunctive system  $\mathfrak{U} \subseteq \text{Eq}_{\Sigma}^4$  as well as  $\mathbf{R}$  and  $\mathbf{S}$  are relative sub-varieties of  $\mathbf{K}$ . Then, so is  $\mathbf{R} \cap \parallel \cup \mathbf{S}$ . In particular, relative sub-varieties of  $\mathbf{K}$  form a distributive lattice.*

*Proof.* Take any  $\mathcal{J}, \mathcal{J} \subseteq \text{Tm}_{\Sigma}^{\omega}$  with  $(\mathbf{R} \mid \mathbf{S}) = (\mathbf{K} \cap \text{Mod}(\mathcal{J} \mid \mathcal{J}))$ , in which case  $(\mathbf{R} \cap \parallel \cup \mathbf{S}) = (\mathbf{K} \cap \text{Mod}((\mathcal{J} \cup \mathcal{J}) \parallel \bigcup\{\mathfrak{U}[x_i/\phi_i, x_{2+i}/\psi_i]_{i \in 2} \mid (\bar{\phi} \mid \bar{\psi}) \in ((\mathcal{J} \mid \mathcal{J})[x_j/x_{(2-j)+(0|1)}]_{j \in \omega})\}))$ , and so the distributivity of unions with intersections completes the argument.  $\square$

This, by (2.10), (2.9) and Lemma 3.14, immediately yields:

**Corollary 3.28.** *Let  $\mathbf{K}$  be a [finite] class of finite  $\Sigma$ -algebras with a disjunctive system  $\mathfrak{U} \subseteq \text{Eq}_{\Sigma}^4$  and  $\mathbf{P}$  the pre-variety generated by  $\mathbf{K}$ . Suppose  $\mathbf{P}$  is a variety. Then,  $\text{SI}(\mathbf{P}) = \mathbf{IS}_{>1}\mathbf{K}$ , in which case  $\mathbf{S} \mapsto (\mathbf{S} \cap \mathbf{S}_{\{>1\}}\mathbf{K})$  and  $\mathbf{R} \mapsto \mathbf{IP}^{\text{SD}}\mathbf{R}$  are inverse to one another isomorphisms between the lattices of sub-varieties of  $\mathbf{P}$  and relative ones of  $\mathbf{S}_{\{>1\}}\mathbf{K}$ , and so they are distributive [and finite].*

Likewise, by (2.10), (2.9), Theorem 3.7 (as well as [20, Remark 2.4] and Lemma 3.27), we immediately have:

**Corollary 3.29.** *Let  $\mathbf{K}$  be a [finite] class of [finite]  $\Sigma$ -algebras with a (finite) implicative system  $\mathfrak{U} \subseteq \text{Eq}_{\Sigma}^4$  and  $\mathbf{P}$  the pre-variety generated by  $\mathbf{K}$ . Suppose  $\mathbf{P}$  is a variety. Then,  $(\text{SI} \mid \text{Si})(\mathbf{P}) = \mathbf{P}_{>1}^{\bar{1}} = \mathbf{IS}_{>1}\mathbf{K}$ , in which case  $\mathbf{S} \mapsto (\mathbf{S} \cap \mathbf{S}_{\{>1\}}\mathbf{K})$  and  $\mathbf{R} \mapsto \mathbf{IP}^{\text{SD}}\mathbf{R}$  are inverse to one another isomorphisms between the [finite] (distributive) lattices of sub-varieties of  $\mathbf{P}$  and relative ones of  $\mathbf{S}_{\{>1\}}\mathbf{K}$ .*

## 4. MORGAN-STONE LATTICES VERSUS DISTRIBUTIVE ONES

From now on, we deal with the signatures  $\Sigma_{+[.01]}^{(-)} \triangleq (\Sigma_+(\cup\{\neg\})[\cup\{\perp, \top\}])$ , [bounded] {distributive} lattices being supposed to be  $\Sigma_{+[.01]}$ -algebras with their variety denoted by  $[\mathbf{B}]\{\mathbf{D}\}\mathbf{L}$  and the chain [bounded] distributive lattice with carrier  $n \in (\omega \setminus 2)$  and the natural ordering on this denoted by  $\mathfrak{D}_{n[.01]}$ , in which case  $\epsilon_2^n \triangleq \{(0, 0), \langle 1, n-1 \rangle\}$  is an embedding of  $\mathfrak{D}_{2[.01]}$  into  $\mathfrak{D}_{n[.01]}$ , while, for each  $i \in 2$ ,  $\epsilon_{3;i}^4 \triangleq (\chi_3^{3 \setminus (2-i)} \times \chi_3^{3 \setminus (1+i)})$  is an embedding of  $\mathfrak{D}_{3[.01]}$  into  $\mathfrak{D}_{2[.01]}^2$ . First, taking the Prime Ideal Theorem, (2.8), (2.10) and Corollary 3.11 into account, we immediately have the following well-known fact (cf. [8] as to REDPC for  $[\mathbf{B}]\mathbf{DL}$ ):

**Lemma 4.1.** *Let  $\mathfrak{A} \in [\mathbf{B}]\mathbf{L}$  and  $F \subseteq A$ . Suppose  $F$  is either a prime filter of  $\mathfrak{A}$  or in  $\{\emptyset, A\}$ . Then, [unless  $F \in \{\emptyset, A\}$ ]  $h \triangleq \chi_A^F \in \text{hom}(\mathfrak{A}, \mathfrak{D}_{2[.01]})$  [and  $h[A] = 2$ ], in which case  $[\mathbf{B}]\mathbf{DL} = \mathbf{IP}^{\text{SD}}\mathfrak{D}_{2[.01]}$ , and so  $[\mathbf{B}]\mathbf{DL}$  is the semi-simple [pre-/quasi-]variety generated by  $\mathfrak{D}_{2[.01]}$  with  $(\text{Si} \mid \text{SI})([\mathbf{B}]\mathbf{DL}) = \mathbf{ID}_{2[.01]}$  and REDPC scheme  $\mathcal{U}_{V_1}^{(x_0)}$ .*

A [bounded] (De) Morgan-Stone {(D)MS} lattice is any  $\Sigma_{+[.01]}^-$ -algebra, whose  $\Sigma_{+[.01]}$ -reduct is a [bounded] distributive lattice and which satisfies the  $\Sigma_{+}^-$ -identities:

$$(4.1) \quad \neg(x_0 \wedge x_1) \approx (\neg x_0 \vee \neg x_1),$$

$$(4.2) \quad x_0 \lesssim \neg\neg x_0,$$

in which case, by (4.1) [and (4.2)[ $x_0/\top$ ]], it satisfies the  $\Sigma_{+}^-$ -quasi-identity [and the  $\Sigma_{+[.01]}^-$ -identity]:

$$(4.3) \quad (x_0 \lesssim x_1) \rightarrow (\neg x_1 \lesssim \neg x_0),$$

$$(4.4) \quad \neg\neg\top \approx \top],$$

and so the  $\Sigma_{+[.01]}^-$ -identities:

$$(4.5) \quad \neg(x_0 \vee x_1) \approx (\neg x_0 \wedge \neg x_1),$$

$$(4.6) \quad \neg\neg\neg x_0 \approx \neg x_0,$$

$$(4.7) \quad \neg\perp \approx \top],$$

their variety being denoted by  $[\mathbf{B}](\mathbf{D})\mathbf{MSL}$ . Then, bounded Morgan-Stone lattices, satisfying the  $\Sigma_{+[.01]}^-$ -identity:

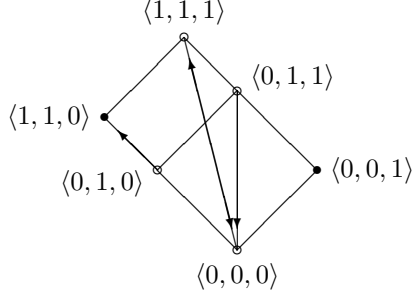
$$(4.8) \quad \neg\top \approx \perp,$$

are nothing but (De) Morgan-Stone {MS} algebras [2] (cf. [23]), their variety being denoted by (D)MSA. An  $a \in A$  is called {a} (negatively-)idempotent {element of an  $\mathfrak{A} \in \mathbf{MSL}$ }, if  $\{(\neg^{\mathfrak{A}})a\}$  forms a subalgebra of  $\mathfrak{A}$ , i.e.,  $\neg^{\mathfrak{A}}(\neg^{\mathfrak{A}})a = (\neg^{\mathfrak{A}})a$ , with their set denoted by  $\mathfrak{S}_{(\neg^{\mathfrak{A}})}^{\mathfrak{A}}$ , Morgan-Stone lattices with carrier of cardinality no less than  $2(\{-1\})$  and with({out non-}negatively-)idempotent elements being said to be ({totally} negatively-)idempotent.

*Remark 4.2.* By (4.1), (4.5), (4.6), Corollary 3.11 and Theorem 3.5,  $\mathcal{U}_{\{x_0, \neg x_0, \neg\neg x_0\}}^{(x_0, \neg x_0, \neg\neg x_0)}$  is an REDPC scheme for  $[\mathbf{B}]\mathbf{MS}(\mathbf{L}/\mathbf{A})$ .  $\square$

**4.1. Subdirectly-irreducibles.** Let  $\mathfrak{MS}_6$  be the  $\Sigma_{+}^-$ -algebra with  $(\mathfrak{MS}_6 \upharpoonright \Sigma_{+}^-) \triangleq ((\mathfrak{D}_2^2 \upharpoonright (2^2 \setminus \{(1, 0)\})) \times \mathfrak{D}_2)$  and  $\neg^{\mathfrak{MS}_6} \bar{a} \triangleq \langle 1 - a_2, 1 - a_2, 1 - a_1 \rangle$ , for all  $\bar{a} \in \mathbf{MS}_6$  (the Hasse diagram of its lattice reduct with its [non-]idempotent elements marked by [non-]solid circles and arrows reflecting action of its operation  $\neg$  on its non-idempotent elements is depicted at Figure 1), in which case it is routine to check to be a Morgan-Stone lattice, and so are both  $\mathfrak{MS}_5 \triangleq (\mathfrak{MS}_6 \upharpoonright (\mathbf{MS}_6 \setminus \{(0, 0, 1)\}))$



FIGURE 1. The Morgan-Stone lattice  $\mathfrak{MS}_6$ .

and  $\mathfrak{MS}_2 \triangleq (\mathfrak{MS}_5 \upharpoonright \{(i, 1, 0) \mid i \in 2\})$  as well as, for each  $j \in 2$ ,  $\mathfrak{MS}_{4;j} \triangleq (\mathfrak{MS}_{5+j} \upharpoonright (MS_{5+j} \setminus (((j+1) \times \{1\}) \times \{1-j\})))$ . Likewise, let  $(\mathfrak{DM}|\mathfrak{S})_{4|3}$  be the  $\Sigma_-$ -algebra with  $((\mathfrak{DM}|\mathfrak{S})_{4|3} \upharpoonright \Sigma_-) \triangleq \mathfrak{D}_{2|3}^2$  and  $\neg(\mathfrak{DM}|\mathfrak{S})_{4|3} \triangleq (((\pi_1 \upharpoonright 2) \circ (2^2 \setminus \Delta_2)) \times ((\pi_0 \upharpoonright 2) \circ (2^2 \setminus \Delta_2))) \upharpoonright \chi_3^1$ , in which case  $\epsilon_{4|3}^{6|5} \triangleq (((\pi_0 \upharpoonright 2^2) \times (\pi_0 \upharpoonright 2^2)) \times (\pi_1 \upharpoonright 2^2)) \upharpoonright (\epsilon_{3;0}^4 \times \chi_3^{3|1})$  is an embedding of  $(\mathfrak{DM}|\mathfrak{S})_{4|3}$  into  $(\mathfrak{MS}|\mathfrak{MS})_{6|5}$ . Finally, for any  $n \in (\{3, 4\} \upharpoonright \{2\})$ , let  $(\mathfrak{R}|\mathfrak{B})_n$  be the  $\Sigma_-$ -algebra with  $((\mathfrak{R}|\mathfrak{B})_n \upharpoonright \Sigma_-) \triangleq \mathfrak{D}_n$  and  $\neg(\mathfrak{R}|\mathfrak{B})_n \triangleq \{(m, n-1-m) \mid m \in n\}$ , in which case  $\epsilon_2^{3|4}$  is an embedding of  $\mathfrak{B}_2$  into  $\mathfrak{R}_{3|4}$ , while, for every  $l \in 2$ ,  $\epsilon_{3;l}^4$  is an embedding of  $\mathfrak{R}_3$  into  $\mathfrak{DM}_4$ , and so  $\epsilon_{3;l}^4 \circ \epsilon_4^6$  is that into  $\mathfrak{MS}_{4;(1-l)}$ . Moreover,  $\{MS_6, MS_5, MS_2, \text{img}(\epsilon_2^3 \circ \epsilon_3^5)\} \cup (\bigcup \{MS_{4;k}, \text{img}(\epsilon_{3;k}^4 \circ \epsilon_4^6) \mid k \in 2\})$  are exactly the carriers of members of  $\mathbf{S}_{>1}\mathfrak{MS}_6$ , in which case these are isomorphic to those of the skeleton  $\mathbf{MS} \triangleq (\{\mathfrak{MS}_\ell \mid \ell \in \{6, 5, 2\}\} \cup \{\mathfrak{MS}_{4;k} \mid k \in 2\} \cup \{\mathfrak{DM}_4, \mathfrak{R}_3, \mathfrak{S}_3, \mathfrak{B}_2\})$ , and so this is that of  $\mathbf{IS}_{>1}\mathfrak{MS}_6$  with the embeddability *partial* ordering  $\preceq$  between members of  $\mathbf{MS}$ , for these are all finite. And what is more,  $D_6 \triangleq (MS_6 \cap \pi_0^{-1}[\{1\}])$  is a prime filter of  $\mathfrak{MS}_6 \upharpoonright \Sigma_+$ , while  $\Omega \triangleq \{x_0, \neg x_0, \neg\neg x_0\}$  is an equality determinant for  $\langle \mathfrak{MS}_6, D_6 \rangle$ , in which case, by [19, Lemma 11],  $\mathcal{U}_\Omega \triangleq \{(\tau(x_i) \wedge \rho(x_{2+j})) \lesssim (\tau(x_{1-i}) \vee \rho(x_{3-j})) \mid i, j \in 2, \tau, \rho \in \Omega\}$  is a disjunctive system for  $\mathfrak{MS}_6$ , and so, for  $\mathbf{IS}\mathfrak{MS}_6$ .

*Remark 4.3.* Elements of  $\mathcal{PF}_4 \triangleq \{2^2 \cap \pi_i^{-1}[\{1\}] \mid i \in 2\}$  are exactly all prime filters of  $\mathfrak{D}_2^2$ , while  $\{x_0, \neg x_0\}$  is an equality determinant for  $\mathbf{M} \triangleq (\{\mathfrak{DM}_4\} \times \mathcal{PF}_4)$ , in which case, by Theorem 3.13,  $\mathcal{U}_{V_1}^{(x_0, \neg x_0)}$  is an implicative system for  $\mathbf{IS}_{\{>1\}}\mathfrak{DM}_4$  {and so, by Corollary 3.12, its members are simple, as it is well-known but shown directly in a more cumbersome way}.  $\square$

**Theorem 4.4.** *For any prime filter  $F$  of the  $\Sigma_+$ -reduct of any  $\mathfrak{A} \in \mathbf{MSL}$  there is an  $h \in \text{hom}(\mathfrak{A}, \mathfrak{MS}_6)$  with  $(\ker h) \subseteq (\ker \chi_A^F)$ , in which case  $\mathbf{MSL}$  is the [pre-/quasi-]variety generated by  $\mathfrak{MS}_6$  with REDPC scheme  $\mathcal{U}_\Omega^{(x_0, \neg x_0, \neg\neg x_0)}$ , and so  $\mathbf{SI}(\mathbf{MSL}) = \mathbf{IMS}$ .*

*Proof.* Let  $f \triangleq \chi_A^F$ ,  $G \triangleq (\neg^{\mathfrak{A}})^{-1}[(\neg^{\mathfrak{A}})^{-1}[F]]$ ,  $H \triangleq (A \setminus (\neg^{\mathfrak{A}})^{-1}[F])$  and  $h \triangleq (f \times \chi_A^G) \times \chi_A^H$ , in which case, by (2.1) and (4.6),  $(\ker f) \supseteq (((\ker f) \cap (\ker \chi_A^G)) \cap (\ker \chi_A^H)) = (\ker h) \subseteq (\neg^{\mathfrak{A}} \circ h)$ , while, by (4.1) and (4.5),  $G|H$  is either a prime filter of  $\mathfrak{A} \upharpoonright \Sigma_+$  or in  $\{\emptyset, A\}$ , whereas, by (4.2),  $F \subseteq G$ , and so, by (2.2),  $\pi_0(h(a)) \leq \pi_1(h(a))$ , for all  $a \in A$ . Then, by (2.7), Lemma 4.1 and the Homomorphism Theorem,  $h$  is a surjective homomorphism from  $\mathfrak{A}$  onto the  $\Sigma_+$ -algebra  $\mathfrak{B}$  with  $(\mathfrak{B} \upharpoonright \Sigma_+) \triangleq (\mathfrak{D}_2^3 \upharpoonright h[A])$  as well as  $\neg^{\mathfrak{B}} \triangleq (h^{-1} \circ \neg^{\mathfrak{A}} \circ h)$ , in which case  $B \subseteq MS_6$ , since  $\pi_0(h(a)) \leq \pi_1(h(a))$ , for all  $a \in A$ , and so  $\mathfrak{B} = (\mathfrak{MS}_6 \upharpoonright h[A])$ , as, for all  $a \in A$ ,  $(\neg^{\mathfrak{A}} a \in G) \Leftrightarrow (\neg^{\mathfrak{A}} a \in F) \Leftrightarrow (a \notin H)$ , in view of (4.6), as well as  $(\neg^{\mathfrak{A}} a \in H) \Leftrightarrow (\neg^{\mathfrak{A}} \neg^{\mathfrak{A}} a \notin F) \Leftrightarrow (a \notin G)$ . Hence,  $h \in \text{hom}(\mathfrak{A}, \mathfrak{MS}_6)$  and  $(\ker h) \subseteq (\ker f)$ .

Thus, the Prime Ideal Theorem, (2.8), Corollary 3.28 and Remark 4.2 complete the argument.  $\square$

The  $\Sigma_+^-$ -reduct of any  $\mathfrak{A} \in \mathbf{MS}$ , being a finite lattice, has zero/unit  $a/b$ , in which case we have the bounded Morgan-Stone lattice  $\mathfrak{A}_{01}$  with  $(\mathfrak{A}_{01} \upharpoonright \Sigma_+^-) \triangleq \mathfrak{A}$  and  $(\perp/\top)^{\mathfrak{A}_{01}} \triangleq (a/b)$ , and so, for all  $\mathfrak{C} \in \mathbf{MS}_{01} \triangleq \{\mathfrak{B}_{01} \mid \mathfrak{B} \in \mathbf{MS}\}$  and  $\mathfrak{D} \in \mathbf{MS}_{-2,01} \triangleq (\mathbf{MS}_{01} \setminus \{\mathfrak{MS}_{2,01}\})$ ,  $((\mathfrak{D} \upharpoonright \Sigma_+^-) \preceq (\mathfrak{C} \upharpoonright \Sigma_+^-)) \Rightarrow (\mathfrak{D} \preceq \mathfrak{C})$ . Then, since  $\mathfrak{MS}_{2,01} \notin \mathbf{MSA} \supseteq (\mathbf{ISM}_{6,01}) \supseteq \mathbf{MS}_{-2,01}$ , while surjective lattice homomorphisms preserve lattice bounds (if any), whereas expansions by constants alone preserve congruences, by (2.8), (2.9) and Theorem 4.4, we immediately get:

**Corollary 4.5.** *Let  $\mathbf{K} \triangleq (\emptyset \setminus \{\mathfrak{MS}_{2,01}\})$ . Then,  $\mathbf{V} \triangleq (\mathbf{BMSL} \mid \mathbf{MSA})$  is the [pre-/quasi-]variety generated by  $\{\mathfrak{MS}_{6,01}, \mathfrak{MS}_{2,01}\} \setminus \mathbf{K}$  with  $\text{SI}(\mathbf{V}) = \mathbf{I}(\mathbf{MS}_{01} \setminus \mathbf{K})$  and REDPC scheme  $\mathcal{U}_\Omega^{(x_0, \neg x_0, \neg \neg x_0)}$ .*

This subsumes [2] and also yields a uniform insight into REDPC for Stone and De Morgan algebras, originally given by separate distinct schemes in [12, 21] and a bit enhanced in Corollary 4.7.

**4.2. The lattice of sub-varieties.** [Bounded/] Morgan-Stone lattices[/algebras], satisfying either of the following equivalent — in view of (4.2) —  $\Sigma_+^-$ -identities:

$$(4.9) \quad (\neg \neg x_0 \vee \neg x_0) \approx \parallel \lesssim (x_0 \vee \neg x_0),$$

are called [bounded/] (nearly) De Morgan lattices[/algebras], their variety being denoted by  $[\mathbf{B}/](\mathbf{N})\mathbf{DM}(\mathbf{L}/\mathbf{A})$ . Likewise, those, satisfying the  $\Sigma_+^-$ -identity:

$$(4.10) \quad (x_0 \wedge \neg x_0) \lesssim x_1,$$

are nothing but [bounded/] Stone lattices[/algebras] [cf., e.g., [7]], their variety being denoted by  $[\mathbf{B}/]\mathbf{S}(\mathbf{L}/\mathbf{A})$ . Then, members of  $[[\mathbf{B}/]\mathbf{B}(\mathbf{L}/\mathbf{A}) \triangleq ([\mathbf{B}]\mathbf{DM}(\mathbf{L}/\mathbf{A}) \cap [\mathbf{B}]\mathbf{S}(\mathbf{L}/\mathbf{A}))$  are exactly [bounded/] Boolean lattices[/algebras]. Further, [bounded/] Morgan-Stone lattices[/algebras], satisfying “either of the former”|“the latter” of the following  $\Sigma_+^-$ -identities:

$$(4.11) \quad (\neg \neg x_0 \wedge \neg x_0) \approx \parallel \lesssim (x_0 \wedge \neg x_0),$$

$$(4.12) \quad \neg \neg x_0 \lesssim (x_0 \vee (\neg \neg x_1 \vee \neg x_1)),$$

“in which case they satisfy the  $\Sigma_+^-$ -quasi-identities:

$$(4.13) \quad (\neg x_0 \lesssim x_0) \leftarrow \parallel \rightarrow (\neg x_0 \lesssim \neg \neg x_0),$$

in view of (4.2)” are said to be *quasi-[pseudo-strong]*, their variety being denoted by  $[\mathbf{B}/](\mathbf{Q}|\mathbf{P})\mathbf{SMS}(\mathbf{L}/\mathbf{A})$ . Then, members of  $[\mathbf{B}/]\mathbf{SMS}(\mathbf{L}/\mathbf{A}) \triangleq ([\mathbf{B}/]\mathbf{QSMS}(\mathbf{L}/\mathbf{A}) \cap [\mathbf{B}/]\mathbf{PSMS}(\mathbf{L}/\mathbf{A})) \supseteq ([\mathbf{B}/]\mathbf{DM}(\mathbf{L}/\mathbf{A}) \cup [\mathbf{B}/]\mathbf{S}(\mathbf{L}/\mathbf{A}))$  are said to be *strong*, in which case, by (4.2) and the uniqueness of relative complements in distributive lattices:

$$(4.14) \quad ([\mathbf{B}]\{\mathbf{Q}\}\mathbf{SMSL} \cap [\mathbf{B}]\mathbf{NDML}) = \mathbf{DML}.$$

Furthermore, [bounded/] ([quasi-[pseudo-]strong]) {weakly} Kleene(-Morgan)-Stone lattices [/algebras] are [bounded/] ([quasi-[pseudo-]strong]) De-Morgan(-Stone) lattices[/algebras] satisfying the following  $\Sigma_+^-$ -identity:

$$(4.15) \quad ((\neg \neg x_2 \wedge)(x_0 \wedge \neg x_0)) \lesssim ((x_2 \vee)(\neg x_1 \vee \{\neg \neg\}x_1)),$$

their variety being denoted by

$$\begin{aligned} & [\mathbf{B}/](\llbracket [\mathbf{Q}|\mathbf{P}]\mathbf{S} \rrbracket)\{\mathbf{W}\}\mathbf{K}(\langle \mathbf{M} \rangle \mathbf{S})(\mathbf{L}/\mathbf{A}) \supseteq (\emptyset \cup ([\mathbf{B}/]\mathbf{S}(\mathbf{L}/\mathbf{A}))) \\ & \quad \cup \{[\mathbf{B}/](\llbracket [\mathbf{Q}|\mathbf{P}]\mathbf{S} \rrbracket)\mathbf{K}(\langle \mathbf{M} \rangle \mathbf{S})(\mathbf{L}/\mathbf{A})\} \\ & \quad \quad \quad (([\mathbf{B}/]\mathbf{DM}(\mathbf{L}/\mathbf{A}) \cup [\mathbf{B}/](\llbracket [\mathbf{Q}|\mathbf{P}]\mathbf{S} \rrbracket)\{\mathbf{W}\}\mathbf{K}(\mathbf{S})(\mathbf{L}/\mathbf{A}))) \end{aligned}$$

{in view of (4.2)}. Likewise, members of

$$[\mathbf{B}/]\mathbf{NK}(\mathbf{L}/\mathbf{A}) \triangleq ([\mathbf{B}/]\{\mathbf{W}\}\mathbf{KS}(\mathbf{L}/\mathbf{A}) \cap [\mathbf{B}/]\mathbf{NDM}(\mathbf{L}/\mathbf{A}))$$

are called *[bounded/] nearly Kleene lattices/[algebras]*. Next, the variety of totally negatively-idempotent [bounded] Morgan-Stone lattices, being relatively axiomatized by the  $\Sigma_+^-$ -identity:

$$(4.16) \quad \neg\neg x_0 \approx \neg x_0,$$

is denoted by  $[\mathbf{B}]\mathbf{TNIMSL}$ . Likewise, the variety of one-element [bounded/] Morgan-Stone lattices/[algebras], being relatively axiomatized by the  $\Sigma_+^-$ -identity:

$$(4.17) \quad x_0 \approx x_1,$$

is denoted by  $[\mathbf{B}/]\mathbf{OMS}(\mathbf{L}/\mathbf{A})$ . Further, members of  $[\mathbf{B}/](\mathbf{M}\{\mathbf{W}\}\mathbf{K})\mathbf{S}(\mathbf{L}/\mathbf{A})$ , satisfying following  $\Sigma_+^-$ -identity:

$$(4.18) \quad ((\neg x_0 \wedge \neg\neg x_0) \wedge \neg\neg x_1) \lesssim ((\neg x_0 \wedge x_0) \vee \neg x_1),$$

are said to be *almost quasi-strong*, their variety being denoted by

$$[\mathbf{B}/]\mathbf{AQS}(\mathbf{M}\{\mathbf{W}\}\mathbf{K})\mathbf{S}(\mathbf{L}/\mathbf{A}) \supseteq ([\mathbf{B}/]\mathbf{QS}(\mathbf{M}\{\mathbf{W}\}\mathbf{K})\mathbf{S}(\mathbf{L}/\mathbf{A}) \cup ([\mathbf{B}]\mathbf{TNIMSL}/\{\emptyset\})).$$

Then, members of

$$[\mathbf{B}/]\mathbf{AS}(\mathbf{M}\{\mathbf{W}\}\mathbf{K})\mathbf{S}(\mathbf{L}/\mathbf{A}) \triangleq ([\mathbf{B}/]\mathbf{AQS}(\mathbf{M}\{\mathbf{W}\}\mathbf{K})\mathbf{S}(\mathbf{L}/\mathbf{A}) \cap [\mathbf{B}/]\mathbf{PS}(\mathbf{M}\{\mathbf{W}\}\mathbf{K})\mathbf{S}(\mathbf{L}/\mathbf{A})) \supseteq ([\mathbf{B}/]\mathbf{S}(\mathbf{M}\{\mathbf{W}\}\mathbf{K})\mathbf{S}(\mathbf{L}/\mathbf{A}) \cup ([\mathbf{B}]\mathbf{TNIMSL}/\{\emptyset\}))$$

are said to be *almost strong*. Likewise, members of  $[\mathbf{B}/](\mathbf{M}\{\mathbf{W}\}\mathbf{K})\mathbf{S}(\mathbf{L}/\mathbf{A})$ , satisfying the following  $\Sigma_+^-$ -identity:

$$(4.19) \quad (\neg\neg x_0 \wedge \neg\neg x_1) \lesssim (x_0 \vee \neg x_1),$$

are called *[bounded/] almost “De Morgan” “{weakly} Kleene” lattices/[algebras]*, their variety being denoted by  $[\mathbf{B}/]\mathbf{A}(\mathbf{DM}\{\mathbf{W}\}\mathbf{K})(\mathbf{L}/\mathbf{A}) \supseteq ([\mathbf{B}/](\mathbf{DM}\{\mathbf{W}\}\mathbf{K})(\mathbf{L}/\mathbf{A}) \cup ([\mathbf{B}]\mathbf{TNIMSL}/\{\emptyset\}))$ . Finally, [bounded/] Morgan-Stone lattices/[algebras], satisfying the optional/non-optional version of the following  $\Sigma_+^-$ -identity:

$$(4.20) \quad (\neg x_0 \vee [\neg\neg]x_0) \gtrsim x_1,$$

are called *[bounded/] almost Stone/Boolean lattices/[algebras]*, their variety being denoted by  $[\mathbf{B}/]\mathbf{A}(\mathbf{S}|\mathbf{B})(\mathbf{L}/\mathbf{A})$ .

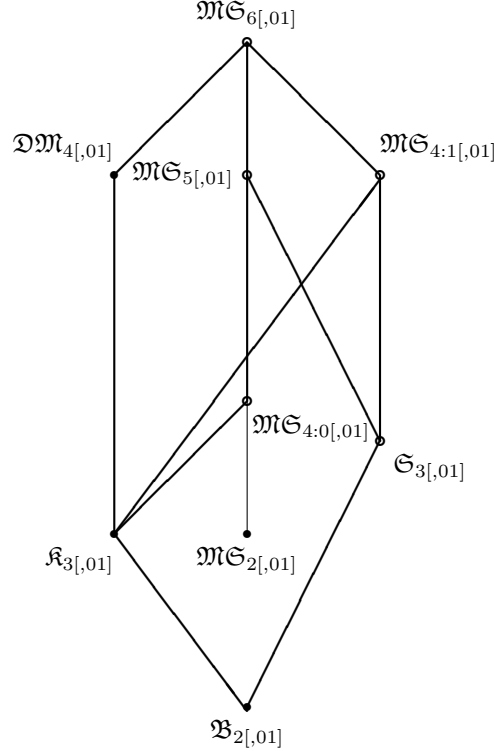
Let

$$\mathcal{MS}_{[01]}[\mathfrak{A}] \triangleq (\{[(4.8), ](4.9), ((4.9)), (4.10), (4.11), (4.12), (4.15), \{(4.15)\}, \langle(4.15)\rangle, \{\{(4.15)\}\}, (4.18), (4.19), (4.20), \lceil(4.20)\rceil, (4.16)\} \cap \mathcal{E}(\mathfrak{A}))$$

[where  $\mathfrak{A} \in \mathbf{MS}_{[01]}$ ].

TABLE 1. Identities of  $\mathcal{MS}_{[01]}$  true in members of  $\mathbf{MS}_{[01]}$ .

$\mathfrak{MS}_{6,[01]}$	$\emptyset \cup \{(4.8)\}$
$\mathfrak{MS}_{5,[01]}$	$\{[(4.8), ](4.12), \{(4.15)\}, \langle(4.15)\rangle, \{\{(4.15)\}\}\}$
$\mathfrak{MS}_{4:0,[01]}$	$\{[(4.8), ](4.9), (4.12), (4.15), \{(4.15)\}, \langle(4.15)\rangle, \{\{(4.15)\}\}\}$
$\mathfrak{MS}_{4:1,[01]}$	$\{[(4.8), ](4.11), (4.15), \{(4.15)\}, \langle(4.15)\rangle, \{\{(4.15)\}\}, (4.18)\}$
$\mathfrak{MS}_{4,[01]}$	$\mathcal{MS}_{[01]} \setminus \{(4.15), \{(4.15)\}, (4.10), (4.20), \lceil(4.20)\rceil, (4.16)\}$
$\mathfrak{MS}_{2,[01]}$	$\mathcal{MS}_{[01]} \setminus \{[(4.8), ](4.9), (4.11), (4.10)\}$
$\mathfrak{K}_{3,[01]}$	$\mathcal{MS}_{[01]} \setminus \{(4.10), (4.20), \lceil(4.20)\rceil, (4.16)\}$
$\mathfrak{S}_{3,[01]}$	$\mathcal{MS}_{[01]} \setminus \{(4.9), ((4.9)), (4.19), (4.20), (4.16)\}$
$\mathfrak{B}_{2,[01]}$	$\mathcal{MS}_{[01]} \setminus \{(4.16)\}$

FIGURE 2. The poset  $\langle \mathcal{MS}_{[0,1]}, \preceq \rangle$  [with merely thick lines].

**Lemma 4.6.** For any  $\mathfrak{A} \in \mathcal{MS}_{[0,1]}$ ,  $\mathcal{MS}_{[0,1]}(\mathfrak{A})$  is given by Table 1. In particular, the poset  $\langle \mathcal{MS}_{[0,1]}, \preceq \rangle$  is given by Figure 2 with (non-)simple/ $\mathcal{U}_{\{x_0, \neg x_0, \neg \neg x_0\}}^{\langle x_0, \neg x_0, \neg \neg x_0 \rangle}$ -implicative members marking (non-)solid circles-nodes [and merely thick lines].

*Proof.* Clearly, for any line of Table 1, the identities of the second column of it are true in the algebra of the first one. Conversely,

$$\begin{aligned}
\mathcal{MS}_{(5|6)[,0,1]} &\not\models ((4.15)|\{\{(4.15)\}\})[x_i/\langle 1 - \min(1, i), 1 | \max(1 - i, i - 1), \\
&\quad \min(1, i) \rangle]_{i \in \{2|3\}}, \\
\mathcal{S}_{3[,0,1]} &\not\models (((4.9)|\{(4.9)\})|((4.19)|\{(4.20)\})) [x_i/(1 + i)]_{i \in \{1|2\}}, \\
\mathcal{DM}_{4[,0,1]} &\not\models ((4.15)|\{\{(4.15)\}\})[x_i/\langle \langle i, i, 1 - i \rangle \rangle]_{i \in \{2\}}, \\
\mathcal{MS}_{4.1[,0,1]} &\not\models (4.12)[x_0/\langle 0, 1, 1 \rangle, x_1/\langle 0, 0, 1 \rangle], \\
\mathcal{MS}_{4.0[,0,1]} &\not\models (4.18)[x_i/\langle i, 1, i \rangle]_{i \in \{2\}}, \\
\mathcal{K}_{3[,0,1]} &\not\models ((4.10)|(\lceil (4.20) \rceil | (4.20))) [x_0/1, x_1/\langle 0|2 \rangle], \\
(\mathcal{B}|\mathcal{MS})_{2[,0,1]} &\not\models (4.16|((4.9)|\{(4.11)\})) [x_0/\langle 0|\langle 0, 1, 0 \rangle \rangle], \\
\mathcal{MS}_{2,0,1} &\not\models (4.8).
\end{aligned}$$

Moreover, by Remark 4.2,  $\mathcal{U}_{\Omega}^{\langle x_0, \neg x_0, \neg \neg x_0 \rangle}$  is an REDPC scheme for  $\text{MSL} \supseteq \text{MS}$ , in which case, by Corollary 3.4, any simple member  $\mathfrak{A}$  of it is  $\mathcal{U}_{\Omega}^{\langle x_0, \neg x_0, \neg \neg x_0 \rangle}$ -implicative, and so all those members of  $\text{MS}$ , which are embeddable into  $\mathfrak{A}$ , being then  $\mathcal{U}_{\Omega}^{\langle x_0, \neg x_0, \neg \neg x_0 \rangle}$ -implicative as well, are simple too. On the other hand,

$$(4.21) \quad \chi_3^{3 \setminus 1} = (\epsilon_3^5 \circ \pi_2) \in \text{hom}(\mathcal{S}_{3[,0,1]}, \mathcal{B}_{2[,0,1]}),$$

in which case  $(\ker \chi_3^{3 \setminus 1}) \in (\text{Co}(\mathcal{S}_{3[,0,1]}) \setminus \{\Delta_3, 3^2\})$ , and so  $\mathcal{S}_{3[,0,1]}$  is not simple. Likewise,  $h \triangleq \{ \langle \bar{a}, [\frac{a_0 + a_1 + a_2 + 1}{2}] \rangle \mid \bar{a} \in \text{MS}_{4.0} \} \in \text{hom}(\mathcal{MS}_{4.0[,0,1]}, \mathcal{K}_{3[,0,1]})$ , in which

case  $(\ker h) \in (\text{Co}(\mathfrak{MS}_{4:0[01]}) \setminus \{\Delta_{MS_{4:0}}, MS_{4:0}^2\})$ , and so  $\mathfrak{MS}_{4:0[01]}$  is not simple. Thus, the fact that varieties are abstract and hereditary, the simplicity of two-element algebras, the equality (4.11) = ((4.10)[ $x_0/\neg x_0, x_1/(x_0 \wedge \neg x_0)$ ], Remark 4.3 and the truth of the identity (4.9)( $\neg x_0 \approx \neg x_1$ ) in  $(\mathfrak{DM}|\mathfrak{MS})_{4|2}$  end the proof.  $\square$

**Corollary 4.7.** *Sub-varieties of  $[B/]MS(L[A])$  form the non-chain distributive lattice with  $29[(+11)/(-9)]$  elements, whose Hasse diagram with [both thick and] thin lines is depicted at Figure 3, any (non-)solid circle-node of it being marked by a (non-)semi-simple|filtral| $\langle \mathcal{U}_{\{x_0, \neg x_0, \neg \neg x_0\}}^{(x_0, \neg x_0)} \rangle$ -implicative variety  $V \subseteq [B/]MS(L[A])$ , numbered from  $1[(+0/20)]$  to  $29[(+11)]$  according to Table 2 with  $\mathbb{k} \triangleq (9 \cdot (1[0/0]))$  [as well as  $\ell \triangleq (29 \cdot (0/1))$ ] and  $MS_{V[01]} \triangleq \max_{\preceq}((MS_{[-2,01]}[UK]) \cap V)$ , where  $K \triangleq (\{\mathfrak{MS}_{2[01]}\}[\emptyset])$ , given by the third column, in which case  $SI(V) = \mathbf{IS}_{>1}MS_{V[01]}$ , and so  $V$  is the (pre-||quasi-)variety generated by  $MS_{V[01]}$ , while  $[B/]SMSL$  is that generated by  $\{SI\}([B/]DML \cup [B/]SL)$  with REDPC scheme  $\mathcal{U}_{\{x_0, \neg x_0\}}^{(x_0, \neg x_0)}$ , whereas any disjunctive sub-pre-variety of  $[B/]MS(L[A])$  is equational, and so is any quasi-equational//finitely implicative one.*

*Proof.* We use Lemma 4.6 tacitly. Then, the intersections of  $MS_{[-2,01]}[UK]$  with the  $29[(+11)/(-9)]$  sub-varieties of  $[B/]MS(L[A])$  involved are exactly all lower cones of the poset  $\langle MS_{[-2,01]}[UK], \preceq \rangle$ , i.e., the sets appearing in the third column of Table 2 are exactly all anti-chains of the poset. So, (2.8), (2.9), (4.1), (4.5), Theorems 3.7, 3.9, 4.4, Corollaries 3.11, 4.5, Lemma 3.14, [20, Remark 2.4], the truth of the  $\Sigma_+^-$ -quasi-identities in  $\{(\bigcup_{i \in 2}\{(x_2 \wedge x_i) \approx (x_{1-i} \vee x_3), (x_2 \wedge \neg x_i) \approx (\neg x_{1-i} \vee x_3)\}) \rightarrow ((x_2 \wedge \neg \neg x_j) \approx (\neg \neg x_{1-j} \vee x_3)) \mid j \in 2\}$  in  $\{\mathfrak{DM}_4, \mathfrak{S}_3\}$  and the fact that pre-varieties are abstract and hereditary complete the argument.  $\square$

It is in this sense that SMSL is the implicational/[quasi]equational join of DML and SL. The lattice of its sub-quasi-varieties is found in the next Section. This task (as well as that solved in [17]) cannot be solved with using tools elaborated in [20]

TABLE 2. Maximal subdirectly-irreducibles of varieties of [bound-  
ed/] Morgan-Stone lattices[algebras].

1[+ℓ]	$[B/]MS(L[A])$	$\{\mathfrak{MS}_{6[01]}\}[UK]$
2[+ℓ]	$[B/]PS(WK)MS(L[A])$	$\{\mathfrak{MS}_{5[01]}, \mathfrak{DM}_{4[01]}\}[UK]$
3[+1][+ℓ]	$[B/]WK[M]S(L[A])$	$\{\mathfrak{MS}_{5[01]}, \mathfrak{MS}_{4:1[01]}, \mathfrak{DM}_{4[01]}\}[UK]$
5[+ℓ]	$[B/]PSWKS(L[A])$	$\{\mathfrak{MS}_{5[01]}\}[UK]$
6[+1][+ℓ]	$[B/]K[M]S(L[A])$	$\{\mathfrak{MS}_{4:i[01]} \mid i \in 2\} \cup \{\mathfrak{DM}_{4[01]}\}[UK]$
8[+1][+ℓ]	$[B/]PSK[M]S(L[A])$	$\{\mathfrak{MS}_{4:0[01]}, \mathfrak{S}_{3[01]}, \mathfrak{DM}_{4[01]}\}[UK]$
10[+ℓ]	$[B/]NDM(L[A])$	$\{\mathfrak{MS}_{4:0[01]}, \mathfrak{DM}_{4[01]}\}[UK]$
11[+ℓ]	$[B/]NK(L[A])$	$\{\mathfrak{MS}_{4:0[01]}\}[UK]$
12	$[B/]TNIMSL$	$\{\mathfrak{MS}_{2[01]}\}$
22[-ℓ]	$[B/][A]QSMS(L[A])$	$\{\mathfrak{MS}_{4:1[01]}, \mathfrak{DM}_{4[01]}\}[UK]$
23[-ℓ]	$[B/][A]QS\{W\}KS(L[A])$	$\{\mathfrak{MS}_{4:1[01]}\}[UK]$
24[-ℓ]	$[B/][A]SMS(L[A])$	$\{\mathfrak{S}_{3[01]}, \mathfrak{DM}_{4[01]}\}[UK]$
25[-ℓ]	$[B/][A]DM(L[A])$	$\{\mathfrak{DM}_{4[01]}\}[UK]$
26[-ℓ]	$[B/][A]S\{W\}KS(L[A])$	$\{\mathfrak{S}_{3[01]}, \mathfrak{K}_{3[01]}\}[UK]$
27[-ℓ]	$[B/][A]\{W\}K(L[A])$	$\{\mathfrak{K}_{3[01]}\}[UK]$
28[-ℓ]	$[B/][A]S(L[A])$	$\{\mathfrak{S}_{3[01]}\}[UK]$
29[-ℓ]	$[B/][A]B(L[A])$	$\{\mathfrak{B}_{2[01]}\}[UK]$
21	$[B/]OMS(L[A])$	$\emptyset$

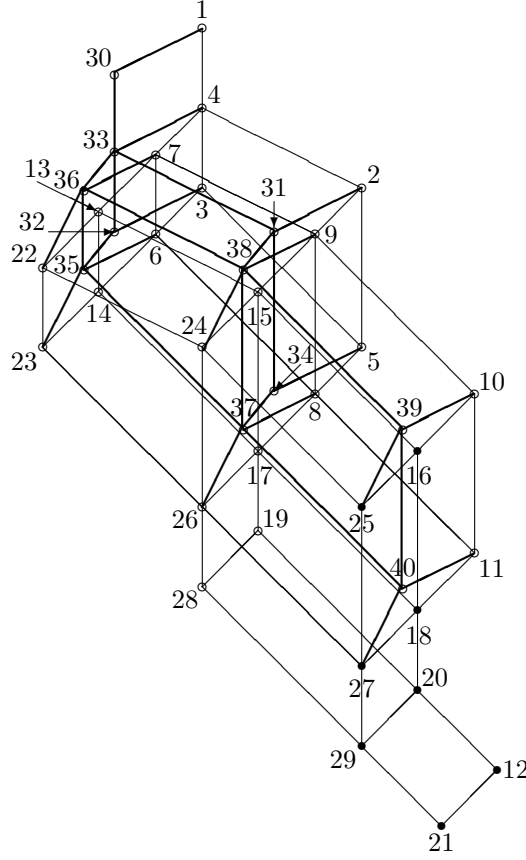


FIGURE 3. The lattice of varieties of [bounded/] Morgan-Stone lattices[algebras].

because of Proposition 5.11 therein. And what is more, despite of implicativity of {sub-varieties of} [B](A)DML and Remark 3.1, we have:

*Remark 4.8.* Clearly,  $\theta \triangleq (\Delta_3 \cup (\{1\} \times 3)) \subseteq (3^2 \setminus (\{0, 2\}^2 \setminus \Delta_{\{0,2\}}))$  forms a subalgebra of  $\mathfrak{K}_{3[0,1]}^2$ , in which case, if  $\mathfrak{K}_{3[0,1]}$  had a dual discriminator  $\delta$ , then we would have  $2 = \delta^{\mathfrak{K}_{3[0,1]}}(1, 0, 2) \theta \delta^{\mathfrak{K}_{3[0,1]}}(0, 0, 2) = 0$ , and so, by Theorem 4.4 and Corollary 2.7, no sub-variety of [B]MSL containing “the non-simple subdirectly-irreducible”  $\mathfrak{K}\mathfrak{S}_{3[0,1]}$  (viz., including [B](K|S)L; cf. Corollary 4.7) is {dual} discriminator.  $\square$

On the other hand, the majority term  $\mu_+$  for the variety of lattices, being a dual discriminator for  $\mathfrak{D}_2$ , is that for  $\{\mathfrak{B}_{2[0,1]}, \mathfrak{M}\mathfrak{S}_{2[0,1]}\}$ , in which case, by Corollary 4.7, sub-varieties of [B]ABL are dual  $\mu_+$ -discriminator, and so, by Remark 4.8, these are exactly all dual ( $\mu_+$ -)discriminator sub-varieties of [B]MSL. Nevertheless, since  $\neg x_0 \approx \top$  is true in  $\mathfrak{M}\mathfrak{S}_{2,01}$ , this is term-wise-definitionally equivalent to  $\mathfrak{D}_{2,01}$  generating the variety BDL (cf., e.g., [1] or Lemma 4.1), in its turn, being well-known (e.g., due to [4] {cf. [20, Lemma 2.10]} and existence of a three-element subdirect square of  $\mathfrak{D}_{2,01}$  with carrier  $2^2 \setminus \{\langle 0, 1 \rangle\}$ , though  $3 \neq 1$  is odd), in which case  $\mathfrak{M}\mathfrak{S}_{2[0,1]}$  has no congruence-permutation term, for, otherwise,  $\mathfrak{D}_{2,01}$  would have one, and so, by Corollaries 2.7 and 4.7, [B]BL is the only discriminator sub-variety of [B]MSL.

## 5. QUASI-VARIETIES OF STRONG MORGAN-STONE LATTICES

Given any  $K \subseteq [B]MSL$ ,  $(N)IK$  stands for the class of (non-)idempotent members of  $K$  (in which case it is the relative sub-quasi-variety of  $K$ , relatively axiomatized by the  $\Sigma_+^-$ -quasi-identity:

$$(5.1) \quad (\neg x_0 \approx x_0) \rightarrow (x_0 \approx x_1),$$

and so a quasi-variety, whenever  $K$  is so).

**Lemma 5.1.** *Any (non-one-element finitely-generated)  $\mathfrak{A} \in [B]MSL$  is non-idempotent iff  $\text{hom}(\mathfrak{A}, \mathfrak{B}_{2[0,1]}) \neq \emptyset$ , in which case  $I[B]SMSL \subseteq [B]DML$ , and so  $[B]SMSL = (NI[B]SMSL \cup [B]DML)$ . In particular,  $NIMS_{[0,1]} = \{\mathfrak{S}_{3[0,1]}, \mathfrak{B}_{2[0,1]}\}$ .*

*Proof.* The “if” part is by the fact that  $\mathfrak{B}_{2[0,1]}$  has no idempotent element. (Conversely, assume  $\text{hom}(\mathfrak{A}, \mathfrak{B}_{2[0,1]}) = \emptyset$ , in which case, by (4.21),  $\text{hom}(\mathfrak{A}, \mathfrak{S}_{3[0,1]}) = \emptyset$ , and so, for any  $h \in \text{hom}(\mathfrak{A}, \{\mathfrak{M}\mathfrak{S}_{6[0,1]}, \mathfrak{M}\mathfrak{S}_{2[0,1]}\})$ ,  $(\text{img } h) \not\subseteq (\text{img } \epsilon_3^5)$ , for, otherwise, we would have  $(h \circ (\epsilon_3^5)^{-1}) \in \text{hom}(\mathfrak{A}, \mathfrak{S}_{3[0,1]}) = \emptyset$ . Take any  $\bar{a} \in A^*$  such that  $\mathfrak{A}$  is generated by  $\text{img } \bar{a}$ . Let  $n \triangleq (\text{dom } \bar{a}) \in \omega$  and  $\bar{b} \triangleq \langle \neg^{\mathfrak{A}} \neg^{\mathfrak{A}} a_j \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} a_j \rangle_{j \in n}$ , in which case there is some  $i \in n$  such that  $h(a_i) \notin (\text{img } \epsilon_3^5)$ , and so  $h(b_i) \in \{ \langle m, m, 1 - m \rangle \mid m \in 2 \}$ . Put [by induction on any  $k \in n$ ]  $c_{1[+k]} \triangleq ((b_{0[+k]} \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} c_k) \wedge^{\mathfrak{A}} c_k)$ , in which case  $h(c_{1[+k]})$  is in  $\{ \langle i, i, j \rangle \mid \langle i, j \rangle \in (2^2 \setminus \langle 0, 0 \rangle) \}$ , for  $h(b_{0[+k]})$  is so, and so, by induction on any  $l \in ((n+1) \setminus (i+1)) \ni n$ , we see that  $h(c_l)$  is in  $\{ \langle m, m, 1 - m \rangle \mid m \in 2 \}$ , for  $h(b_i)$  is so. Then,  $h(\neg^{\mathfrak{A}} c_n) = h(c_n)$ , in which case, by (2.8) and Theorem 4.4 [resp., Corollary 4.5],  $\neg^{\mathfrak{A}} c_n = c_n$ , and so  $\mathfrak{A}$ , being non-one-element, is idempotent.) Finally, (2.8), (4.21) and Corollary 4.7 complete the argument.  $\square$

This, by (2.8), Corollary 4.7, (2.1), (2.7) with  $I = 2$  and the locality of quasi-varieties, immediately yields:

**Corollary 5.2.** *For any variety  $V \subseteq [B]MSL$  {such that either  $[B](S|B)L \subseteq V$ },  $NIV$  is the pre-/quasi-variety generated by*

$$\emptyset \cup \{ \mathfrak{A} \times \mathfrak{B}_{2[0,1]} \mid \mathfrak{A} \in (MS_{V_{[0,1]}} \setminus \{[(\mathfrak{S}|\mathfrak{B})_{(3|2)[0,1]}\})\} \cup (MS_{V_{[0,1]}} \cap \{[(\mathfrak{S}|\mathfrak{B})_{(3|2)[0,1]}\})\} \},$$

*in which case  $NI[B]MSL$  is the one generated by  $\{\mathfrak{M}\mathfrak{S}_{6[0,1]} \times \mathfrak{B}_{2[0,1]}\}$ , while*

$$NI[B]\langle S \rangle (DM||K) \langle S \rangle L$$

*is the one generated by  $\{(\mathfrak{D}\mathfrak{M}||\mathfrak{K})_{(4||3)[0,1]} \times \mathfrak{B}_{2[0,1]} \langle \mathfrak{S}_{3[0,1]} \rangle\}$ , whereas*

$$NI[B](TNI \wr O)MSL = [B]OMSL,$$

*and so any (non-one-element)  $\mathfrak{A} \in [B]MSL$  is non-idempotent iff  $\text{hom}(\mathfrak{A}, \mathfrak{B}_{2[0,1]}) \neq \emptyset$ .*

Likewise, Lemma 5.1 and [17, Proof of Lemma 4.9] immediately yield:

**Corollary 5.3.**  $\mathfrak{K}_3$  is embeddable into any member of  $SKSL \setminus NISKSL$ .

**Corollary 5.4.**  $NI[B]MSL \cup [B]TNIMSL$  is the sub-quasi-variety of  $[B]MSL$  relatively axiomatized by the  $\Sigma_+^-$ -quasi-identity:

$$(5.2) \quad (\neg x_0 \approx x_0) \rightarrow (x_0 \approx \neg x_1)$$

*and is the pre-/quasi-variety generated by  $\{\mathfrak{M}\mathfrak{S}_{6[0,1]} \times \mathfrak{B}_{2[0,1]}, \mathfrak{M}\mathfrak{S}_{2[0,1]}\}$ .*

*Proof.* Clearly, (5.2) = (5.1 $[x_1/\neg x_1]$ ) is true in both  $NI[B]MSL$  and  $\mathfrak{M}\mathfrak{S}_{2[0,1]}$ . Conversely, any  $\mathfrak{A} \in I[B]MSL$ , satisfying (5.2), has an idempotent element  $a$ , in which case, for any  $b \in A$ , as  $\mathfrak{A} \models (5.2)[x_0/a, x_1/(\neg^{\mathfrak{A}})b]$ , we have  $\neg^{\mathfrak{A}} b = a (= \neg^{\mathfrak{A}} \neg^{\mathfrak{A}} b)$ , and so  $\mathfrak{A} \in [B]TNIMSL$ . Then, Corollaries 4.7 and 5.2 complete the argument.  $\square$

Likewise, we have:

**Corollary 5.5.** *For any variety  $V \subseteq [B]MSL$  such that  $V \not\subseteq [B]\{W\}KSL$ , the class  $NIV \cup (V \cap [B]\{W\}KSL)$  is the sub-quasi-variety of  $V$  relatively axiomatized by the  $\Sigma_+^-$ -quasi-identity:*

$$(5.3) \quad (\neg x_0 \approx x_0) \rightarrow (x_0 \lesssim ((\neg\neg)x_1 \vee \neg x_1))$$

and is the pre-/quasi-variety generated by  $MS_{(V \cap [B]\{W\}KSL)_{[0,1]}} \cup \{\mathfrak{A} \times \mathfrak{B}_{2[0,1]} \mid \mathfrak{A} \in (MS_{V_{[0,1]}} \setminus \{\mathfrak{S}_{3[0,1]}, \mathfrak{B}_{2[0,1]}\})\}$ . In particular,  $NI[B]\langle S \rangle DM\langle S \rangle L \cup \langle S \rangle K\langle S \rangle L$  is the sub-quasi-variety of  $[B]\langle S \rangle DM\langle S \rangle L$  relatively axiomatized by either of (5.3) and is the pre-/quasi-variety generated by  $\{\mathfrak{DM}_{4[0,1]} \times \mathfrak{B}_{2[0,1]}, \mathfrak{K}_{3[0,1]} \langle \mathfrak{S}_{3[0,1]} \rangle\}$ .

*Proof.* Clearly, (5.3) is satisfied in  $NIV \cup (V \cap [B]\{W\}KSL)$ . Conversely, consider any  $\mathfrak{A} \in IV$  satisfying (5.3) and any  $a, b \in A$ , in which case there is some  $c \in A$  such that  $\neg^{\mathfrak{A}}c = c$ , and so, as  $\mathfrak{A}(5.3)[x_0/c, x_1/(ab)]$ , we have  $c \leq^{\mathfrak{A}} (\neg^{\mathfrak{A}}(ab) \vee^{\mathfrak{A}} \{\neg^{\mathfrak{A}}\neg^{\mathfrak{A}}\}(ab))$ . Then, by (4.2), (4.3) and (4.5) {as well as (4.6)}, we get  $(a \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}}a) \leq^{\mathfrak{A}} c$ , in which case  $\mathfrak{A} \in (V \cap [B]\{W\}KSL)$ , and so Corollaries 4.7 and 5.2 complete the argument.  $\square$

This, by Lemma 5.1 and [17, Case 8 of Proof of Theorem 4.8], immediately yields:

**Corollary 5.6.**  $\mathfrak{DM}_4$  is embeddable into any member of  $\{S\}DM\{S\}L$  not satisfying (5.3).

Members of  $[B]\{\langle [Q|P]S \rangle\}[W]K\{S\}L$ , satisfying the  $\Sigma_+^-$ -quasi-identity:

$$(5.4) \quad \{\neg x_0 \lesssim x_0, (x_0 \wedge \neg x_1) \lesssim (\neg x_0 \vee x_1)\} \rightarrow (\neg x_1 \lesssim (\neg\neg)x_1),$$

are called (*weakly-*)regular, their quasi-variety being denoted by

$$(W)R[B]\{\langle [Q|P]S \rangle\}[W]K\{S\}L \\ (= \{\langle [ \supseteq ] \rangle\}(R[B]\{\langle [Q|P]S \rangle\}[W]K\{S\}L \cup ([B]OMSL\{\langle [ [B]TNIMSL ] \rangle\}))$$

in view of (4.13)\{\langle [ (4.2) ] \rangle\}.

Given any [bounded] Morgan-Stone lattice  $\mathfrak{A} \in [B]\{\langle [Q|P]S \rangle\}(W)K\{S\}L$ , by (4.1), (4.3) and (4.5) [as well as ((4.2) and) (4.15)],  $(\mathcal{J}\mathcal{F})_{(W)}^{\mathfrak{A}} \triangleq \{a \in A \mid (\neg^{\mathfrak{A}}\neg^{\mathfrak{A}})a (\leq \mid \geq)^{\mathfrak{A}} \neg^{\mathfrak{A}}a\} \supseteq \{b (\wedge \vee)^{\mathfrak{A}} \neg^{\mathfrak{A}}b \mid b \in A\} \neq \emptyset$ , for  $A \neq \emptyset$ , is [an]a ideal|filter of  $\mathfrak{A} \uparrow \Sigma_+$  such that  $\neg^{\mathfrak{A}}[(\mathcal{J}\mathcal{F})_{(W)}^{\mathfrak{A}}] \subseteq (\mathcal{F}\mathcal{J})_{(W)}^{\mathfrak{A}}$  [in which case  $\mathfrak{R}_{(W)}^{\mathfrak{A}} \triangleq ((\mathcal{F}_{(W)}^{\mathfrak{A}} \times \{1\}) \cup (\mathcal{J}_{(W)}^{\mathfrak{A}} \times \{0\}))$ ] forms a subalgebra of  $\mathfrak{A} \times \mathfrak{B}_{2[0,1]}$  such that, for every  $d \in \mathfrak{R}_{(W)}^{\mathfrak{A}}$ ,  $(d_1 = 1) \Rightarrow (d_0 \in \mathcal{F}_{(W)}^{\mathfrak{A}})$ , and so, by Corollary 4.7, the (*weak*) regularization  $\mathfrak{R}_{(W)}(\mathfrak{A}) \triangleq ((\mathfrak{A} \times \mathfrak{B}_{2[0,1]}) \upharpoonright \mathfrak{R}_{(W)}^{\mathfrak{A}})$  of  $\mathfrak{A}$  is in  $(W)R[B]\{\langle [Q|P]S \rangle\}(W)K\{S\}L$ . Then,  $(\pi_0 \upharpoonright \mathfrak{R}^{\mathfrak{S}_{3[0,1]}}) \in \text{hom}(\mathfrak{R}(\mathfrak{S}_{3[0,1]}), \mathfrak{S}_{3[0,1]})$  is bijective, so, by Corollary 4.7,  $\mathfrak{S}_{3[0,1]} \in R[B]SKSL$ . Likewise,  $(\epsilon_2^4 \parallel \{\langle i, \langle \chi_4^{4 \setminus 3}(i) + \chi_4^{4 \setminus 1}(i), \chi_4^{4 \setminus 2}(i) \rangle \rangle \mid i \in 4\}) \in \text{hom}((\mathfrak{B} \parallel \mathfrak{K})_{(2 \parallel 4)[0,1]}, \mathfrak{K}_{4[0,1]} \parallel \mathfrak{R}(\mathfrak{K}_{3[0,1]}))$  is injective||bijective, so, by Corollary 4.7,  $(\mathfrak{B} \parallel \mathfrak{K})_{(2 \parallel 4)[0,1]} \in R[B]KL$ .

**Lemma 5.7.**

$$(W)R[B]\{\langle [Q|P]S \rangle\}[W]K\{S\}L \subseteq (NI[B]\{\langle [Q|P]S \rangle\}[W]K\{S\}L \cup [B]TNIMSL).$$

*Proof.* Consider any  $\mathfrak{A} \in (W)R[B]\{\langle [Q|P]S \rangle\}[W]K\{S\}L$  and any  $a, b \in A$  such that  $\neg^{\mathfrak{A}}a = a$ , in which case, as, for any  $c \in \{b, \neg^{\mathfrak{A}}b\}$ ,  $\mathfrak{A} \models (4.1 \parallel 5.4)[x_0/a, x_1/(c \parallel (a \wedge^{\mathfrak{A}} c))]$  (and  $\mathfrak{A} \models (4.5)[x_0/\neg^{\mathfrak{A}}a, x_1/\neg^{\mathfrak{A}}c]$ ), we have  $\neg^{\mathfrak{A}}c \leq^{\mathfrak{A}} (\neg^{\mathfrak{A}}a \vee^{\mathfrak{A}} \neg^{\mathfrak{A}}c) = \neg^{\mathfrak{A}}(a \wedge^{\mathfrak{A}} c) \leq^{\mathfrak{A}} (\neg^{\mathfrak{A}}\neg^{\mathfrak{A}})(a \wedge^{\mathfrak{A}} c) = (a \wedge^{\mathfrak{A}} (\neg^{\mathfrak{A}}\neg^{\mathfrak{A}})c) \leq^{\mathfrak{A}} (\neg^{\mathfrak{A}}\neg^{\mathfrak{A}})c$ , and so, as  $\mathfrak{A} \models (4.2(\parallel 4.6))[x_0/b]$ , we get both  $b \leq \neg^{\mathfrak{A}}\neg^{\mathfrak{A}}b \leq^{\mathfrak{A}} (\neg^{\mathfrak{A}}\neg^{\mathfrak{A}})\neg^{\mathfrak{A}}b = \neg^{\mathfrak{A}}b$ , when  $c = \neg^{\mathfrak{A}}b$ , and  $\neg^{\mathfrak{A}}b \leq^{\mathfrak{A}} (\neg^{\mathfrak{A}}\neg^{\mathfrak{A}})b$ , when  $c = b$ . Then,  $\neg^{\mathfrak{A}}b = (\neg^{\mathfrak{A}}\neg^{\mathfrak{A}})b$ , in which case, as  $\mathfrak{A} \models (4.15(\parallel 4.15))$



$[x_0/(a\|(-^{\mathfrak{A}})b), x_1/(b\|a)]$ , we have  $(-^{\mathfrak{A}})b \leq^{\mathfrak{A}} a \leq^{\mathfrak{A}} (-^{\mathfrak{A}})b$ , i.e.,  $a = (-^{\mathfrak{A}})b$ , and so (by Corollary 5.4)  $\mathfrak{A}$  is (either) non-idempotent (or totally negatively-idempotent).  $\square$

**Corollary 5.8.**  $\mathfrak{K}_4$  is embeddable into any  $\mathfrak{A} \in (\text{NIQSMSL} \setminus \text{SL}) \supseteq (\text{RQSKSL} \setminus \text{SL})$ .

*Proof.* Then, there are some  $a, b \in A$  such that  $c \triangleq (a \wedge^{\mathfrak{A}} -^{\mathfrak{A}} a) \neq d \triangleq (b \wedge^{\mathfrak{A}} c) \leq^{\mathfrak{A}} c$ , in which case, applying (4.1) and (4.3) [twice], we have  $[-^{\mathfrak{A}} -^{\mathfrak{A}} d \leq^{\mathfrak{A}} -^{\mathfrak{A}} -^{\mathfrak{A}} c \leq^{\mathfrak{A}} -^{\mathfrak{A}} c \leq^{\mathfrak{A}} -^{\mathfrak{A}} d$ , and so, by (4.2) and (4.11), we get  $-^{\mathfrak{A}} -^{\mathfrak{A}} (c|d) = (c|d)$ . In this way, as  $c \neq d$ , by (5.1), we have  $-^{\mathfrak{A}} c \neq c$ , in which case we get  $-^{\mathfrak{A}} d \neq -^{\mathfrak{A}} c$ , and so  $\{ \langle 0, d \rangle, \langle 1, c \rangle, \langle 2, -^{\mathfrak{A}} c \rangle, \langle 3, -^{\mathfrak{A}} d \rangle \}$  is an embedding of  $\mathfrak{K}_4$  into  $\mathfrak{A}$ . Finally, Lemma 5.7 completes the argument.  $\square$

**Theorem 5.9.** Let  $\mathbf{V} \triangleq [\mathbf{B}]\{(\mathbf{Q}(\|\mathbf{P}\))\mathbf{S})\}(\mathbf{W})\mathbf{K}\{\mathbf{S}\}\mathbf{L}$  and  $\mathbf{K} \triangleq (\emptyset\{\cup(\text{MS}_{\mathbf{V}[0,01]} \cap \{\mathfrak{S}_{3[0,01]}\}(\cup(\emptyset(\|\{\mathfrak{M}\mathfrak{S}_{2[0,01]}\}))))\})$ . Then,  $\mathbf{QV} \triangleq (\mathbf{W})\mathbf{R}[\mathbf{B}]\{(\mathbf{Q}(\|\mathbf{P}\))\mathbf{S})\}(\mathbf{W})\mathbf{K}\{\mathbf{S}\}\mathbf{L}$  is the pre-/quasi-variety generated by  $\mathfrak{R}_{((\mathbf{W}))}[\text{MS}_{\mathbf{V}[0,01]} \setminus \mathbf{K}] \cup \mathbf{K}$ , so  $\mathbf{R}[\mathbf{B}]\{\mathbf{S}\}\mathbf{K}\{\mathbf{S}\}\mathbf{L}$  is the one generated by  $\{\mathfrak{K}_{4[0,01]}, \mathfrak{S}_{3[0,01]}\}$ .

*Proof.* Consider any finitely-generated  $\mathfrak{A} \in (\mathbf{Q} \setminus ([\mathbf{B}]\text{OMSL}(\cup[\mathbf{B}]\text{TNIMSL})))$ . Take any  $\bar{a} \in A^+$  such that  $\mathfrak{A}$  is generated by  $\text{img } \bar{a}$ . Let  $n \triangleq (\text{dom } \bar{a}) \in (\omega \setminus 1)$  and  $b \triangleq (\wedge_+^{\mathfrak{A}} \langle -^{\mathfrak{A}} -^{\mathfrak{A}} a_m \vee^{\mathfrak{A}} -^{\mathfrak{A}} a_m \rangle_{m \in n})$ , in which case, by (4.1), (4.5) and (4.15), we have  $-^{\mathfrak{A}} b \leq^{\mathfrak{A}} b$ . Consider any  $\mathfrak{B} \in \mathbf{K}' \triangleq \{\mathfrak{M}\mathfrak{S}_{6[0,01]}, \mathfrak{M}\mathfrak{S}_{2,01}\}$  and  $h \in \text{hom}(\mathfrak{A}, \mathfrak{B})$ . Let  $(I|J) \triangleq \{i \in n \mid h(a_i) \notin (\mathcal{F}|J)_{(\mathbf{W})}^{\mathfrak{B}}\}$ ,  $(\iota|j) = |(I|J)|$  and  $\bar{k}|\bar{\ell}$  any bijection from  $\iota|j$  onto  $I|J$ . We prove, by contradiction, that there is some  $g \in \text{hom}(\mathfrak{A}, \mathfrak{B}_{2[0,01]})$  such that  $g[\text{img}((\bar{k}|\bar{\ell}) \circ \bar{a})] = \{0|1\}$ . For suppose that, for every  $g \in \text{hom}(\mathfrak{A}, \mathfrak{B}_{2[0,01]})$ , there is either some  $i \in \iota$  or some  $j \in j$  such that  $g(a_{(\bar{k}|\bar{\ell})_{i|j}}) = (1|0)$ , in which case, as, by Lemmas 5.1 and 5.7,  $\text{hom}(\mathfrak{A}, \mathfrak{B}_{2[0,01]}) \neq \emptyset$ , we have  $(I \cup J) \neq \emptyset$ , and so we are allowed to put  $c \triangleq (\vee_+^{\mathfrak{A}} ((\bar{k} \circ \bar{a} (\circ -^{\mathfrak{A}} \circ -^{\mathfrak{A}})) * (\bar{\ell} \circ \bar{a} \circ -^{\mathfrak{A}})))$ . Then,  $\pi_{0|2}(h((-^{\mathfrak{A}} -^{\mathfrak{A}})c)) = 0$ , in which case (by (4.6))  $\pi_0(h(-^{\mathfrak{A}} c)) = 1$ , and so  $-^{\mathfrak{A}} c \not\leq^{\mathfrak{A}} (-^{\mathfrak{A}} -^{\mathfrak{A}})c$ , for  $(h \circ \pi_0) \in \text{hom}(\mathfrak{A}|_{\Sigma_+}, \mathfrak{D}_2)$ . Now, consider any  $\mathfrak{C} \in \mathbf{K}'$  and  $f \in \text{hom}(\mathfrak{A}, \mathfrak{C})$ , in which case  $(\mathfrak{C}|(\text{img } f)) \in \mathbf{V} \not\cong \mathfrak{D}\mathfrak{M}_{4[0,01]}$ , in view of Corollary 4.7, and so  $(\text{img } \epsilon_4^6) \not\subseteq (\text{img } f)$ , i.e.,  $\mathfrak{S}^{\mathfrak{M}\mathfrak{S}_6} = \epsilon_4^6[2^2 \setminus \Delta_2] \not\subseteq (\text{img } f)$ . Consider the following complementary cases:

- $(\text{img } f) \subseteq (\text{img } \epsilon_3^5)$ ,  
in which case, by (4.21),  $e \triangleq (f \circ (\epsilon_3^5)^{-1} \circ \chi_3^{3 \setminus 2}) \in \text{hom}(\mathfrak{A}, \mathfrak{B}_{2[0,01]})$ , and so, by the assumption to be disproved,  $\pi_{1|2}(f(c)) = e(c) = 1$ . Then,  $f(b \wedge^{\mathfrak{A}} -^{\mathfrak{A}} c) = \langle 0, 0, 0 \rangle \leq^{\mathfrak{C}} f(-^{\mathfrak{A}} b \vee^{\mathfrak{A}} c)$ .
- $(\text{img } f) \not\subseteq (\text{img } \epsilon_3^5)$ ,  
in which case there is some  $m \in n$  such that  $f(a_m) \notin (\text{img } \epsilon_3^5) \not\subseteq \mathfrak{S}^{\mathfrak{M}\mathfrak{S}_6}$ , in which case  $f(b) \in \mathfrak{S}^{\mathfrak{M}\mathfrak{S}_6}$ , and so  $f(b \wedge^{\mathfrak{A}} -^{\mathfrak{A}} c) \leq^{\mathfrak{C}} f(b) = f(-^{\mathfrak{A}} b) \leq^{\mathfrak{C}} f(-^{\mathfrak{A}} b \vee^{\mathfrak{A}} c)$ .

Thus, anyway,  $f(b \wedge^{\mathfrak{A}} -^{\mathfrak{A}} c) \leq^{\mathfrak{C}} f(-^{\mathfrak{A}} b \vee^{\mathfrak{A}} c)$ , in which case, by (2.8) and Theorem 4.4 [resp., Corollary 4.5],  $(b \wedge^{\mathfrak{A}} -^{\mathfrak{A}} c) \leq^{\mathfrak{A}} (-^{\mathfrak{A}} b \vee^{\mathfrak{A}} c)$ , and so  $\mathfrak{A} \not\models (5.4)[x_0/b, x_1/c]$ . This contradiction to the (weak) regularity of  $\mathfrak{A}$  definitely shows that, for each  $\mathfrak{D} \in \text{MS}_{\mathbf{V}[0,01]} \subseteq \text{ISK}'$  and every  $h' \in \text{hom}(\mathfrak{A}, \mathfrak{D})$ , there is some  $g' \in \text{hom}(\mathfrak{A}, \mathfrak{B}_2)$  such that  $(\text{img } f') \subseteq \mathfrak{R}_{(\mathbf{W})}^{\mathfrak{D}}$ , where  $f' \triangleq (h' \times g')$ , in which case, by (2.7),  $f' \in \text{hom}(\mathfrak{A}, \mathfrak{R}_{(\mathbf{W})}(\mathfrak{D}))$ , while, by (2.1),  $(\ker f') \subseteq (\ker h')$ , and so the locality of quasi-varieties, (2.8) and Corollary 4.7 complete the argument.  $\square$

Thus, the apparatus of (weak) regularizations of [bounded] (weakly) Kleene-Stone lattices involved here yields a more transparent and immediate insight/proof into/to [20, Proposition 4.7].

**Lemma 5.10.**  $\mathfrak{K}_3 \times \mathfrak{B}_2$  is embeddable into any  $\mathfrak{A} \in (\text{NISKSL} \setminus \text{RSKSL})$ .

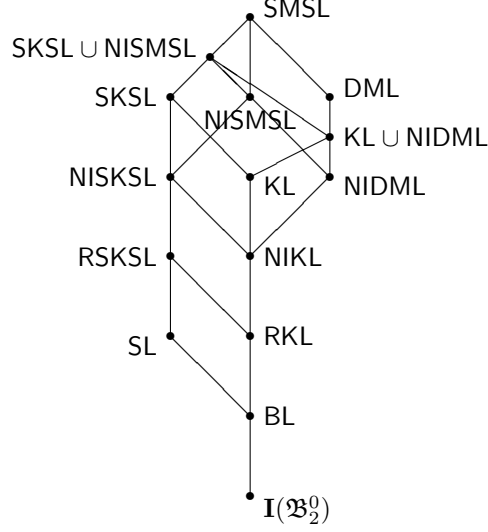


FIGURE 4. The lattice of pre-/quasi-varieties of strong Morgan-Stone lattices.

*Proof.* Then, by (4.1), (4.3), (4.5) and (4.6), there are some  $a, b \in A$  such that  $(c|d) \triangleq \neg^{\mathfrak{A}} \neg^{\mathfrak{A}}(a|b) (\geq | \not\leq)^{\mathfrak{A}} \neg^{\mathfrak{A}}(c|d)$  and  $(c \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}} d) \leq^{\mathfrak{A}} (\neg^{\mathfrak{A}} c \vee^{\mathfrak{A}} d)$ , in which case, using (4.1), (4.5) and (4.6), by induction on construction of any  $\varphi \in \text{Tm}_{\Sigma_+}^2$ , we get  $\neg^{\mathfrak{A}} \neg^{\mathfrak{A}} \varphi^{\mathfrak{A}}(c, d) = \varphi^{\mathfrak{A}}(c, d)$ , and so the subalgebra  $\mathfrak{B}$  of  $\mathfrak{A}$  generated by  $\{c, d\}$  is a non-idempotent Kleene lattice such that  $\mathfrak{B} \not\models (5.4)[x_0/c, x_1/d]$ . Hence,  $\mathfrak{K}_3 \times \mathfrak{B}_2$  being embeddable into  $\mathfrak{B}$ , by [17, Case 4 of Proof of Theorem 4.8], is so into  $\mathfrak{A}$ .  $\square$

**Lemma 5.11.**  $\mathfrak{DM}_4 \times \mathfrak{B}_2$  is embeddable into any  $\mathfrak{A} \in (\text{NISMSL} \setminus \text{SKSL})$ .

*Proof.* Then, there are some  $a, b \in A$  such that, by (4.2),  $c \triangleq \neg^{\mathfrak{A}} \neg^{\mathfrak{A}}(a \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}} a) \not\leq^{\mathfrak{A}} d \triangleq (\neg^{\mathfrak{A}} b \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} \neg^{\mathfrak{A}} b)$ , in which case, by (4.1), (4.5) and (4.6), we have both  $\neg^{\mathfrak{A}}(c|d) (\geq | \leq)^{\mathfrak{A}} (c|d) = \neg^{\mathfrak{A}} \neg^{\mathfrak{A}}(c|d)$ , and so, by induction on construction of any  $\varphi \in \text{Tm}_{\Sigma_+}^2$ , we get  $\neg^{\mathfrak{A}} \neg^{\mathfrak{A}} \varphi^{\mathfrak{A}}(c, d) = \varphi^{\mathfrak{A}}(c, d)$ . Thus, the subalgebra  $\mathfrak{B}$  of  $\mathfrak{A}$  generated by  $\{c, d\}$  is a non-idempotent De Morgan lattice such that  $\mathfrak{B} \not\models (4.15)[x_0/c, x_1/d]$ , in which case, by the proof of [17, Lemma 4.10],  $\mathfrak{DM}_4 \times \mathfrak{B}_2$  is embeddable into  $\mathfrak{B}$ , and so into  $\mathfrak{A}$ .  $\square$

**Lemma 5.12.** Let  $\mathfrak{A} \in \text{QSMSL}$  and  $a \in A$ . Suppose  $\neg^{\mathfrak{A}} \neg^{\mathfrak{A}} a \neq a$ . Then,  $b \triangleq (\neg^{\mathfrak{A}} a \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}} \neg^{\mathfrak{A}} a) \leq^{\mathfrak{A}} c \triangleq (a \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} a) \leq^{\mathfrak{A}} d \triangleq (\neg^{\mathfrak{A}} a \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} \neg^{\mathfrak{A}} a)$ , while both  $\neg^{\mathfrak{A}} c = b = \neg^{\mathfrak{A}} d$  and  $\neg^{\mathfrak{A}} b = d$ , whereas  $b \neq c \neq d$ , in which case  $\{\langle 0, b \rangle, \langle 1, c \rangle, \langle 2, d \rangle\}$  is an embedding of  $\mathfrak{S}_3$  into  $\mathfrak{A}$ , and so  $\mathfrak{S}_3$  is embeddable into any member of  $(\text{QSMSL} \setminus \text{DML})$ .

*Proof.* In that case, by (4.2),  $b \leq^{\mathfrak{A}} c \leq^{\mathfrak{A}} d$ , while, by (4.1), (4.5) and (4.6), both  $\neg^{\mathfrak{A}} c = b = \neg^{\mathfrak{A}} d$  and  $\neg^{\mathfrak{A}} b = d$ , whereas  $c \neq d$ , for, otherwise, since  $\mathfrak{A} \models (4.2|4.11)[x_0/a]$ ,  $\{b, \neg^{\mathfrak{A}} a, a, \neg^{\mathfrak{A}} \neg^{\mathfrak{A}} a, d\}$  would be a pentagon of the distributive lattice  $\mathfrak{A}|\Sigma_+$ , and so  $b \neq c$ , for otherwise, we would have  $c = b = \neg^{\mathfrak{A}} c = \neg^{\mathfrak{A}} b = d$ .  $\square$

**Theorem 5.13.** Sub-pre/quasi-varieties of SMSL form the fifteen-element non-chain distributive lattice depicted at Figure 4.

*Proof.* We use Corollary 4.7 tacitly. Clearly,  $\mathfrak{DM}_4 \times \mathfrak{B}_2$  is not in SKSL, for  $\mathfrak{DM}_4$  is not so, while  $\pi_0|(2^2 \times \Delta_2)$  is a surjective homomorphism from the former onto the latter, in which case, by Corollary 5.5,  $\text{SKSL} \subsetneq (\text{SKSL} \cup \text{NISMSL}) \subsetneq \text{SMSL}$ , for

$\text{SMSL} \ni \mathfrak{D}\mathfrak{M}_4 \not\equiv (5.3)[x_i/\langle i, 1-i \rangle]_{i \in 2}$ . Likewise,  $\mathfrak{S}_3 \notin \text{DML}$ , so, by Corollaries 5.2, 5.5 and Theorem 5.9,  $(\text{KL} \cup \text{NIDML}) \subsetneq (\text{SKSL} \cup \text{NISMSL})$ ,  $\text{NIDML} \subsetneq \text{NISMSL}$ ,  $\text{NIKL} \subsetneq \text{NISKSL}$  and  $\text{RKL} \subsetneq \text{RSKSL}$ , while, by Corollary 5.2,  $\text{NIKL} \ni (\mathfrak{K}_3 \times \mathfrak{B}_2) \not\equiv (5.4)[x_0/\langle \langle 0, 1 \rangle, \langle 1, 1 \rangle \rangle, x_1/\langle \langle \langle 0, 0 \rangle, \langle 1, 1 \rangle \rangle]$ , so, by Lemma 5.7,  $\text{RSKSL} \subsetneq \text{NISKSL}$ , as well as  $\text{KL} \ni \mathfrak{K}_3 \not\equiv (5.1)[x_0/\langle 0, 1 \rangle, x_1/\langle 0, 0 \rangle]$ , so  $\text{NISKSL} \subsetneq \text{SKSL}$ . Finally, by Theorem 5.9,  $\mathfrak{S}_3 \in \text{RSKSL} \ni \mathfrak{K}_4 \not\equiv (4.10)[x_i/(1-i)]_{i \in 2}$ , so  $\text{SL} \subsetneq \text{RSKSL}$ . Thus, by Lemma 5.1, Corollaries 5.2, 5.5, Theorem 5.9 and [17, Theorem 4.8], the fifteen quasi-varieties involved are pair-wise distinct and do form the lattice depicted at Figure 4. Now, consider any pre-variety  $\text{P} \subseteq \text{SMSL}$  such that  $\text{P} \not\subseteq \text{DML}$ , in which case, by Lemma 5.12,  $\mathfrak{S}_3 \in \text{P}$ , and so  $\text{SL} \subseteq \text{P}$ , as well as the following exhaustive cases:

- (1)  $\text{P} \not\subseteq (\text{SKSL} \cup \text{NISMSL})$ ,  
in which case, by Corollaries 5.5 and 5.6,  $\mathfrak{D}\mathfrak{M}_4 \in \text{P} \ni \mathfrak{S}_3$ , and so  $\text{P} = \text{SMSL}$ .
- (2)  $\text{P} \subseteq (\text{SKSL} \cup \text{NISMSL})$  but neither  $\text{P} \subseteq \text{SKSL}$  nor  $\text{P} \subseteq \text{NISMSL}$ ,  
in which case  $(\text{SKSL}|\text{NISMSL}) \not\subseteq (\text{P} \cap (\text{NISMSL}|\text{SKSL}))$ , and so, by Lemma| Corollary 5.11|5.3  $((\mathfrak{D}\mathfrak{M}_4 \times \mathfrak{B}_2)|\mathfrak{K}_3) \in \text{P} \ni \mathfrak{S}_3$ . Then, by Corollary 5.5,  $\text{P} = (\text{SKSL} \cup \text{NISMSL})$ .
- (3)  $\text{P} \subseteq \text{NISMSL}$  but  $\text{P} \not\subseteq \text{SKSL}$ ,  
in which case, by Lemma 5.11,  $(\mathfrak{D}\mathfrak{M}_4 \times \mathfrak{B}_2) \in \text{P} \ni \mathfrak{S}_3$ , and so, by Corollary 5.2,  $\text{P} = \text{NISMSL}$ .
- (4)  $\text{P} \subseteq \text{SKSL}$  but  $\text{P} \not\subseteq \text{NISMSL}$ ,  
in which case, by Corollary 5.3,  $\mathfrak{K}_3 \in \text{P} \ni \mathfrak{S}_3$ , and so  $\text{P} = \text{SKSL}$ .
- (5)  $\text{P} \subseteq \text{NISKSL}$  but  $\text{P} \not\subseteq \text{RSKSL}$ ,  
in which case, by Lemma 5.10,  $(\mathfrak{K}_3 \times \mathfrak{B}_2) \in \text{P} \ni \mathfrak{S}_3$ , and so, by Corollary 5.2,  $\text{P} = \text{NISKSL}$ .
- (6)  $\text{P} \subseteq \text{RSKSL}$  but  $\text{P} \not\subseteq \text{SL}$ ,  
in which case, by Corollary 5.8,  $\mathfrak{K}_4 \in \text{P} \ni \mathfrak{S}_3$ , and so, by Theorem 5.9,  $\text{P} = \text{RSKSL}$ .
- (7)  $\text{P} \subseteq \text{SL}$ ,  
in which case  $\text{P} = \text{SL}$ .

In this way, [17, Theorem 4.8] completes the argument.  $\square$

This, by Corollaries 4.7, 5.2, 5.5 and Theorem 5.9, immediately yields:

**Corollary 5.14.** *Any [pre-/quasi-]variety  $\text{P} \subseteq \text{SMSL}$  such that  $\text{P} \not\subseteq \text{DML}$  is generated by  $(\text{P} \cap \text{DML}) \cup \text{SL}$ .*

### 5.1. Relatively simple relatively subdirectly-representable pre-varieties of strong Morgan-Stone lattices and algebras.

**Lemma 5.15.** *Let  $\text{P} \subseteq [\text{B}]\text{SMSL}$  be a pre-variety. Then,  $(\text{Si}_{\text{P}}(\mathbb{Q}) \cap \text{NI}[\text{B}]\text{SMSL}) \subseteq \mathfrak{I}\mathfrak{B}_{2[0,1]} \subseteq [\text{B}]\text{BL} \subseteq [\text{B}]\text{KL} \subseteq [\text{B}]\text{DML}$ .*

*Proof.* Consider any  $\mathfrak{A} \in (\text{Si}_{\mathbb{Q}}(\mathbb{Q}) \cap [\text{B}]\text{NISMSL})$ , in which case  $|\mathfrak{A}| > 1$  [viz.,  $\perp^{\mathfrak{A}} \neq \top^{\mathfrak{A}}$ ], and so [as, by (4.7) and (4.8),  $\{ \langle 0, 0, \perp^{\mathfrak{A}} \rangle, \langle 1, 1, \top^{\mathfrak{A}} \rangle \} \in \text{hom}(\mathfrak{B}_{2,01}, \mathfrak{A})$  is injective], by Corollary 4.7 and Theorem 5.13,  $\mathfrak{B}_{2[0,1]} \in \mathbb{Q}$ . Then, by Corollary 5.2, since  $\mathfrak{B}_{2[0,1]}$  has no proper subalgebra, there is some  $h \in \text{hom}^{\text{S}}(\mathfrak{A}, \mathfrak{B}_{2[0,1]}) \neq \emptyset$ , in which case, by (2.5), as  $(\text{img } h) = \Delta_2$  is not a singleton,  $A^2 \neq (\ker h) = h_2^{-1}[\Delta_{B_2}] \in \text{Co}_{\mathbb{Q}}(\mathfrak{A}) \subseteq \{A^2, \Delta_A\}$ , and so  $h$  is injective, as required, in view of Corollary 4.7.  $\square$

**Theorem 5.16.** *Any relatively semi-simple relatively subdirectly-representable (more specifically, implicative) pre-variety  $\text{P} \subseteq [\text{B}]\text{SMSL}$  is a sub-variety of  $[\text{B}]\text{DML}$ , in which case it is  $\mathcal{U}_{V_1|\Omega, wp(\Omega)}^{\varnothing}$ -implicative, and so “{relatively} {finitely-}semi-simple”/“ $[\mathcal{U}_{V_1|\Omega, wp(\Omega)}^{\varnothing}]$ -implicative sub-{pre-}varieties of  $[\text{B}]\text{SMSL}$  are exactly sub-varieties of  $[\text{B}]\text{DML}$ .*

*Proof.* In that case,  $\mathsf{P}$  is generated by  $\mathsf{K} \triangleq \text{Si}_{\mathsf{P}}(\mathsf{P})$ , and so, by Lemmas 5.1 and 5.15,  $\mathsf{P} \subseteq [\mathsf{B}]\text{DML}$ . Consider the following complementary cases:

- $\mathsf{K} = \emptyset$ ,  
in which case  $\mathsf{P} = [\mathsf{B}]\text{OMSL}$ .
- $\mathsf{K} \neq \emptyset$ .

Consider the following complementary subcases:

- $\mathsf{K} \subseteq \text{NI}[\mathsf{B}]\text{SMSL}$ ,  
in which case, by Footnote 1 and Lemma 5.15,  $\mathsf{K} = \mathbf{IB}_{2,[01]}$ , and so, by Corollary 4.7,  $\mathsf{P} = [\mathsf{B}]\text{BL}$ .
- $\mathsf{K} \not\subseteq \text{NI}[\mathsf{B}]\text{SMSL}$ .

Consider the following complementary subcases:

- \*  $\mathsf{K} \subseteq ([\mathsf{B}]\text{SKSL} \cup \text{NI}[\mathsf{B}]\text{SMSL})$ ,  
in which case  $\text{IK} \subseteq [\mathsf{B}]\text{KL}$ , and so, by Lemma 5.15,  $\mathsf{P} \subseteq [\mathsf{B}]\text{KL}$ .  
Conversely, take any  $\mathfrak{A} \in (\text{IK} \neq \emptyset$ , in which case  $(\mathfrak{A} \upharpoonright \Sigma_+^-) \in (\text{ISKSL}$ , and so, by Corollary 5.3, there is an embedding  $e$  of  $\mathfrak{K}_3$  into  $\mathfrak{A} \upharpoonright \Sigma_+^-$ . Then, [as  $a \triangleq e(\langle 0, 1 \rangle) = \neg^{\mathfrak{A}} a$ , by (4.7) and (4.8),  $\{\langle 0, 0, \perp^{\mathfrak{A}} \rangle, \langle 0, 1, a \rangle, \langle 1, 1, \top^{\mathfrak{A}} \rangle\}$  is an embedding of  $\mathfrak{K}_{3,01}$  into  $\mathfrak{A}$ , in which case]  $\mathfrak{K}_{3,[01]} \in \mathsf{Q}$ , and so, by Corollary 4.7,  $\mathsf{P} = [\mathsf{B}]\text{KL}$ .
- \*  $\mathsf{K} \not\subseteq ([\mathsf{B}]\text{SKSL} \cup [\mathsf{B}]\text{NISMML})$ .

Take any  $\mathfrak{B} \in (\mathsf{K} \setminus ([\mathsf{B}]\text{SKSL} \cup [\mathsf{B}]\text{NISMML})) \neq \emptyset$ , in which case, by Corollaries 5.5 and 5.6, there is an embedding  $f$  of  $\mathfrak{DM}_4$  into  $\mathfrak{B} \upharpoonright \Sigma_+^-$ , and so  $\mathfrak{DM}_4 \in \mathsf{P}$  in the  $[\ ]$ -non-optional case. [By contradiction, prove that  $\mathfrak{DM}_{4,01} \in \mathsf{P}$ . For suppose  $\mathfrak{DM}_{4,01} \notin \mathsf{P}$ , in which case it is not embeddable into  $\mathfrak{B}$ , and so, by (4.7) and (4.8), both  $f(\langle 0|1, 0|1 \rangle) \neq (\perp | \top)^{\mathfrak{A}}$ . Then, by (4.7) and (4.8),  $g \triangleq (((\pi_0 \upharpoonright (2^2 \times \{\langle 0, 1 \rangle\})) \circ f) \cup \{\langle \langle 0, 0 \rangle, \langle 0, 0 \rangle, \perp^{\mathfrak{B}} \rangle, \langle \langle 1, 1 \rangle, \langle 1, 1 \rangle, \top^{\mathfrak{B}} \rangle\})$  is an embedding of  $\mathfrak{DM}_6 \triangleq ((\mathfrak{DM}_{4,01} \times \mathfrak{K}_{3,01}) \upharpoonright (\text{dom } g))$  into  $\mathfrak{B}$ , while  $\Delta_{K_3} \times \Delta_{K_3}$  is that of  $\mathfrak{K}_{3,01}$  into  $\mathfrak{DM}_6$ , whereas both  $\pi_{0 \upharpoonright 1}[DM_6] = (DM \parallel K)_{4 \parallel 3}$ , in which case  $\{\mathfrak{DM}_6, \mathfrak{K}_{3,01}\} \subseteq \mathsf{P} \not\supseteq \mathfrak{DM}_{4,01}$ , and so, by the Homomorphism Theorem, (2.5) and the filtrality of BDML with its simple members  $\mathfrak{DM}_{4,01}$  and  $\mathfrak{K}_{3,01}$ , being due to Corollary 4.7, since  $\text{Fi}(2) = \{\wp(N, 2) \mid N \in \wp(2)\}$ ,  $\text{Co}_{\{\mathsf{P}\}}(\mathfrak{DM}_6) = (\{\ker(\pi_i \upharpoonright DM_6) \mid i \in (2 \setminus \{1\})\}) \cup \{\Delta_{DM_6}, DM_6^2\}$ ). In this way, since  $\Delta_{DM_6} \subsetneq \ker(\pi_1 \upharpoonright DM_6) \subsetneq DM_6^2$ , for  $6 \neq 3 \neq 1$ ,  $\mathfrak{DM}_6 \in (\text{SI}_{\mathsf{P}}(\mathsf{P}) \setminus \text{Si}_{\mathsf{P}}(\mathsf{P}))$ , contrary to the relative semi-simplicity of  $\mathsf{P}$ .] Thus, by Corollary 4.7,  $\mathsf{P} = [\mathsf{B}]\text{DML}$ .

This, by Corollary 4.7 (and Corollary 3.4), completes the argument.  $\square$

## 6. CONCLUSIONS

Perhaps, the most acute problem remained open concerns the lattice of quasi-varieties of *all* (at least, quasi-strong) MS lattices. In this connection, perhaps, a most acute open issue what is a “reasonable” class generating  $\mathsf{R}(\text{PS})\text{KSL} \not\supseteq \mathfrak{R}_{\mathsf{W}}(\mathfrak{MG}_5) \in \text{NIPSKSL}$ , for  $\mathfrak{R}_{\mathsf{W}}(\mathfrak{MG}_5)(5.4)[x_i / \langle 1 - i, 1, 0, 1 \rangle]_{i \in 2}$ . After all, an interesting (though purely methodological) point remained open is whether the *optional* version of Corollary 5.14 can be proved directly *prior* proving Corollaries 4.7, 5.2, 5.5 as well as Theorems 5.9 and 5.13, in which case these would immediately ensue from the main results of [17]. Likewise, it would be interesting to find equational proofs (like that of (4.14)) of the rather curious inclusions such as  $[\mathsf{B}/]\text{NDM}(\text{L}/\text{A}) \subseteq [\mathsf{B}/]\text{PSMS}(\text{L}/\text{A}) \subseteq [\mathsf{B}/]\text{WKMS}(\text{L}/\text{A})$  and  $[\mathsf{B}/]\text{QSWKS}(\text{L}/\text{A}) \subseteq [\mathsf{B}/]\text{QSKS}(\text{L}/\text{A})$ , just ensuing from Corollary 4.7.

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