On (θ, φ)- Derivations on Lie Ideal of Semiprime Rings

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July 27, 2021

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Abstract. Let R be a semiprime ring of characteristic $\neq 2$. Earlier the properties of Lie rings of derivations in commutative differentially prime rings R was investigated by many authors. In recent manuscript we find the conditions on semiprime rings R, when the left (θ,φ)-derivations is acting on Lie ideal of R. In particular we prove that if A is a nonzero Lie ideal and a subring of a characteristic $\neq 2$ of a semiprime ring R and d is a (θ,φ)-derivation of R satisfaiying the condition $d(ab) = d(ba) \forall a, b \in A$, then $A \subseteq Z(R)$ and $[R,R] \subseteq Z(R)$.

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2000 *Mathematics Subject Classification*: Primary 16W25, 16N60; Secondary 16B60
Key words : Derivation, prime and semiprime rings, Lie Ideal

1. Introduction

Let R be an associative ring with identity (with respect to the addition "+" and the multiplication "·") and

$Z(R) = \{z \in R : zx = xz \forall x \in R\}$ denotes the center of R.

Recall that R is prime if $aRb = 0$ implies that $a = 0$ or $b = 0$. A ring R is said to be semiprime if $aRa = 0$ implies that $a = 0$. The ring R is 2-torsion free if whenever $2a = 0$, with $a \in R$, then $a = 0$.

An additive mapp $d : R \to R$ is called a derivation of R if

$$d(ab) = d(a)b + ad(b) \forall a, b \in R.$$

The set of all derivations of R denoted by $\mathrm{Der}\, R$, $[a, b] = ab - ba$ is called a Lie commutator of $a, b \in R$, $[R, R]$ the commutator subgroup,

$annT = \{r \in R : rT = 0 = Tr\}$ the annihilator of $T \subseteq R$.

An additive map d is called (θ, φ) derivation if

$$d(ab) = d(a)\theta(b) + \varphi(a)d(b) \forall a, b \in R$$

such that $\theta, \varphi : R \to R$ are two maps of R. An additive map $\delta : R \to R$ is called a left derivation if

$$\delta(ab) = a\delta(b) + b\delta(a) \forall a, b \in R.$$

In the same manner the additive maps (θ, φ) of R is called a left θ, φ-derivation if $\delta(ab) = \theta(a)\delta(b) + \varphi(b)\delta(a) \forall a, b \in R$.

All other definitions are standard and it can be found in [**?**, **?**, **?**, **?**] and [**?**].

Recall that an additive subgroup A of R^+ is called a Lie ideal of R if $[a, r] \in A$.

In [**?**] H.E. Bell and L.C. Kappe proved that if d is a derivation on a semiprime ring, such that d is endomorphism or anti-endomorphism on R, then the derivation d must equal zero.

Further, if a derivation d is acting as a homomorphism or anti-homomorphism on a nonzero right ideal of a prime ring R, then the derivation d must equal zero also. In addition, many authors have extended results of [**?**].

In this way Yengul and Argac [**?**] proved that these results is true for α-derivations of prime rings and semiprime rings and O. Golbasi and N. Aydin [**?**] have extended these results which such that if$d$ is (θ,φ)-derivations which acts as a homomorphism or an anti-homomorphism on a prime ring R, then $d = 0$ on R.

Therefore, M. Asharf [**?**] concering (θ,φ)-derivations d which is acting as a homomorphism or an anti-homomorphism on a nonzero ideal A of a prime ring R. Also in [**?**] M. Ashraf, N. Rehman studied this result for a left (θ,φ)- derivation d which is acting as a homomorphism or an anti-homomorphism on a nonzero ideal A of a prime ring R.

There are different results related to the property of commutativity of a ring and the existence specific types of derivations of a ring R.
In [**?**] proved for a semiprime ring R if there exists a nonzero ideal A of R and d is derivation satisfying the condition $d(ab) = d(ba)\forall a, b \in A$, then $A \subseteq Z(R)$.

Furthermore, in[**?**] proved that if d is (θ,φ)-derivation acting as a homomorphism or an anti-homomorphism on a nonzero left ideal A of a semiprime ring R, then $d = 0$.

Finally, this problem has been activety studied by many authors as [**?**, **?**, **?**, **?**, **?**, **?**] and others

2. Preliminaries

we will state some lemmas, which helps us to prove the main results, also we will prove that if A is a nonzero Lie ideal and a subring of a characteristic $\neq 2$ of a semiprime ring R and d is a (θ,φ)- derivation of R satisfaiying the condition $d(ab) = d(ba)\forall a, b \in A$, then $A \subseteq Z(R)$ and $[R, R] \subseteq Z(R)$.

Lemma 2.1. [**?**]
Let R be a prime ring of a characteristic $\neq 2$. Let θ,φ be two automorphisms of R and d is a nonzero (θ,φ)-derivation of R such that $d(ab) = d(ba)\forall a, b \in R$, then R is commutative.

Lemma 2.2. [?, Lemma 2]
If $A \nsubseteq Z(R)$ is a Lie ideal of a prime ring R of a characteristic $\neq 2$ and $a, b \in R$ such that $aAb = 0$, then $a = 0$ or $b = 0$.

Lemma 2.3. [?, Lemma 1.3]
If R is a 2-torsion-free semiprime ring and A is a commutative Lie ideal of R, then $A \subseteq Z(R)$.

Lemma 2.4 ([?]). *If R is a 2-torsion-free prime ring and A a nonzero Lie ideal of R. Assume that θ and φ are two automorphisms of R and (θ, φ)- derivation of R satisfaiying the condition $d(A) = \{0\}$, then $d = 0$ on R or $A \subseteq Z(R)$.*

Lemma 2.5. *Let R be a prime ring and A a nonzero Lie ideal of R. If $r \in R$ such that $Ar = 0$, then $r = 0$.*

Proof.
Let A be an ideal of a prime ring R. and let $[a, r] \in A, \forall a \in A$ and $\forall r \in R$. Then $xar = 0 \forall a \in A$, $\forall r \in R$. This means that $xRr = 0$. Since R is a prime ring, so $x = 0$ or $r = 0, \forall a \in A$. Using that A is a nonzero Lie ideal of R, hence $r = 0$. ☐

3. Derivation on Ideals

Lemma 3.1. [?]
Let R be a semiprime ring and A a right ideal of R, then $Z(A) \subseteq Z(R)$.

Now we can get the same result in lemma (2 -1), when we apply the condition $d(ab) = d(ba) \forall a, b \in A$ a nonzero ideal of R.

Proposition 3.2.
Let R be a semiprime ring of a characteristic $\neq 2$ and A a nonzero ideal of R. Let θ, φ be two automorphisms of R and d be a nonzero (θ, φ)-derivation of R satisfying the condition $d(ab) = d(ba) \forall a, b \in A$, then R is commutative.

Proof.
Suppose that $s \in A$ such that $d(s) = 0$. Now let $s = [a, b]$, then

$$d(t)\theta(s) = \delta(ts) = d(st) = \varphi(s)\delta(t), \forall t \in A.$$

Hence

$$[d(t), s]_{\theta,\varphi} = 0 \forall t \in A. \quad (3-1)$$

From $d(s) = 0$ and A is an ideal of R, then by [?] [Theorem 1], we have $s \in Z(A) \forall s \in A$, since $[a, b] \in Z(A) \forall a, b \in A$, we get

$$[r, [a, b] = 0 \forall a, b, r \in A. \quad (3-2)$$

Now replacing b by ab in (3 -2), we have

$$[r, [a, ab]] = [r, a][a, b] = 0 \forall a, b, r \in A. \quad (3-3)$$

Further, replacing b by ba in (3 -3), we get

$$[r, a][r, br] = [r, a]b[a, r] = 0 \forall a, b, r \in A.$$

Then

$$[r, a]A[a, r] = \{0\} \forall a, r \in A.$$

Since A is an ideal of R, we have
$[r, a]RA[a, r] = \{0\} \forall a, r \in A$. Since R is prime, then
$[r, a] = 0$ or $A[a, r] = 0 \forall a, r \in A$.

Now if $A[a, r] = 0 \forall a, r \in A$ and since A is a nonzero ideal of R, we have $[a, r] = 0 \forall a, r \in A$. Hence A is commutative. Then by Lemma (3 -2), we have $Z(A) \subseteq Z(R)$. So $A \subseteq Z(R)$, hence by Lemma (3 -2) R is commutative.

□

4. Derivation on Lie Ideals

Let R be a semiprime ring of a characteristic $\neq 2$. We will extend Lemma (2 - 1) for a non zero Lie Ideal and a subring A of R.

Theorem 4.1.

Let R be a semiprime ring of a characteristic $\neq 2$, and let A be a non zero Lie ideal and subring of R. Assume that (θ,φ) are two automorphisms of R and Let $\delta : R \to R$ is (θ,φ)-derevation of R such that $\delta(ab) = \delta(ba) \forall a, b \in A$, then $A \subseteq Z(R)$ and $[R, R] \subseteq Z(R)$.

Proof.

Assume that $\delta(c) = 0, c \in A$. Now, if $c = [a, b]$, then $\delta(a)\theta(c) = \delta(ab) = \delta(ca) = \varphi(c)\delta(a), \forall a \in A$. Then

$$[\delta(a), c]_{\theta,\varphi} = 0 \forall a \in A.$$

From $\delta(c) = 0$ and using [**?**, Theorem 1]], we have $c \in Z(A), \forall c \in A$. Since $[a, b] \in Z(A) \forall a, b \in A$, then for all $a, b, d \in A$ we have

$$[d, [a, b]] = 0. \qquad (4-1)$$

Further, replacing b by ab in (4 -1), we have for all $a, b, d \in A$ we have

$$[d, [a, ab] = [d, a][a, b] = 0. \qquad (4-2)$$

Now, replacing b by bd in (4 -2), we have for all $a, b, d \in A$

$$[d, a][a, bd] = [d, a]b[a, d] = 0. \qquad (4-3)$$

Since for all $a, d \in A, [a, d] \in Z(A)$ and for all $a, d \in A$ and from (4 -3) we have $b[d, a]^2 = 0$.

Further, replacing b by $[t, r], t \in A, r \in R$ then we get $\forall a, d, t \in A$ and $r \in R$.

$$[t, r][d, a]^2 = 0.$$

Then

$$0 = tr[d, a]^2 - rt[d, a]^2 = tr[d, a]^2 \forall a, t, d \in A \forall r \in R.$$

Hence for all $a, d \in A$ we write $AR[d, a]^2 = 0$.

Since R is a prime ring, then $\forall a, d \in A$, we have $A = 0$ or $[d, a]^2 = 0$.

On the other hand A is a nonzero Lie ideal of R, then $[d, a]^2 = 0$ for all $a, d \in A$. Also, since A is a semiprime, then $\forall a, d \in A$ we have $[d, a] = 0$, hence A is commutative. Now from Lemma (2.4) we have that $A \subseteq Z(R)$. Since $A \subseteq Z(R) \forall a \in A, r \in R$, then we have $[a, r] = 0$.

Further, replace a by $[a, u], \forall u \in R$, we have $\forall a \in A, r, u \in R$.

$$0 = [[a, u], r] = [au, r] - [ua, r] = a[u, r] - [u, r]a = [a, [u, r]].$$

Thus $\forall a \in A, r, u \in R$ we have

$$[a, [R, R]] = 0.$$

Hence $[R, R] \subseteq Z(R)$.

□

Lemma 4.2. [?]
Let R be a prime ring and A be a nonzero ideal of R. Assume that θ,φ are two automorphisms of R and $\delta : R \to R$ is a left (θ,φ)-derivation of R. Then
1) If δ acts as a homomorphism on A, then $\delta = 0$ on R.
2) If δ acts as an-antihomomorphism on A, then $\delta = 0$ on R.

Lemma 4.3. [?, Lemma 4]
Let R be a prime ring of a characteristic $\neq 2$, and if A is a Lie ideal of R such that $A \nsubseteq Z(R)$ and $a,b \in R$ such that $aAb = \{0\}$. Then $a = 0$ or $b = 0$.

Lemma 4.4. *Let R be a prime ring of a characteristic $\neq 2$ and A is a Lie ideal of R such that $A \nsubseteq Z(R)$. Then there exists an ideal B of R with $[B,R] \subseteq A$, but $[B,R] \nsubseteq Z(R)$.*

Proof.
Since $A \nsubseteq Z(R)$ and $charR \neq 2$, then from [?] $[A,A] \neq 0$ and we get $[B,R] \subseteq A$ where $B = R[A,A]R \neq 0$ is the ideal of R. Thus it is follows $[B,R] \subseteq Z(G)$, since if we suppose that $[B,R] \nsubseteq Z(R)$ then $[B[B,R] = 0$. Hence $B \subseteq Z(G)$. From $B \neq 0$ is an ideal of R,hence $R = Z(R)$.
∎

Lemma 4.5. *Let R be a prime ring of a characteristic $\neq 2$ and A is a Lie ideal of R such that $A \subseteq Z(R)$. If $cAd = 0$. Then $c = 0$ or $d = 0$.*

Proof.
Let B an deal of R with $[B,R] \nsubseteq Z(R)$, but $[B,R] \subseteq A$. Suppose that $a \in A$, $b \in B$ and $r \in R$, $[bca,r] \in [B,R] \subseteq A$. Then

$$0 = c[bca,r]d = c[bc,r]ad + cbc[a,r]d = c(bcr - rbc)ad = cbcrad.$$

Since $c[a,r]d \in cAd = 0$. Hence $cBcRAd = 0$.
If $c \neq 0$ and since R is a prime then, we have $Ad = 0$. Now if $t \in R$ and $a \in A$, then

$$(at - ta) \in A,$$

whence $(at - ta)d = 0$. Hence $atd = 0$, this means $aRd = 0$. Since $A \neq 0$ we have $d = 0$. □

Theorem 4.6.
Let R be a prime ring of a characteristic $\neq 2$. Let A be a nonzero Lie ideal of R such that $a^2 \in A$ $\forall a \in A$.
Suppose that θ, φ are two automorphisms of R and $\delta : R \to R$ is a left (θ, φ)-derivation of R satisfies the following conditions.

1) If δ acts as a homomorphism on A. Then $\delta = 0$ or $A \subseteq Z(R)$.

2) If δ acts as an anti-homomorphism on A. Then $\delta = 0$ or $A \subseteq Z(R)$.

Proof.
1) Suppose that $A \nsubseteq Z(R)$. Now If δ acts as a homomorphism on A, then $\forall a, b \in A$ we have

$$\delta(ab) = \delta(a)\delta(b) = \theta(a)\delta(b) = \varphi(b)\delta(a). \qquad (4-4)$$

Further, replacing a by $2ab$ in (4 -4) and since $charR \neq 2$, then, $\forall a, b \in A$ we have

$$(\theta(a)\delta(b) + \varphi(b)\delta(a)) = \theta(a)\theta(b)\delta(b) + \varphi(b)\delta(a)\delta(b).$$

Hence, $\forall a, b \in A$ we get

$$\theta(a)\delta(a)\delta(b) = \theta(a)\theta(b)\delta(b).$$

Then $\forall a, b \in A$.

$$\theta(a)(\delta(b) - \theta(b)\delta(b)) = 0. \qquad (4-5)$$

Replacing a by $2ac$, where $c \in A$ in (4 -5) and since $charR \neq 2$, then, $\forall a, b, c \in A$ we have

$$\theta(ac)(\delta(b) - \theta(b))\delta(b)) = 0.$$

Thus $\forall a, b, c \in A$ we get

$$ac\theta^{-1}((\delta(b) - \theta(b))\delta(b)) = 0.$$

Then $\forall a, b \in A$ we have

$$aA\theta^{-1}((\delta(b) - \theta(b))\delta(b)) = 0.$$

Using Lemma (2 -2) we have $\forall a, b \in A$

$a = 0$ or $(\delta(b) - \theta(b))\delta(b) = 0$.
Notice that A is a nonzero ideal of R, then $\forall b \in a$ we have

$$\delta^2(b) = \theta(b))\delta(b),$$

since δ is a left (δ, θ)-derivation of R, then $\forall b \in A$ we have

$$\theta(b)\delta(b) - \varphi(b)\delta(b) = \theta(b)\delta(b).$$

So

$$\varphi(b)\delta(b) = 0, \forall b \in A. \qquad (4-6)$$

On Linearizing (4 -6) we get

$$0 = \varphi(b+a)\delta(b+a)$$

$$= (\varphi(b) + \varphi(a))(\delta(b) + \delta(a))$$

$$= \varphi(b)\delta(b) + \varphi(b)\delta(a) + \varphi(a)\delta(b) + \varphi(a)\delta(a)$$

$$= \varphi(b)\delta(a) + \varphi(a)\delta(b)\forall a, b \in A. \qquad (4-7)$$

Now substitute a by $2ab$ in (4 -7) and using that $char R \neq 2$, we have

$$0 = \varphi(b)\delta(b)\delta(a) + \varphi(b)\varphi(a)\delta(b)$$

$$= \varphi(b)\varphi(a)\delta(b).$$

This means that $ba\varphi^{-1}(\delta(b)) = 0 \forall a, b \in A$. That is $bA\varphi^{-1}(\delta(b)) = 0$. By Lemma (2 -2), we get $b = 0$ or $\delta(b) = 0 \forall b \in A = 0$.
Since A is a nonzero Lie ideal of R, we get

$$\delta(b) = 0, \forall b \in A. \qquad (4-8)$$

Now replacing b by $[b, r], r \in R$ in (4 -8) we will get

$$0 = \delta([b, r]) = \delta(br - rb) = \delta(br) - \delta(rb)$$

$$= \theta(b)\delta(r) + \varphi(r)\delta(b) - \theta(r)\delta(b) - \varphi(b)\delta(r)$$

$$= \theta(b)\delta(r) - \varphi(b)\delta(r)$$

$$= (\theta(b) - \varphi(b))\delta(r).$$

So

$$(\theta(A)-\varphi(A))\delta(r)=0, \forall r\in R.$$

Since θ, φ are two automorphisims of R and A is a nonzero Lie ideal of R, we see that $\theta(A), \varphi(A)$ are a nonzero Lie ideal of R. Then $\theta(A)-\varphi(A)$ is a nonzero Lie ideal of R. Now using Lemma (2 -1), we have $\delta(r)=0$, this means $\delta=0$ on R.

2) Suppose that δ acts as an anti-homomorphism on A, then for all $a, b\in A$ we have

$$\delta(ab)=\delta(b)\delta(a)=\theta(a)\delta(b)+\varphi(b)\delta(a). \qquad (4-9)$$

Substituating b by $2ab$ in (4-9) and from that $charR\neq 2$ we have $\forall a, b\in A$, then we get

$$\delta(ab)\delta(a)=\theta(a)\delta(b)\delta(a)+\varphi(b)\delta(a)\delta(a)$$

$$=\theta(a)\delta(b)\delta(a)+\varphi(a)\varphi(b)\delta(a).$$

Then

$$\varphi(a)\delta(b)\delta(a)=\varphi(a)\varphi(b)\delta(a), \forall a, b\in A. \qquad (4-10)$$

Replacing in (4 -10) with b by $2sb$, $s\in A$ and using $charR\neq 2$, we have for all $a, b, s\in A$,

$$\varphi(s)\varphi(b)\delta(a)\delta(b)=\varphi(a)\varphi(s)\varphi(b)\delta(a). \qquad (4-11)$$

Using (4 -10) in (4 -11) we have for all $a, b, s\in A$,

$$[\varphi(a)\ ,\varphi(s)]\varphi(b)\delta(a)=0.$$

Thus for all $a, b, s\in A$, we get

$$[a, s]b\varphi^{-1}(\delta(a))=0,$$

equivalently,

$$[a, s]A\varphi^{-1}(\delta(a))=0.$$

From Lemma (2 -2) we have $=[a, r]=0$ or $\delta(a)=0$. Suppose that

$$B=\{a\in A, [a, s]=0\forall s\in A\},$$

and

$$C = \{a \in A, \delta(a) = 0\}.$$

Then B, C are proper subgroups of A and $A = B \cup C$, hence $A = B$ or $A = C$.

Now if $A = B$, then for all $a, s \in A$, $[a, s] = 0$.

This means that A is commutative. Using lemma (2 -3) we have $A \subseteq Z(R)$ and we get a contradiction.

Then for all $a \in A, \delta(a) = 0$.

Replacing a by $[a, r]$ we have for all $a \in A$ and $r \in R$,

$$0 = \delta([a, r]) = \delta(ar - ra)$$

$$= \theta(a)\delta(r) + \varphi(r)\delta(a) - \theta(r)\delta(a) - \varphi(a)\delta(r)$$

$$\theta(a)\delta(r) + \varphi(a)\delta(r) - (\theta(a) - \varphi(a))\delta(r).$$

Eqivalently

$$(\theta(A) - \varphi(A))\delta(r) = 0.$$

Now using the same technique as in part (1) we get the required.

□

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