Anisotropic elliptic nonlinear problem with L^1-data in weighted Sobolev space variable exponent

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January 22, 2020

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***Abstract*—In this article, we prove an existence result of entropy solutions for anisotropic elliptic obstacle problem associated to the equations of the type :**

$$(\mathcal{P})\quad \begin{cases} A\,u = -\mathbf{div}\,\phi(u) = f \quad \textbf{in} \qquad \Omega \\ u = 0 \qquad \textbf{on} \quad \partial\Omega, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, $A = -\sum_{i=1}^{N} \partial_i\, a_i(x,\ u,\ \nabla u)$ is a Leray-Lions anisotropic operator acting from $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega,\ \overrightarrow{\omega})$ into its dual $W_0^{-1,\overrightarrow{p}'(\cdot)}(\Omega,\ \overrightarrow{\omega}^*)$ and $\phi_i \in C^0(\mathbb{R},\mathbb{R})$, the right hand side f belongs to and $L^1(\Omega)$.

I. Introduction

The study of the obstacle problem originated in the context of elasticity as the equations that models the shape of an elastic membrane which is pushed by an obstacle from one side affecting its shape. The resulting equation for the function whose graph represents the shape of the membrane involves two distinctive regions: in the part of the domain where the membrane does not touch the obstacle, the function will satisfy an elliptic PDE. In the part of the domain where the function touches the obstacle (contact set), the function will be a supersolution of the elliptic PDE. Everywhere, the function is constrained to stay above the obstacle. Obstacle problem is deeply related to the study of minimal surfaces and the capacity of a set in potential theory as well. Applications include the study of fluid filtration in porous media, constrained heating, elastoplasticity, optimal control and financial mathematics...

Let Ω be a bounded open subset of $\mathbb{R}^N (N \geq 2)$ with smooth boundary and let $p_i(\,.\,) \in C_+(\overline{\Omega})$ for $i = 0, 1, \ldots, N$, and consider the exponent vector $\overrightarrow{p}(\cdot) = \{p_1(\cdot), \ldots, p_N(\cdot)\}$, the vector $\overrightarrow{\omega}$ denoting a vector of measurable positive functions, i.e., $\overrightarrow{\omega} = \{\omega_1, \ldots, \omega_N\}$, with ω_i are weight measurable functions for all $i = 1, \cdots, N$.

Let us consider the weighted anisotropic Sobolev space $W^{1,\overrightarrow{p}(\cdot)}(\Omega, \overrightarrow{\omega})$, and A is the Leray-Lions operator acting from $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega,\ \overrightarrow{\omega})$ into its dual $W_0^{-1,\overrightarrow{p}'(\cdot)}(\Omega,\ \overrightarrow{\omega}^*)$ defined by $Au = -\operatorname{div} a(x,\, u,\, \nabla\, u)$.

We consider the obstacle problem associated with the following elliptic equations:

$$\begin{cases} -\sum_{i=1}^{N} \partial_i\, a_i(x,u,\nabla u) - \sum_{i=1}^{N} \partial_i\, \phi_i(u) \;=\; f \quad \text{in}\ \Omega, \\ u = 0 \qquad\qquad \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where $\phi = (\phi_1\,, \cdots, \phi_N)$ belongs to $C^0(\mathbb{R},\mathbb{R})^N$. As regards the second member, we assume that the datum f belongs to L^1 .

The problem (1) does not admit weak solution, because the function ϕ_i does not belongs to $L^1_{loc}(\Omega)$ in general. To defeat this difficulty we use the entropy solutions in this study, the notion of a entropy solution was introduced by P. Benilan et al [7]. The anisotropic elliptic obstacle problem associated elliptic problems the weighted anisotropic Sobolev space (we refer to [1], [2], [5], [8], [12] for more details), and P.-L. Lions [15] in their study of the Boltzmann equation. We mention some works in the direction of the anisotropic space such as [8], [16].

The purpose of this paper is to analyze the existence of entropy solutions for obstacle anisotropic problem (1), in the convex class

$$K_\psi := \Big\{\ u \in W_0^{1,\overrightarrow{p}(x)}(\Omega,\ \overrightarrow{\omega}(x)),\ u \geq \psi \quad \text{a.e in}\ \Omega \Big\},$$

where ψ is a measurable function on Ω such that

$$\psi^+ \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega,\ \overrightarrow{\omega}) \cap L^\infty(\Omega). \tag{2}$$

In recent years this kind of problems still attracting the interest of the researchers, we mention some works in this direction [11], [12], [16]. Moreover the non weighted case $\omega_i \equiv 1$ for any $i \in \{1, ..., N\}$treated by Y. Akdim, C. Allalou and A. Salmani (see. [4]) have proved the existence of entropy solutions for anisotropic elliptic obstacle problem like (1). Boccardo et al. in [10] studied the existence of weak solutions for nonlinear elliptic problem (1) with $Au = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(|\frac{\partial u}{\partial x_i}|^{p_i-2}\frac{\partial u}{\partial x_i}\right)$, $\phi_i(u) = 0$ for $i = 1, \cdots, N$ and the right-hand side is a bounded Radon measure on Ω.

II. Preliminaries

Let Ω be a bounded open subset of $\mathbb{R}^N (N \geq 2)$, we assume that the variable exponent $p(\cdot) : \overline{\Omega} \to [1, \infty[$ is log-Hölder continuous on Ω, that is there is a real constant $c > 0$ such that for all $x, y \in \overline{\Omega}$, $x \neq y$ with $|x-y| < \frac{1}{2}$ one has: $|p(x) - p(y)| \leq \frac{c}{-log\,|x-y|}$ and satisfying $p^- \leq p(x) \leq p^+ < \infty$ where $p^- := \text{ess}\inf_{x \in \overline{\Omega}} p(x)$; $p^+ := \text{ess}\sup_{x \in \overline{\Omega}} p(x)$.

For almost everywhere strictly positive and measurable function $w : \Omega \to \mathbb{R}$ will be called a weight. We shall denote by $L^{p(\,\cdot\,)}(\Omega, w)$ the set of all measurable functions u on Ω such that the norm

$$\|u\|_{p(x), w(x)} = \inf \left\{ \mu > 0 : \int_\Omega w(x) \left| \frac{u}{\mu} \right|^{p(x)} dx \leq 1 \right\},$$

is finite. $L^{p(\,\cdot\,)}(\Omega, w)$ is also called weighted Lebesgue space.

Proposition 1. *[1] the space* $\left(L^{p(x)}(\Omega, w), \|.\|_{p(x), w} \right)$ *is of Banach.*

Throughout the paper, we assume that w_i a weight function for any $i = 1, \ldots, N$, satisfying the conditions:

$$(\,\mathbf{A_1}\,) \qquad w_i \in L^1_{loc}(\Omega); \;\; w_i^{\frac{-1}{p_i(x)-1}} \in L^1_{loc}(\Omega).$$

The reasons why we assume ($\mathbf{A_1}$) can be found in [14].

Proposition 2. *[1] Let Ω be a bounded open subset of $\mathbb{R}^N$, and w_i be a weight function on Ω, for any $i = 1, \ldots, N$, If (A_1) is verified, then for all $i = 1, \ldots, N$ we have $L^{p_i(x)}(\Omega, w_i) \hookrightarrow L^1_{loc}(\Omega)$.*

Lets $p_i(\,.\,) \in C_+(\overline{\Omega})$ and x in Ω, and w_i are weight measurable functions for all $i = 1, \cdots, N$.

We define the following vectors $\overrightarrow{p}(.) = \{p_1(\cdot), \ldots, p_N(.)\}$ and $\overrightarrow{w}(.) = \{w_1(\cdot), \ldots, w_N(.)\}$. We denote $\partial^0 u = u$ and $\partial^i u = \dfrac{\partial u}{\partial x_i}$ for $i = 1, \ldots, N$, and

$$\underline{p} = \min\{\, p_1^-, \;\ldots,\; p_N^-\} \text{ then } \underline{p} > 1. \tag{3}$$

At present, let us consider the weighted anisotropic variable exponent Sobolev space $W^{1,\overrightarrow{p}(.)}(\Omega, \overrightarrow{w}(.))$ is defined as follow $W^{1,\overrightarrow{p}(.)}(\Omega, \overrightarrow{w}(.)) = \Big\{ u \in L^1(\Omega)$ and $\partial^i u \in L^{p_i(x)}(\Omega, w_i), \quad i = 1, ..., N \Big\}$, is a Banach space with respect to norm (see [12])

$$\|u\|_{1, \overrightarrow{p}(\,.\,), \overrightarrow{w}(.)} = \|u\|_{L^1(\Omega)} + \sum_{i=1}^N \|\partial^i u\|_{p_i(.), w_i(.)}. \tag{4}$$

We denote by $C_0^\infty(\Omega)$ the space of all functions with compact support in Ω with continuous derivatives of arbitrary order.

We define the functional space $W_0^{1,\overrightarrow{p}(.)}(\Omega, \overrightarrow{w}(.))$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,\overrightarrow{p}(.)}(\Omega, \overrightarrow{w}(.))$ with respect to the norm (4). Note that $C_0^\infty(\Omega)$ is dense in $W_0^{1,\overrightarrow{p}(.)}(\Omega, \overrightarrow{w}(.))$. By an adapted method of that of Adams [2], and by constructing an isometric isomorphism from $W^{1,\overrightarrow{p}(.)}(\Omega, \overrightarrow{w}(\,.\,))$ into $\prod_{i=1}^N L^{p_i(.)}(\Omega, w_i(.))$, we can show that if $1 \leq p_i(.) < \infty$, the space $\left(W_0^{1,\overrightarrow{p}}(\Omega, \overrightarrow{w}(.)) , \|\cdot\|_{1, \overrightarrow{p}(\,.\,), \overrightarrow{w}(.)} \right)$ is separable and reflexive if $1 < p_i(.) < \infty$, for all $i = 1, \ldots, N$.

For $p_i(.) > 1$, $W^{-1,\overrightarrow{p'}(.)}(\Omega, \overrightarrow{w^*}(.))$ designs its dual where $\overrightarrow{p'}(.)$ is the conjugate of $\overrightarrow{p}(.)$, i.e., $p_i'(.) = \dfrac{p_i(.)}{p_i(.)-1}$ and $\overrightarrow{w^*}(.) = \Big\{ w_i^*(.) = w_i^{1-p_i'(.)}(.) \,,\; i = 1, \ldots, N \Big\}$. We denote p_s the function defined by

$$p_s(x) = \frac{\underline{p}(x)s(x)}{s(x)+1},$$

we have $p_s(x) < p(x)$ a.e. in Ω, and

$$\begin{cases} p_s^*(x) = \frac{Np_s(x)}{N-p_s(x)} & \text{if } p(x)s(x) < N(s(x)+1), \\ p_s^*(x) \;\; \text{arbitrary}, & \text{else if}, \end{cases}$$

Lemma 1. *Let Ω be a smooth bounded open subset of $\mathbb{R}^N$, and suppose that $\inf w_i(.) > 0$ a.e. in Ω for all $i = 1, \ldots, N$. Let (A_1) be satisfied, we have the following continuous and compact embedding*

1) *If $\underline{p}(.) < N$, then $W_0^{1,\overrightarrow{p}(.)}(\Omega, \overrightarrow{w}(.)) \hookrightarrow\hookrightarrow L^{q(\cdot)}(\Omega)$ for all $q(.) \in \left[\underline{p}(.), p_s^*(.)\right[$,*
2) *If $\underline{p}(.) = N$, then $W_0^{1,\overrightarrow{p}(.)}(\Omega, \overrightarrow{w}(.)) \hookrightarrow\hookrightarrow L^{q(\cdot)}(\Omega)$ for all $q(.) \in [\underline{p}(.), \infty[$,*
3) *If $\underline{p}(.) > N$, then $W_0^{1,\overrightarrow{p}(.)}(\Omega, \overrightarrow{w}(.)) \hookrightarrow\hookrightarrow L^\infty(\Omega) \cap C^0(\overline{\Omega})$.*

The proof of this lemma follows from the fact that the embedding

$$W_0^{1,\overrightarrow{p}(.)}(\Omega, \overrightarrow{w}(.)) \subset W_0^{1,\overrightarrow{p_s}(.)}(\Omega) \subset W_0^{1,\underline{p}}(\Omega)$$

is continuous, and in view of the compact embedding theorem $W_0^{1,\overrightarrow{p}(.)}(\Omega, \overrightarrow{w}(.))$ for Sobolev spaces. Moreover, we consider the set

$$\mathcal{T}_0^{1,\overrightarrow{p}(.)}(\Omega, \overrightarrow{w}(.)) := \Big\{ u : \Omega \mapsto \mathbb{R}, \;\text{ measurable, such that } T_k(u) \in W_0^{1,\overrightarrow{p}(.)}(\Omega, \overrightarrow{w}(.)), \;\text{ for any } k > 0 \Big\},$$

where
$$T_k(s) = \begin{cases} s & \text{if } \;|s| \leq k, \\ k\dfrac{s}{|s|} & \text{if } \;|s| > k. \end{cases}$$

III. Basic assumptions and notion of solutions

We assume that $a_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \mapsto \mathbb{R}$ are Carathéodory functions for $i = 1, 2, \ldots, N$, which satisfies the following conditions, for all $s \in \mathbb{R}$, $\xi, \xi' \in \mathbb{R}^N$ and a. e. in $x \in \Omega$, and for $i = 1, \ldots, N$,

$$a_i(x,\; s,\; \xi)\, \xi_i \geq \alpha\, \omega_i\, |\xi_i|^{p_i}, \tag{5}$$

$$|a_i(x,s,\xi)| \leq \beta \omega_i^{\frac{1}{p_i(\cdot)}} (R_i(x) + \omega_i^{\frac{1}{p_i'(\cdot)}} |s|^{\frac{p_i(\cdot)}{p_i'(\cdot)}} + \omega_i^{\frac{1}{p_i'(\cdot)}} |\xi_i|^{p_i(\cdot)-1}), \tag{6}$$

$$(a_i(x,\ s,\ \xi)-a_i(x,\ s,\ \xi'))(\xi_i-\xi_i')>0 \quad \text{for} \quad \xi_i\neq\xi_i', \tag{7}$$

where $R_i(\,.\,)$ is a nonnegative function lying in $L^{p_i'(\,\cdot\,)}(\Omega)$ and $\alpha,\beta>0$. Moreover, we suppose that

$$\phi_i\in C^0(\mathbb{R},\mathbb{R}) \quad \text{for} \quad i=1,\dots,N, \text{ and} \tag{8}$$

$$f\in L^1(\Omega). \tag{9}$$

Lemma 2. *[3] Assume that (5) - (7) hold, let $(u_n)_n$ a sequence in $W_0^{1,\overrightarrow{p}(.)}(\Omega,\ \overrightarrow{w}(.))$ and $u\in W_0^{1,\overrightarrow{p}(.)}(\Omega,\ \overrightarrow{w}(.))$, if $u_n\rightharpoonup u$ weakly in $W_0^{1,\overrightarrow{p}(.)}(\Omega,\ \overrightarrow{w}(.))$, and $\sum_{i=1}^N\int_\Omega(a_i(x,u_n,\nabla u_n)-a_i(x,u_n,\nabla u))(D^iu_n-D^iu)dx\to 0$, then $u_n\longrightarrow u$ strongly in $W_0^{1,\overrightarrow{p}(.)}(\Omega,\ \overrightarrow{w}(.))$.*

Lemma 3. *[5] Let $(u_n)_n$ a sequence from $W_0^{1,\overrightarrow{p}(.)}(\Omega,\ \overrightarrow{w}(.))$ such that $u_n\rightharpoonup u$ weakly in $W_0^{1,\overrightarrow{p}(.)}(\Omega,\ \overrightarrow{w}(.))$. Then $T_k(u_n)\rightharpoonup T_k(u)$ weakly in $W_0^{1,\overrightarrow{p}(.)}(\Omega,\ \overrightarrow{w}(.))$.*

Lemma 4. *[3] If $u\in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega,\overrightarrow{\omega})$ then $\sum_{i=1}^N\int_\Omega\partial_iudx=0$.*

Definition 1. *A measurable function u is said to be an entropy solution for the obstacle problem (1), if $u\in\mathcal{T}_0^{1,\overrightarrow{p}(\cdot)}(\Omega,\overrightarrow{\omega})$ such that $u\geq\psi$ a.e. in Ω and*

$$\sum_{i=1}^N\int_\Omega[\,a_i(x,u,\nabla u)\,\partial_iT_k(u-\varphi)+\phi_i(u)\,\partial_iT_k(u-\varphi)]\,dx \leq\int_\Omega fT_k(u-\varphi)dx$$

for all $\varphi\in K_\psi\cap L^\infty(\Omega)$.

IV. Main results

Theorem 1. *Assuming that $(5)-(9)$ hold, there exists at least one entropy solution u of the problem (1).*

Proof : Step l : Approximate problems

We consider the following approximate problems

$$\begin{cases} u_n\in K_\psi\\ \displaystyle\sum_{i=1}^N\int_\Omega a_i(x,u_n,\nabla u_n)\partial_i(u_n-v)dx\\ \displaystyle+\sum_{i=1}^N\int_\Omega\phi_i^n(\ u_n)\partial_i(u_n-v)dx\leq\int_\Omega f_n\,(u_n-v)dx\\ \qquad\forall\,v\in K_\psi\quad\text{and}\,\forall\,k>0,\end{cases} \tag{10}$$

where $f_n=T_n(f)$ and $\phi_i^n(s)=\phi_i(T_n(s))$.

We define the operators Φ_n of K_ψ to $W_0^{-1,\overrightarrow{p}'(\cdot)}(\Omega,\overrightarrow{\omega}^*)$ by : $\langle\Phi_n\,u,v\rangle=\sum_{i=1}^N\int_\Omega\phi_i(T_n(u))\partial_ivdx$ for all $u\in K_\psi$ and $v\in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega,\overrightarrow{\omega})$.

Lemma 5. *[4] The operator $B_n=A+\Phi_n$ is pseudo-monotone and coercive in the following sense, there exists $v_0\in K_\psi$ such that $\dfrac{\langle B_nv,v-v_0\rangle}{\|v\|_{1,\overrightarrow{p}(\cdot),\overrightarrow{\omega}}}\longrightarrow\infty$ if $\|v\|_{1,\overrightarrow{p}(\cdot),\overrightarrow{\omega}}\to\infty$ for $v\in K_\psi$.*

According to Lemma 5 and Theorem 8.2 chapter 2 in [15], the problem (10) admit a least one solutions.

Step 2 : A priori estimate

Proposition 3. *Suppose that $(5)-(9)$ are hold, and if u_n is a solution of the approximate problem (10). Then the following assertion is valid: there exists a constant C such that $\sum_{i=1}^N\int_\Omega|\partial_iT_k(u_n)|^{p_i(x)}\omega_i(x)dx\leq C(k+1)\;\forall k>0$.*

Proof. Let $v=u_n-\eta T_k(u_n^+-\psi^+)$ where $\eta\geq 0$. Since $v\in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega,\overrightarrow{\omega})$ and for all η small enough, we get $v\in K_\psi$. We take v as test function in problem (10), we obtain

$$\sum_{i=1}^N\int_\Omega a_i(x,\ u_n\,,\nabla u_n)\partial_iT_k(u_n^+-\psi^+)dx$$
$$\leq\int_\Omega f_nT_k(u_n^+-\psi^+)dx+\sum_{i=1}^N\int_\Omega|\phi_i^n(u_n)||\partial_iT_k(u_n^+-\psi^+)|dx.$$

Since $\partial_{,i}T_k(u_n^+-\psi^+)=0$ on the set $\{u_n^+-\psi^+>k\}$, we pose $L:=\{u_n^+-\psi^+\leq k\}$ then

$$\sum_{i=1}^N\int_L a_i(x,\ u_n,\ \nabla u_n)\partial_i(u_n^+-\psi^+)dx$$
$$\leq\int_\Omega f_nT_k(u_n^+-\psi^+)dx+\sum_{i=1}^N\int_L|\phi_i^n(u_n)||\partial_i(u_n^+-\psi^+)|dx,$$

thus, we can write

$$\sum_{i=1}^N\int_L a_i(x,\ u_n^+,\ \nabla u_n^+)\partial_iu_n^+dx\leq\int_\Omega f_nT_k(u_n^+-\psi^+)dx$$
$$+\sum_{i=1}^N\int_L|\phi_i^n(u_n)||\partial_iu_n^+|\,\omega_i^{\frac{-1}{p_i(x)}}(x)\,\omega_i^{\frac{1}{p_i(x)}}(x)\,dx$$
$$+\sum_{i=1}^N\int_L|\phi_i^n(u_n)||\partial_i\psi^+|dx+\sum_{i=1}^N\int_L|a_i(x,u_n^+,\nabla u_n^+)\partial_i\psi^+|dx$$

Using to Young's inequalities, and accordingng to (6), we obtain

$$\sum_{i=1}^{N}\int_L a_i(x,\ u_n,\ \nabla u_n)\partial_i u_n^+ dx \le \int_\Omega f_n T_k(u_n^+ - \psi^+)\,dx$$
$$+ C_1(\alpha)\sum_{i=1}^{N}\int_L |\phi_i^n(T_{k+\|\psi\|_\infty}(u_n))|^{p_i'(x)}\,\omega_i^{\frac{-1}{p_i(x)-1}}(x)\,dx$$
$$+ \frac{\alpha}{6}\sum_{i=1}^{N}\int_L |\partial_i u_n^+|^{p_i(x)}\,\omega_i(x)\,dx$$
$$+ \sum_{i=1}^{N}\int_L |\phi_i^n(T_{k+\|\psi\|_\infty}(u_n))||\partial_i\psi^+|dx$$
$$+ \sum_{i=1}^{N}\frac{\alpha}{6}\int_L R_i(x)|^{p_i'(x)}dx + \sum_{i=1}^{N}\frac{\alpha}{6}\int_L |u_n^+|^{p_i(x)}\omega_i(x)\,dx$$
$$+ \sum_{i=1}^{N}\frac{\alpha}{6}\int_L |\partial_i u_n^+|^{p_i(x)}\,\omega_i(x)\,dx + \sum_{i=1}^{N} C_2(\alpha)\int_L |\partial_i\psi^+|^{p_i(x)}\omega_i(x)dx.$$

Combining (2), (5), (6), (7) and ($\mathbf{A_1}$), we get

$$\sum_{i=1}^{N}\int_{\{u_n^+-\psi^+\le k\}} |\partial_i u_n^+|^{p_i(x)}\,\omega_i(x)\,dx \le Ck + C' \qquad (11)$$

Since $\{x\in\Omega,\ u^+\le k\}\subset\{x\in\Omega,\ u^+-\psi^+\le k+\|\psi^+\|_\infty\}$, then

$$\sum_{i=1}^{N}\int_\Omega |\partial_i T_k(u_n^+)|^{p_i(x)}\,\omega_i(x)\,dx$$
$$= \sum_{i=1}^{N}\int_{\{u^+\le k\}} |\partial_i u_n^+|^{p_i(x)}\,\omega_i(x)\,dx$$
$$\le \sum_{i=1}^{N}\int_{\{u^+-\psi^+\le k+\|\psi^+\|_\infty\}} |\partial_i u_n^+|^{p_i(x)}\,\omega_i(x)\,dx. \qquad (12)$$

Hence, thanks to (11), we get

$$\sum_{i=1}^{N}\int_\Omega |\partial_i T_k(u_n^+)|^{p_i(x)}\,\omega_i(x)\,dx$$
$$\le (k+\|\psi^+\|_\infty)C + C' \quad \forall k>0. \qquad (13)$$

Similarly taking $v = u_n + T_k(u_n^-)$ as test function in approximate problem (10), we have

$$\sum_{i=1}^{N}\int_\Omega |\partial_i T_k(u_n)|^{p_i(x)}\,\omega_i(x)\,dx \le C''(k+1). \qquad (14)$$

By (13) and (14), we obtain

$$\sum_{i=1}^{N}\int_\Omega |\partial T_k(u_n)|^{p_i(x)}\omega_i(x)\,dx \le (k+\|\psi^+\|_\infty+1)C' \ \forall k>0.$$

□

Step 3 : Strong convergence of truncations

Proposition 4. *Let u_n be a solution of approximate problem* (10)*. Then there exists a measurable function u and a subsequence of u_n such that*

$$T_k(u_n)\to T(u) \quad \textit{strongly in} \quad W_0^{1,\overrightarrow{p}(\cdot)}(\Omega,\overrightarrow{\omega}).$$

Proof. According to Proposition 3, we obtain

$$\|T_k(u_n)\|_{W_0^{1,\overrightarrow{p}(\cdot)}(\Omega,\overrightarrow{\omega})} \le C(k+\|\psi^+\|_\infty+1)^{\frac{1}{\underline{p}}}. \qquad (15)$$

Firstly, we will show that $(u_n)_n$ is a Cauchy sequence in measure in Ω. For all $\lambda>0$, we obtain $\{|u_n-u_m|>\lambda\}\subset\{|u_n|>k\}\cup\{|u_m|>k\}\cup\{|T_k(u_n)-T_k(u_m)|>\lambda\}$ which implies that

$$\text{meas}\,\{|u_n-u_m|>\lambda\} \le \text{meas}\{|u_n|>k\} + \text{meas}\{|u_m|>k\}$$
$$+ \text{meas}\,\{|T_k(u_n)-T_k(u_m)|>\lambda\}. \qquad (16)$$

Using Hölder's inequality, Lemma 1 and (15), we have

$$\begin{aligned} \text{k.meas}\{|u_n|>k\} &= \int_{\{|u_n|>k\}} |T_k(u_n)|dx \le \int_\Omega |T(u_n)dx \\ &\le (\text{meas}(\Omega))^{\frac{1}{\underline{p}}}\|T_k(u_n)\|_{L^{\underline{p}}(\Omega)} \\ &\le C(\text{meas}(\Omega))^{\frac{1}{\underline{p}}}\|T_k(u_n)\|_{W_0^{1,\overrightarrow{p}(\cdot)}(\Omega,\overrightarrow{\omega})} \\ &\le C(k+\|\psi^+\|_\infty+1)^{\frac{1}{\underline{p}}}. \end{aligned}$$

Then $\text{meas}\{|u_n|>k\}\le C\left(\frac{1}{k^{-1+\underline{p}}}+\frac{1+\|\psi^+\|_\infty}{k\underline{p}}\right)^{\frac{1}{\underline{p}}}\to 0$ as $k\to\infty$. Which implies that, for all $\varepsilon>0$, there exists k_0 such that $\forall k>k_0$, we get

$$\text{meas}\{|u_n|>k\}\le\frac{\varepsilon}{3} \quad \text{and} \quad \text{meas}\{|u_m|>k\}\le\frac{\varepsilon}{3}. \qquad (17)$$

Since the sequence $(T_k(u_n))_n$ is bounded in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega,\overrightarrow{\omega})$ there exists a subsequence $(T_k(u_n))_n$ such that $T(u_n)$ converges to v_k a.e. in Ω, weakly in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega,\overrightarrow{\omega})$ and strongly in $L^{\underline{p}}(\Omega)$ as n tends to ∞. Then the sequence $(T_k(u_n))_n$ is a Cauchy sequence in measure in Ω, thus for all $\lambda>0$, there exists n_0 such that

$$\text{meas}\{|T_k(u_n)-T_k(u_m)|>\lambda\}\le\frac{\varepsilon}{3}, \qquad \forall n,m\ge n_0. \qquad (18)$$

Using (16), (17) and (18), then $\forall\,\lambda,\varepsilon>0$, we have

$$meas\{|u_n-u_m|>\lambda\}\le\varepsilon \qquad \forall n,m\ge n_0.$$

Hence $(u_n)_n$ is a Cauchy sequence in measure in Ω, then there exists a subsequence denoted by $(u_n)_n$ such that u_n converges to a measurable function u a.e. in Ω and

$$T_k(u_n)\rightharpoonup T(u) \text{ weakly in } W_0^{1,\overrightarrow{p}(\cdot)}(\Omega,\overrightarrow{\omega})\text{and a.e.in}\Omega\,\forall k>0. \qquad (19)$$

Now, we will show that

$$\lim_{n\to\infty}\sum_{i=1}^{N}\int_\Omega (a_i(x,T_k(u_n),\nabla T_k(u_n))-$$
$$a_i(x,T_k(u_n),\nabla T_k(u)))(\partial_i T_k(u_n)-\partial_i T_k(u))dx = 0. \qquad (20)$$

Let consider $v = u_n + T_1(u_n - T_m(u_n))^-$ as test function in approximate problem (10), we have

$$-\sum_{i=1}^{N}\int_\Omega a_i(x,u_n,\nabla u_n)\partial_i T_1(u_n - T_m(u_n))^-\,dx - \sum_{i=1}^{N}\int_\Omega \phi_i^n(u_n)\partial_i T_1(u_n - T_m(u_n))^-\,dx \le -\int_\Omega f_n T_1(u_n - T_m(u_n))^- dx.$$

We pose $L^- := \{-(m+1) \le u_n \le -m\}$, Then

$$\sum_{i=1}^{N}\int_{L^-} a_i(x,u_n,\nabla u_n)\partial_i u_n\,dx + \sum_{i=1}^{N}\int_{L^-} \phi_i(u_n)\partial_i u_n\,dx \le -\int_\Omega f_n T_1(u_n - T_m(u_n))^-\,dx. \quad (21)$$

We pose $\Phi_i^n(s) = \int_0^s \phi_i^n(t)\chi_{L^-}dt$. Then using the Green's formula, we obtain

$$\sum_{i=1}^{N}\int_{L^-} \phi_i(u_n)\partial_{,i} u_n\,dx = \sum_{i=1}^{N}\int_\Omega \partial_i \Phi_i^n(u_n)\,dx = 0.$$

Then, we have

$$\sum_{i=1}^{N}\int_{L^-} a_i(x,u_n,\nabla u_n)\partial_i u_n\,dx \le -\int_\Omega f_n T_1(u_n - T_m(u_n))^- dx \quad (22)$$

According to Lebesgue's theorem, we get

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_\Omega f_n T_1(u_n - T_m(u_n))^-\,dx = 0$$

Then, we get

$$\lim_{m\to\infty}\limsup_{n\to\infty}\sum_{i=1}^{N}\int_{L^-} a_i(x,u_n,\ \nabla u_n)\partial_i u_n\,dx = 0. \quad (23)$$

Similarly, we take $v = u_n - \eta T_1(u_n - T_m(u_n))^+$ as test function in approximate problem (10), we pose $L^+ := \{m \le u_n \le m+1\}$, then

$$\lim_{m\to\infty}\limsup_{n\to\infty}\sum_{i=1}^{N}\int_{L^+} a_i(x,\ u_n,\ \nabla u_n)\partial_i u_n\,dx = 0. \quad (24)$$

We define the following function of one real variable:

$$h_m(s) = \begin{cases} 1 & \text{if } |s| \le m \\ 0 & \text{if } |s| \ge m+1 \\ m+1-|s| & \text{if } m \le |s| \le m+1, \end{cases}$$

where $m > k$. Let consider $\varphi = u_n - \eta(T_k(u_n) - T(u))^+ h_m(u_n)$ as test function in approximate problem (10), we obtain

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega a_i(x,\ u_n,\ \nabla u_n)\partial_i(T_k(u_n) - T(u))^+ h_m(u_n)dx \\ &+ \sum_{i=1}^{N}\int_\Omega a_i(x,\ u_n\ ,\ \nabla u_n)(T_k(u_n) - T_k(u))^+\partial_i u_n h_m'(u_n)\,dx \\ &+ \sum_{i=1}^{N}\int_\Omega \phi_i^n(u_n)\partial_i(T_k(u_n) - T_k(u))^+ h_m(u_n)\,dx \\ &+ \sum_{i=1}^{N}\int_\Omega \phi_i^n(u_n)\partial_i u_n(T_k(u_n) - T_k(u))^+ h_m'(u_n)\,dx \\ &\le \int_\Omega f_n(T_k(u_n) - T_k(u))^+ h_m(u_n)\,dx.\end{aligned} \quad (25)$$

Using (23) and (24), we get the second integral in (25) converges to zero as n and m tend to ∞. Since $h_m(u_n) = 0$ if $|u_n| > m+1$, we obtain

$$\sum_{i=1}^{N}\int_\Omega \phi_i^n(u_n)\partial_i(T_k(u_n) - T_k(u))^+ h_m(u_n)dx = \sum_{i=1}^{N}\int_\Omega \phi_i(T_{m+1}(u_n))h_m(u_n)\partial_i(T_k(u_n) - T_k(u))^+\,dx. \quad (26)$$

By Lebesgue's theorem, we get $\phi_i^n(T_{m+1}(u_n))h_m(u_n) \to \phi_i(T(u))h_m(u)$ in $L^{p_i'}(\Omega,\omega_i^*)$ and $\partial_i T_k(u_n) \rightharpoonup \partial_i T(u)$ weakly in $L^{p_i}(\Omega,\omega_i(x))$ as n goes to ∞, then the third integral in (25) converges to zero as n and m tend to ∞.

Combining (5), (23), (24) and Lebesgue's theorem, we have

$$\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^{N}\int_{L^-} |\partial_i u_n|^{p_i(x)}(T_k(u_n) - T_k(u))^+\,\omega_i(x)\,dx = 0, \quad (27)$$

and

$$\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^{N}\int_{L^+} |\partial_i u_n|^{p_i(x)}(T_k(u_n) - T_k(u))^+\,\omega_i\,dx = 0. \quad (28)$$

We conclude that

$$\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^{N}\int_\Omega a_i(x,\ u_n,\ \nabla u_n)\partial_i(T_k(u_n) - T_k(u))^+ h_m(u_n)dx \le 0,$$

which implies that, if we take $L_k := \{T_k(u_n) - T_k(u) \geq 0, |u_n| \leq k\}$ and $L'_k := \{T_k(u_n) - T_k(u) \geq 0, |u_n| > k\}$

$$\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^{N}\int_{L_k} a_i(x,u_n,\nabla u_n)\partial_i(T_k(u_n)-T_k(u))h_m(u_n)\,dx$$
$$-\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^{N}\int_{L'_k} a_i(x,u_n,\nabla u_n)\partial_i T_k(u)h_m(u_n)\,dx \leq 0.$$

Since $h_m(u_n) = 0$ in $\{|u_n| > m+1\}$, we obtain

$$\sum_{i=1}^{N}\int_{L'_k} a_i(x,\ u_n\ ,\nabla u_n)\partial_i T_k(u)h_m(u_n)dx$$
$$= \sum_{i=1}^{N}\int_{L'_k} a_i(x,\ T_{m+1}(u_n),\ \nabla T_{m+1}(u_n))\partial_i T_k(u)h_m(u_n)dx.$$

Since $(a_i(x,T_{m+1}(u_n),\nabla T_{m+1}(u_n)))_{n\geq 0}$ is bounded in $L^{p'_i(\cdot)}(\Omega,\omega_i^*)$ we have $a_i(x,\ T_{m+1}(u_n),\ \nabla T(u))$ converges to Y_m^i weakly in $L^{p'_i}(\Omega,\omega_i^*)$. Hence

$$\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^{N}\int_{L'_k} a_i(x,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\partial_i T_k(u)h_m(u_n)dx$$
$$= \lim_{m\to\infty}\sum_{i=1}^{N}\int_{\{|u|>k\}} Y_m^i\partial_i T_k(u)h_m(u)\,dx = 0,$$

as results

$$\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^{N}\int_{\{T_k(u_n)-T_k(u)\geq 0\}} a_i(x,\ T_k(u_n),\ \nabla T_k(u_n))$$
$$\partial_i(T_k(u_n)-T_k(u))h_m(u_n)\,dx \leq 0. \quad (29)$$

Moreover, we have $a_i(x,T_k(u_n),\nabla T_k(u))h_m(u_n) \to a_i(x,T_k(u),\nabla T_k(u))h_m(u)$ in $L^{p'_i(\cdot)}(\Omega,\omega_i^*)$ and $\partial_i(T_k(u_n)-T_k(u))$ converges to 0 weakly in $L^{p_i(\cdot)}(\Omega,\omega_i)$, then

$$\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^{N}\int_{\{T_k(u_n)-T_k(u)\geq 0\}} a_i(x,\ T_k(u_n),\ \nabla T_k(u))$$
$$\partial_i(T_k(u_n)-T_k(u))h_m(u_n)\,dx = 0. \quad (30)$$

According to (7), (29) and (30), we deduce

$$\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^{N}\int_{\{T_k(u_n)-T_k(u)\geq 0\}}(a_i(x,T_k(u_n),\nabla T_k(u_n))$$
$$-a_i(x,T_k(u_n),\nabla T_k(u)))\partial_i(T_k(u_n)-T_k(u))h_m(u_n)dx = 0. \quad (31)$$

Similarly, we choose $\varphi = u_n + (T_k(u_n)-T_k(u))^- h_m(u_n)$ as test function in approximate problem (10), we have,

$$\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^{N}\int_{\{T_k(u_n)-T_k(u)\leq 0\}}(a_i(x,T_k(u_n),\nabla T_k(u_n))$$
$$-a_i(x,T_k(u_n),\nabla T_k(u)))\partial_i(T_k(u_n)-T_k(u))h_m(u_n)dx = 0. \quad (32)$$

Using (31) and (32), we have

$$\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^{N}\int_{\Omega}(a_i(x,T_k(u_n),\nabla T_k(u_n))$$
$$-a_i(x,T_k(u_n),\nabla T_k(u)))\partial_i(T_k(u_n)-T_k(u))h_m(u_n)dx = 0. \quad (33)$$

Now, we show

$$\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^{N}\int_{\Omega}(a_i(x,T_k(u_n),\nabla T_k(u_n)) - a_i(x,T_k(u_n)$$
$$,\nabla T_k(u)))\partial_i(T_k(u_n)-T_k(u))(1-h_m(u_n))dx = 0. \quad (34)$$

Let $\varphi = u_n + T_k(u_n)^-(1-h_m(u_n))$ as test function in approximate problem (1), we obtain

$$\begin{aligned}
&-\sum_{i=1}^{N}\int_{\Omega} a_i(x,\ u_n,\ \nabla u_n)\partial_i T_k(u_n)^-(1-h_m(u_n))dx\\
&+\sum_{i=1}^{N}\int_{\Omega} a_i(x,\ u_n,\ \nabla u_n)\partial_i u_n T_k(u_n)^- h'_m(u_n)dx\\
&-\sum_{i=1}^{N}\int_{\Omega}\phi_i(u_n)\partial_i T_k(u_n)^-(1-h_m(u_n))dx\\
&+\sum_{i=1}^{N}\int_{\Omega}\phi_i(u_n)\partial_i u_n T_k(u_n)^- h'_m(u_n)dx\\
&\leq -\int_{\Omega} f_n T_k(u_n)^-(1-h_m(u_n))dx.
\end{aligned} \quad (35)$$

Thanks to (23) and (24), we obtain

$$\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^{N}\int_{\Omega} a_i(x,\ u_n,\ \nabla u_n)\partial_i u_n T_k(u_n)^- h'_m(u_n)dx = 0.$$

Then the second integral in (35) converges to zero as n and m goes to ∞. Since $\partial_i T_k(u_n)^- \rightharpoonup \partial_i T_k(u)^-$ in $L^{p_i(\cdot)}(\Omega,\omega_i)$ and $\phi_i(T_k(u_n))(1-h_m(u_n)) \to \phi_i(T_k(u))(1-h_m(u))$ strongly in $L^{p'_i(\cdot)}(\Omega,\omega_i^*)$, we get

$$\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^{N}\int_{\Omega}\phi_i(u_n)\partial_i T_k(u_n)^-(1-h_m(u_n))dx$$
$$= \lim_{m\to\infty}\sum_{i=1}^{N}\int_{\Omega}\phi_i(T_k(u))\partial_i T_k(u)^-(1-h_m(u))dx.$$

In view to Lebesgue's theorem, we have

$$\lim_{m\to\infty}\sum_{i=1}^{N}\int_{\Omega}\phi_i(T_k(u))\partial_i T_k(u)^-(1-h_m(u))dx = 0.$$

Hence the third integral in (35) converges to zero as n and m tends to ∞.

We set $\Phi_i^n(t) = \int_0^t \phi_i(s)T_k(s)^- h'_m(s)ds$, in sight to Green's Formula, we obtain

$$\sum_{i=1}^N \int_\Omega \phi_i^n(u_n)\partial_i u_n T_k(u_n)^- h'_m(u_n)\,dx = \sum_{i=1}^N \int_\Omega \partial_i \Phi_i^n(u_n)\,dx = 0.$$

Then the fourth integral in (35) converges to zero as n and m tend to ∞. Using to Lebesgue's theorem, we get the integral on the right hand in (35) converges to zero as n and m goes to ∞. We Conclude

$$\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^N \int_{\{u_n\le 0\}} a_i(x,u_n,\nabla u_n)\partial_i T_k(u_n)(1-h_m(u_n))dx = 0. \quad (36)$$

Following this, for η small enough, we choose $\varphi = u_n - \eta T_k(u_n^+ - \psi^+)(1-h_m(u_n))$ as test function in approximate problem (10), we have

$$\begin{aligned}
&\sum_{i=1}^N \int_\Omega a_i(x,u_n,\nabla u_n)\partial_i T_k(u_n^+ - \psi^+)(1-h_m(u_n))dx \\
&- \sum_{i=1}^N \int_\Omega a_i(x,u_n,\ \nabla u_n)\partial_i u_n T_k(u_n^+ - \psi^+)h'_m(u_n)dx \\
&+ \sum_{i=1}^N \int_\Omega \phi_i^n(u_n)\partial_i T_k(u_n^+ - \psi^+)(1-h_m(u_n))dx \\
&- \sum_{i=1}^N \int_\Omega \phi_i^n(u_n)\partial_i u_n T_k(u_n^+ - \psi^+)h'_m(u_n)dx \\
&\le \int_\Omega f_n T_k(u_n^+ - \psi^+)(1-h_m(u_n))dx.
\end{aligned} \quad (37)$$

Thanks to Hölder's inequality, (5), (23) and (24), we get

$$\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^N \int_\Omega \phi_i^n(u_n)\partial_i u_n T_k(u_n^+ - \psi^+)h'_m(u_n)dx = 0.$$

Using the Young's inequality we have

$$\begin{aligned}
&\sum_{i=1}^N \int_\Omega a_i(x,\ u_n,\ \nabla u_n)\partial_i T_k(u_n^+ - \psi^+)(1-h_m(u_n))dx \le \\
&\sum_{i=1}^N \int_{L^-} a_i(x,\ u_n,\ \nabla u_n)\partial_i u_n T_k(u_n^+ - \psi^+)dx \\
&+ \int_\Omega f_n T_k(u_n^+ - \psi^+)(1-h_m(u_n))dx \\
&+ \sum_{i=1}^N \int_{\{u_n^+ - \psi^+ \le k\}} \phi_i^n(u_n)\partial_i u_n^+(1-h_m(u_n))dx \\
&+ \sum_{i=1}^N \int_{\{u_n^+ - \psi^+ \le k\}} \phi_i^n(u_n)\partial_i \psi^+(1-h_m(u_n))dx
\end{aligned} \quad (38)$$

Thank to (23), we obtain the first integral on the right hand converges to zero as n and m tend to ∞. By Lebesgue's theorem, we have the second integral in the right hand converges to zero as n and m tend to ∞. Since

$$\begin{aligned}
&\sum_{i=1}^N \int_{\{u_n^+ - \psi^+ \le k\}} \phi_i^n(u_n)\partial_i u_n^+(1-h_m(u_n))dx \\
&= \sum_{i=1}^N \int_\Omega \phi_i^n(T_{\{k+\|\psi^+\|_{L^\infty(\Omega)}\}}(u_n)) \\
&\quad \partial_i T_{\{k+\|\psi^+\|_{L^\infty(\Omega)}\}}(u_n^+)(1-h_m(u_n))dx.
\end{aligned} \quad (39)$$

As $\partial_i T_{\{k+\|\psi^+\|_{L^\infty(\Omega)}\}}(u_n^+) \rightharpoonup \partial_i T_{\{k+\|\psi^+\|_{L^\infty(\Omega)}\}}(u^+)$ weakly in $L^{p_i(\cdot)}(\Omega,\omega_i)$ and $\phi_i^n(T_{\{k+\|\psi^+\|_{L^\infty(\Omega)}\}}(u_n))(1-h_m(u_n)) \to \phi_i(T_{\{k+\|\psi^+\|_{L^\infty(\Omega)}\}}(u))(1-h_m(u))$ strongly in $L^{p'_i(\cdot)}(\Omega,\omega_i^*)$ we obtain

$$\begin{aligned}
&\sum_{i=1}^N \int_\Omega \phi_i^n(T_{\{k+\|\psi^+\|_{L^\infty(\Omega)}\}}(u_n))\partial_i T_{\{k+\|\psi^+\|_{L^\infty(\Omega)}\}}(u_n^+) \\
&(1-h_m(u_n))dx = \sum_{i=1}^N \int_\Omega \phi_i(T_{\{k+\|\psi^+\|_{L^\infty(\Omega)}\}}(u)) \\
&\partial_i T_{\{k+\|\psi^+\|_{L^\infty(\Omega)}\}}(u)(1-h_m(u))dx + \varepsilon(n)\ .
\end{aligned}$$

Using the Lebesgue's theorem, we have

$$\lim_{m\to\infty}\sum_{i=1}^N \int_\Omega \phi_i(T_{\{k+\|\psi^+\|_{L^\infty(\Omega)}\}}(u))\partial_i T_{\{k+\|\psi^+\|_{L^\infty(\Omega)}\}}(u)(1-h_m(u))dx = 0. \quad (40)$$

Hence, we get the third integral converges to zero as n and m tend to ∞. Similarly as (36), we have

$$\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^N \int_{\{u_n>0\}} a_i(x,u_n,\nabla u_n)\partial_i T_k(u_n)(1-h_m(u_n))dx = 0. \quad (41)$$

According to (36) and (41), we obtain

$$\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^N \int_\Omega a_i(x,u_n,\nabla u_n)\partial_i T_k(u_n)(1-h_m(u_n))dx = 0. \quad (42)$$

Furthermore, we have

$$\begin{aligned}
&\sum_{i=1}^N \int_\Omega (a_i(x,T_k(u_n),\nabla T_k(u_n)) - a_i(x,T_k(u_n),\nabla T_k(u))) \\
&(\partial_i T_k(u_n) - \partial_i T_k(u))dx = \sum_{i=1}^N \int_\Omega (a_i(x,T_k(u_n),\nabla T_k(u_n)) \\
&- a_i(x,T_k(u_n),\nabla T_k(u)))(\partial_i T_k(u_n) - \partial_i T_k(u))h(u_n)dx \\
&+ \sum_{i=1}^N \int_\Omega (a_i(x,T_k(u_n),\nabla T_k(u_n)))\partial_i T_k(u_n)(1-h_m(u_n))dx
\end{aligned}$$

$$
-\sum_{i=1}^{N}\int_{\Omega}(a_i(x,T_k(u_n),\nabla T_k(u_n)))\partial_i T_k(u)(1-h_m(u_n))dx
$$

$$
-\sum_{i=1}^{N}\int_{\Omega}(a_i(x,T_k(u_n),\ \nabla T_k(u)))(\partial_i T_k(u_n)-\partial_i T_k(u))
$$

$$
(1-h_m(u_n))dx.
$$

Combining (33) and (42), the first and the second integrals on the right hand side converge to zero as n and m tend to ∞.

As $(a_i(x,T_k(u_n),\nabla T_k(u_n)))_n$ is bounded in $L^{p_i'(.)}(\Omega,\omega_i^*)$ and $\partial_i T_k(u)(1-h(u_n))$ converge to zero in $L^{p_i(.)}(\Omega,\omega_i)$ as n and m tend to ∞, then the third integral on the right hand side converge to zero as n and m tend to ∞. Where

$$
\begin{aligned}
a_i(x,T_k(u_n),\nabla T_k(u_n))(1-h(u)) \\
\longrightarrow a_i(x,T_k(u),\nabla T_k(u))(1-h(u)) \quad (43)
\end{aligned}
$$

strongly in $L^{p_i'(.)}(\Omega,\omega_i^*)$ and $\partial_i T_k(u_n)\rightharpoonup \partial_i T(u)$ weakly in $L^{p_i(.)}(\Omega,\omega_i)$ we obtain the fourth integral on the right hand side converge to zero as n and m tend to ∞. Then, we obtain (20).

Thanks to (19), (20) and Lemma 2, we have

$$
T_k(u_n)\rightarrow T(u)\ \text{ strongly in } W_0^{1,\overrightarrow{p}(\cdot)}(\Omega,\overrightarrow{\omega})\,\text{and a.e. in}\ \Omega\ \forall\, k>0.
$$

$\square$

Step 4 : Passing to the limit. Let $\varphi\in K_\psi\cap L^\infty(\Omega)$, we chose $v=u_n-T_k(u_n-\varphi)$ as test function in approximate problem (10), we have

$$
\sum_{i=1}^{N}\int_{\Omega}a_i(x,u_n,\nabla u_n)\partial_i T_k(u_n-\varphi)dx+\sum_{i=1}^{N}\int_{\Omega}\phi_i^n(u_n)
$$

$$
\partial_i T_k(u_n-\varphi)dx\le\int_{\Omega}f_n T_k(u_n-\varphi)dx,
$$

which implies that,

$$
\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_{k+\|\varphi\|_\infty}(u_n),\nabla T_{k+\|\varphi\|_\infty}(u_n))\partial_i T_k(u_n-\varphi)dx
$$

$$
+\sum_{i=1}^{N}\int_{\Omega}\phi_i(T_{k+\|\varphi\|_\infty}(u_n))\partial_i T_k(u_n-\varphi)dx\le\int_{\Omega}f_n T_k(u_n-\varphi)dx.
$$

As $T_k(u_n)\rightarrow T(u)$ strongly in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega,\overrightarrow{\omega})$ and a.e. in Ω for all $k>0$, we obtain

$$
\begin{aligned}
a_i(x,T_{k+\|\varphi\|_\infty}(u_n),\nabla T_{k+\|\varphi\|_\infty}(u_n))\rightharpoonup \\
a_i(x,T_{k+\|\varphi\|_\infty}(u),\nabla T_{k+\|\varphi\|_\infty}(u))
\end{aligned}
$$

weakly in $L^{p_i'(.)}(\Omega,\omega_i^*)$

$$
\phi_i(T_{k+\|\varphi\|_\infty}(u_n))\rightarrow\phi_i(T_{k+\|\varphi\|_\infty}(u))\ \text{ strongly in } L^{p_i'(.)}(\Omega,\omega_i^*)
$$

and $\partial_i T_k(u_n-\varphi)\rightarrow\partial_i T_k(u-\varphi)$ strongly in $L^{p_i}(\Omega,\omega_i)$ we can pass to limit in

$$
\begin{cases}
u_n\in K_\psi\\
\displaystyle\sum_{i=1}^{N}\int_{\Omega}a_i(x,u_n,\nabla u_n)\partial_i\,T_k(u_n-\varphi)dx\\
+\sum_{i=1}^{N}\int_{\Omega}\phi_i^n(u_n)\partial_i T_k(u_n-\varphi)dx\le\int_{\Omega}f_n\ T_k(u_n-\varphi)dx\\
\forall\varphi\in K_\psi\cap L^\infty(\Omega)\quad\text{and}\ \forall\,k>0,
\end{cases}
$$

this completes the proof of theorem 1.

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