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Abstract: This paper teaches us how to build a *Hyperbolic Lattice-Grid*. From this Hyperbolic Lattice-Grid, we introduce the *Hyperbolic Sieve of Primes and Products xy*. Then we apply these properties to organize all the products xy in an array.
Keywords: Sieve of Primes, elementary number theory.
2010 Mathematics Subject Classification: 11N35, 11N36, 11A05, 11A51.

1. Introduction

In this chapter, we will introduce a new Hyperbolic Sieve of Primes and Products xy. The basis of that Hyperbolic Sieve is the lattice-grid. The dots in the lattice-grid assume hyperbolic values of the form $Y[y] = xy$.

For this lattice-grid, we will give the name of *Hyperbolic Lattice Grid*.

The *Hyperbolic Lattice-Grid* allows us not only to sieve the Primes numbers, but it will also be possible to organize in the XY plane all the products of the form xy with no repetition of the Integers divisors x and y.

2. Constructing the Hyperbolic Lattice-Grid

Let's construct the Hyperbolic Lattice-Grid in an XY-plane.

1. First, we draw all the circles of the form $x^2 + y^2 = n$ where $n \in \mathbb{N}$. So, each circle will have a radius given by $r = \sqrt{n}$.
2. Second, for each $r = \sqrt{n}$, we draw all the lines of the forms $x = r = \sqrt{n}$ and $y = r = \sqrt{n}$.

These two steps produce our Hyperbolic Lattice-Grid.

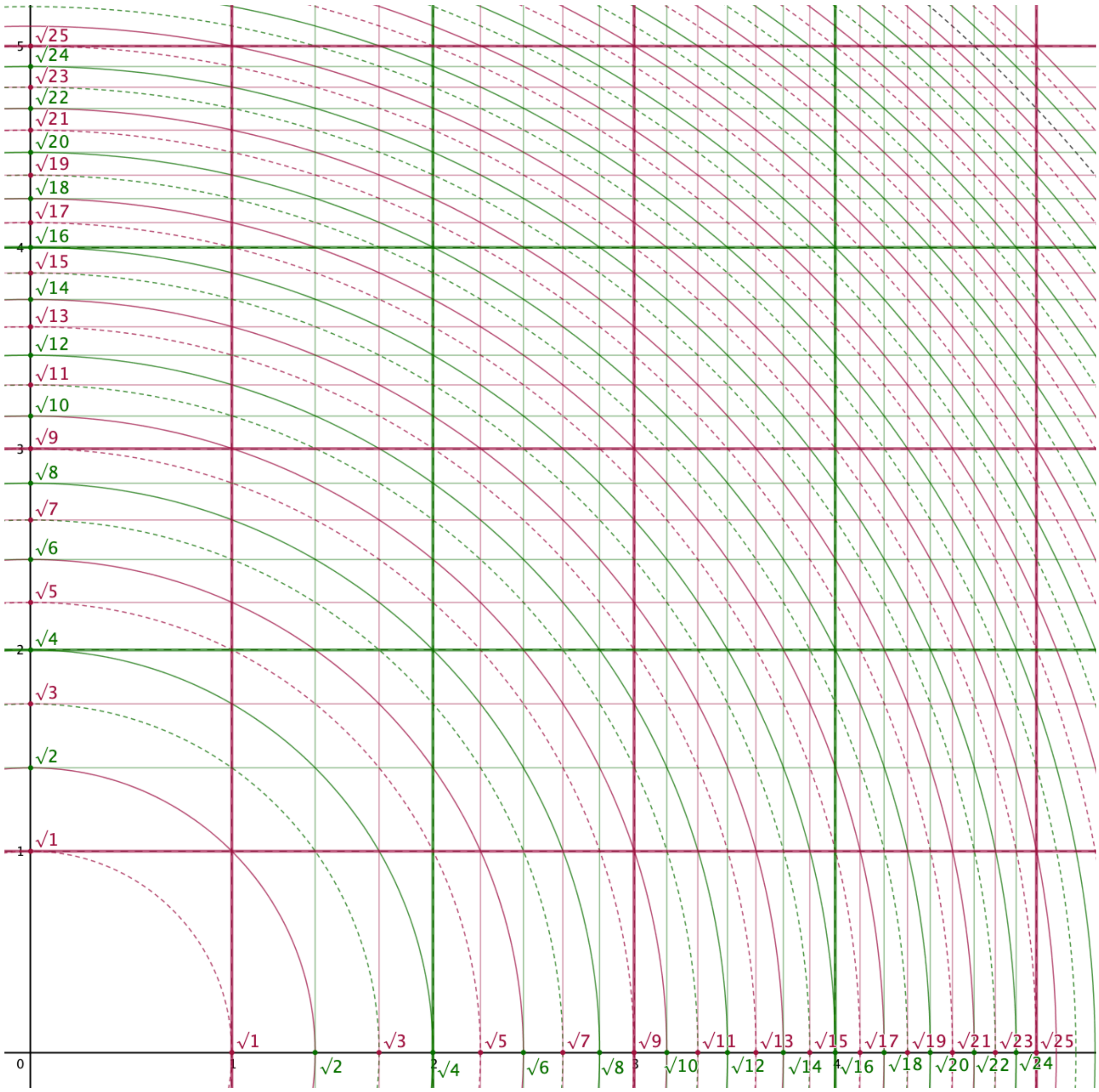

Figure 1. The Hyperbolic Lattice-Grid construction.

Note that, in each dot generated by the crossing between the vertical lines, horizontal lines, and the circles, we can assign a value $Y[y]$ to the dot that is the product of its coordinates (x, y):

$$Y[y] = xy$$

See the figure below:

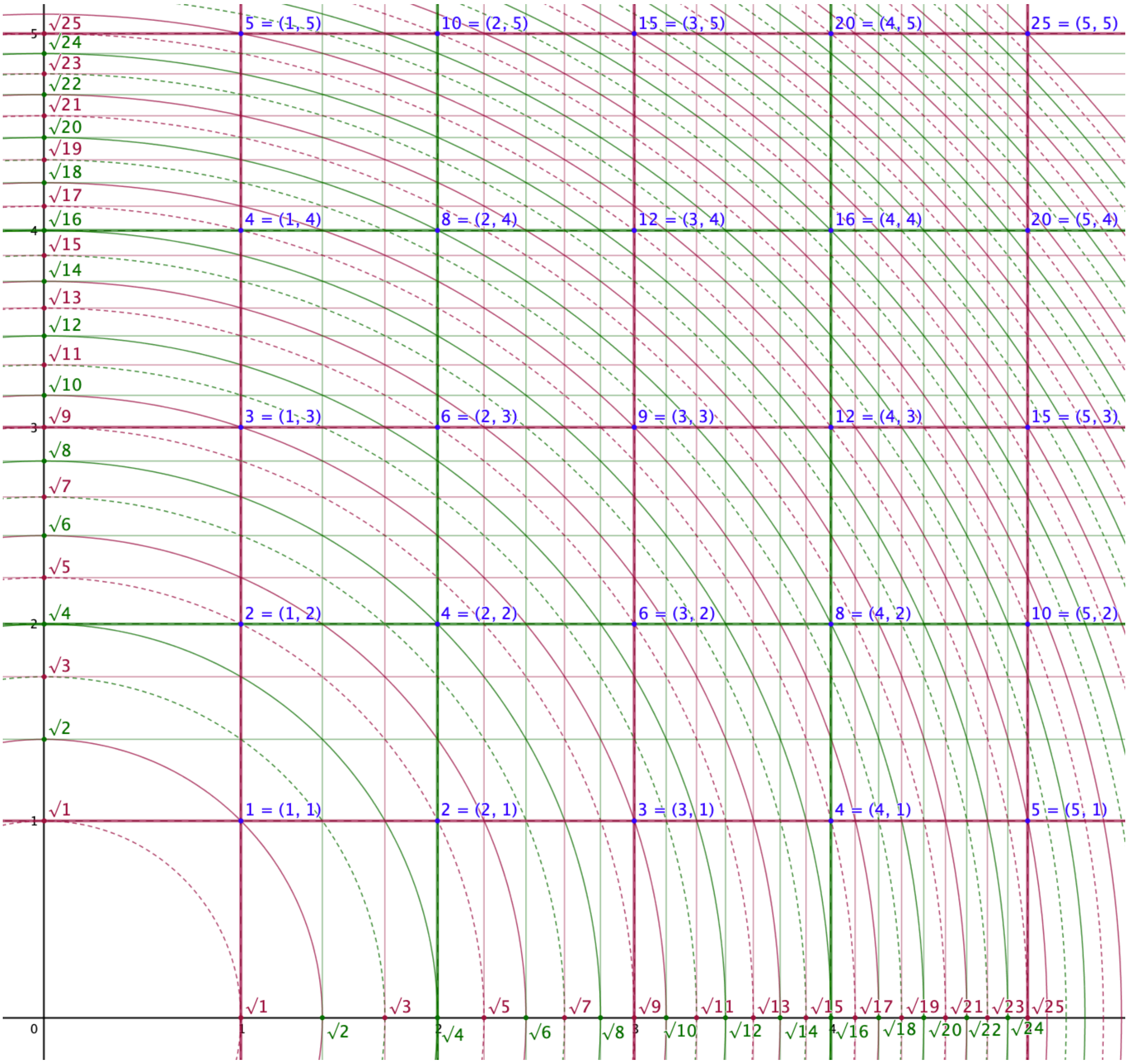

Figure 2. The Hyperbolic Lattice-Grid with its $Y[y] = xy$ integers dots values in blue.

Any dot has the value equivalent to the product of its coordinates x and y. For example, Integer 4 has three dots in this 1st quadrant:

1. $(x, y) = \left(\sqrt{1}, \sqrt{16}\right) = (1,4)$, and $Y[y] = 1 * 4 = 4$.
2. $(x, y) = \left(\sqrt{4}, \sqrt{4}\right) = (2,2)$, and $Y[y] = 2 * 2 = 4$.
3. $(x, y) = \left(\sqrt{16}, \sqrt{1}\right) = (4,1)$, and $Y[y] = 4 * 1 = 4$.

Because all the plane is a hyperbolic plane, we can now draw the hyperbolas that are the result of the products $Y[y] = xy = Integer$.

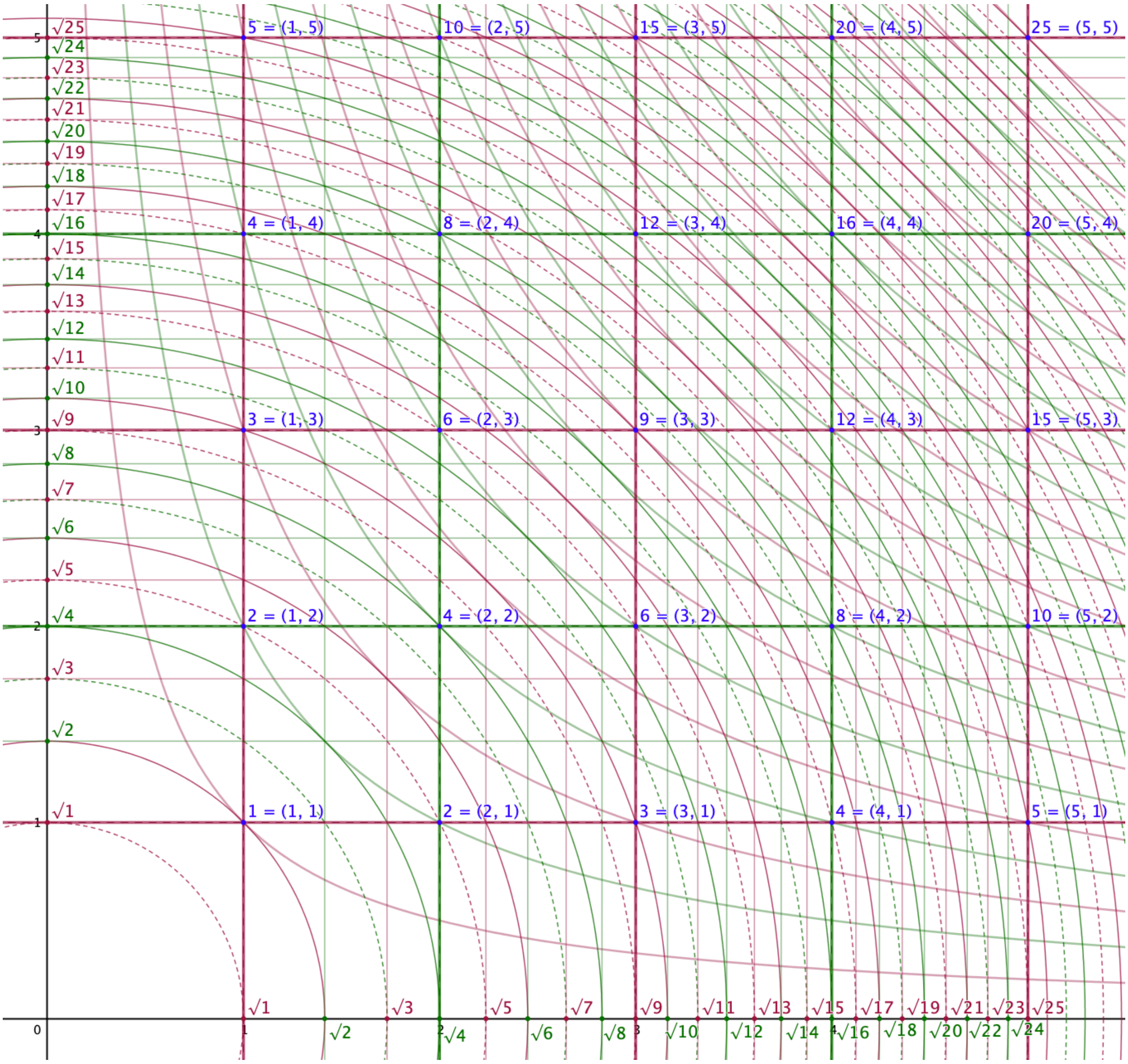

Figure 3. The Hyperbolic Lattice-Grid with its hyperbolas Y[y]=$xy = Integer$ in the 1st quadrant.

Circles with radius $r = \sqrt{|Even\ numbers|}$ crosses hyperbolas of the form $Y[y] = xy = (Integer)$. They are represented by full lines circles.

Circles with radius $r = \sqrt{|Odd\ numbers|}$ crosses hyperbolas of the form $Y[y] = xy = (Integer \pm 0.5) = \left(\frac{Odd}{2}\right)$. They are represented by dotted lines circles.

3. The Hyperbolic Sieve of Primes in Hyperbolic Lattice-Grid

All the hyperbolas $Y[y] = xy$ will cross the diagonal line $x = y$. The diagonal line $x = y$ represents the Square numbers sequence A000290 $Y[y] = x^2 = y^2$.

Because all dots of the Hyperbolic Lattice-Grid have the value $Y[y] = xy$, so the diagonal line $x = y$ will have the dots $Y[y] = x^2 = y^2$.

The diagonal line $x = y$ is a symmetry line of the Hyperbolic Lattice-Grid in the first quadrant of the XY plane.

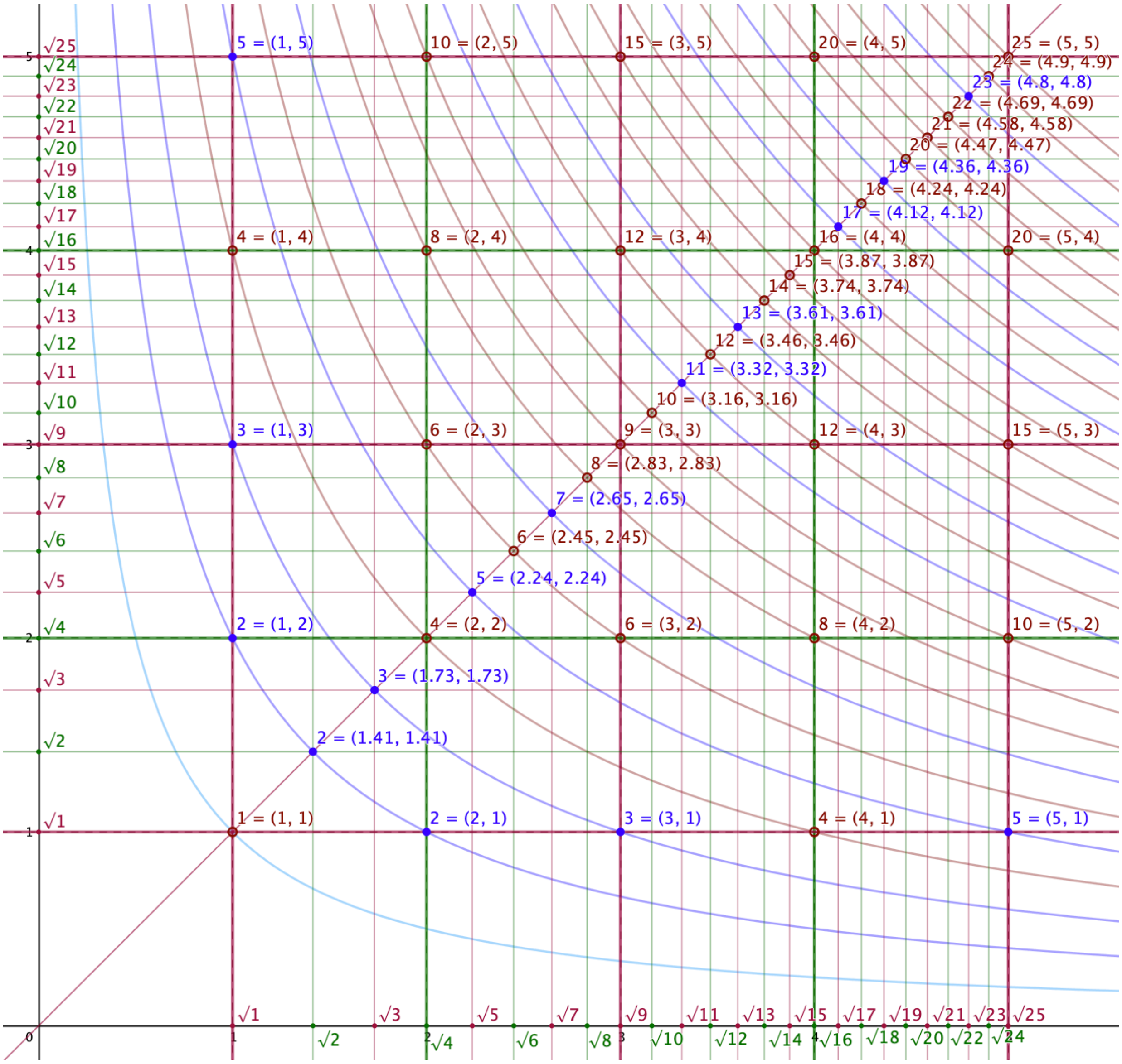

Figure 4. The *Hyperbolic Sieve of Primes* in the 1st quadrant. The Prime numbers on the diagonal line $x = y$ are those that have the hyperbolas passing through only two other dots with Integer coordinates of the form $(1, Prime)$ and $(Prime, 1)$.

Each positive Integer number has a unique representation in the diagonal line $x = y$ in the 1st quadrant.

- Integer 0 is the dot $(x, y) = (\sqrt{0}, \sqrt{0}) = (0,0)$ in the diagonal line $x = y$. It is not a Prime nor a Composite because there are infinitely many dots with Integer coordinates of the form $(x, 0)$ and $(0, y)$ in which the hyperbola $Y[y] = xy = 0$ crosses.
- Integer 1 is the dot $(x, y) = (\sqrt{1}, \sqrt{1}) = (1,1)$ in the diagonal line $x = y$. It is not a Prime nor a Composite because the hyperbola $Y[y] = xy = 1$ crosses in only 1 dot with Integer coordinates: $(1,1)$.
- Integer 2 is the dot $(x, y) = (\sqrt{2}, \sqrt{2})$ in the diagonal line $x = y$. It is a Prime because the hyperbola $Y[y] = xy = 2$ crosses in 2 dots with Integer coordinates: $(2,1)$ and $(1,2)$.
- Integer 3 is the dot $(x, y) = (\sqrt{3}, \sqrt{3})$ in the diagonal line $x = y$. It is a Prime because the hyperbola $Y[y] = xy = 3$ crosses in 2 dots with Integer coordinates: $(3,1)$ and $(1,3)$.
- Integer 4 is the dot $(x, y) = (\sqrt{4}, \sqrt{4}) = (2,2)$ in the diagonal line $x = y$. It is a Composite because the hyperbola $Y[y] = xy = 4$ crosses in more than 2 dots with Integer coordinates: $(4,1)$, $(2,2)$ and $(1,4)$.
- Integer 5 is the dot $(x, y) = (\sqrt{5}, \sqrt{5})$ in the diagonal line $x = y$. It is a Prime because the hyperbola $Y[y] = xy = 5$ crosses in 2 dots with Integer coordinates: $(5,1)$ and $(1,5)$.
- Integer 6 is the dot $(x, y) = (\sqrt{6}, \sqrt{6})$ in the diagonal line $x = y$. It is a Composite because the hyperbola $Y[y] = xy = 6$ crosses in more than 2 dots with Integer coordinates: $(6,1)$, $(3,2)$, $(2,3)$ and $(1,6)$.
- Integer 7 is the dot $(x, y) = (\sqrt{7}, \sqrt{7})$ in the diagonal line $x = y$. It is a Prime because the hyperbola $Y[y] = xy = 7$ crosses in 2 dots with Integer coordinates: $(7,1)$ and $(1,7)$.
- Integer 8 is the dot $(x, y) = (\sqrt{8}, \sqrt{8})$ in the diagonal line $x = y$. It is a Composite because the hyperbola $Y[y] = xy = 8$ crosses in more than 2 dots with Integer coordinates: $(8,1)$, $(4,2)$, $(2,4)$ and $(1,8)$.
- Integer 9 is the dot $(x, y) = (\sqrt{9}, \sqrt{9}) = (3,3)$ in the diagonal line $x = y$. It is a Composite because the hyperbola $Y[y] = xy = 9$ crosses in more than 2 dots with Integer coordinates: $(9,1)$, $(3,3)$ and $(1,9)$.
- And so on.

4. Right-isosceles triangles at Hyperbolic Lattice-Grid

Let's construct infinitely many right-isosceles triangles at hyperbolic lattice-grid where:

1. The 2 equal sides of each triangle will be at XY-axis.
2. The hypotenuse is the line between two Zeros dots equidistant from the intersection between the two XY axes.

See the picture below:

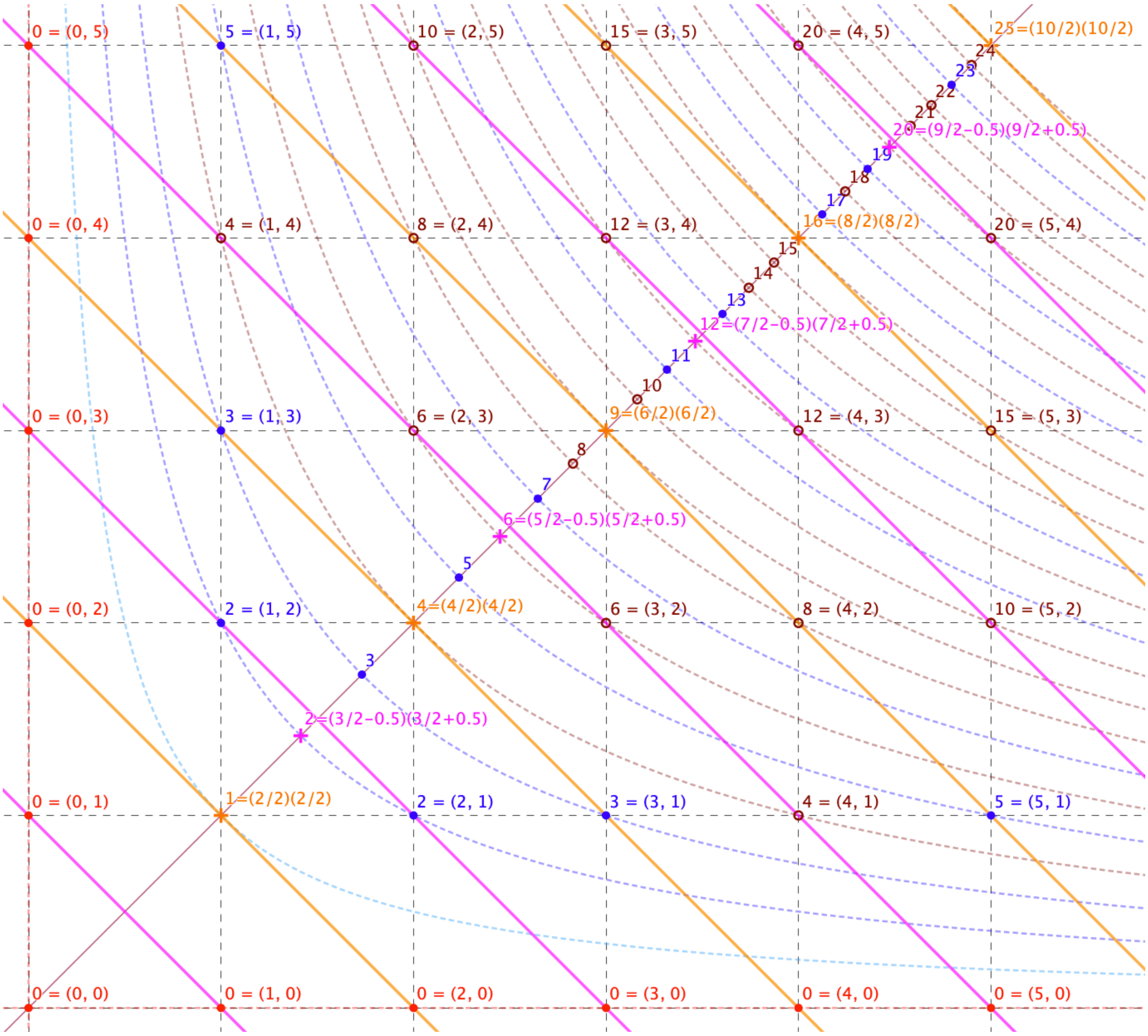

Figure 5. Right-isosceles triangles at hyperbolic lattice-grid.

Let a be the side of the isosceles triangle and c be the value of the hypotenuse. All hypotenuses c are diagonals -45°.

$$c^2 = 2a^2$$
$$c = \sqrt{2}a$$

The Square numbers line is the diagonal 45° $x = y$.

At the midpoint of the hypotenuses, the crossings with the diagonal line of the Square numbers occur.

The midpoint of the hypotenuse c has coordinates $(x, y) = \left(\frac{a}{2}, \frac{a}{2}\right)$.

So, at hyperbolic lattice-grid, the midpoint of the hypotenuses will have a value given by $\left(\frac{a}{2}\right)^2$.

If $a = Even$, the hypotenuses are in orange color and $\frac{a}{2} = Integer$. So, $\left(\frac{a}{2}\right)^2$ is a product of two integers and they will cross the Square hyperbolas in one dot.

If $a = Odd$, the hypotenuses are in pink color and $\frac{a}{2} = Integer + \frac{1}{2}$. So, $\left(\frac{a}{2}\right)^2$ is not a product of two integers and they will cross the Oblong hyperbolas in two dots.

$$\left(\frac{a}{2}\right)^2 - \left(\frac{a}{2}\right) = \frac{a}{2}\left(\frac{a}{2} - 1\right) = \frac{2k+1}{2}\left(\frac{2k+1}{2} - 1\right) = \left(k + \frac{1}{2}\right)\left(k - \frac{1}{2}\right)$$

And,

$$\left(k + \frac{1}{2}\right)\left(k - \frac{1}{2}\right) = \left(k - \frac{1}{2}\right)\left(k + \frac{1}{2}\right) = \left\lfloor\frac{a}{2}\right\rfloor\left\lceil\frac{a}{2}\right\rceil = \left\lceil\frac{a}{2}\right\rceil\left\lfloor\frac{a}{2}\right\rfloor$$

4.1 Conclusion I:

For each right-isosceles triangle of side a, the largest possible internal product of the triangle is $xy = \left(\frac{a}{2}\right)^2$.

- All the $xy > \left(\frac{a}{2}\right)^2$ products on hyperbolic lattice-grid occur outside the right-isosceles triangle with side a.
- Not all internal xy products on the right-isosceles triangle are covered by it every time the xy product occurs on the hyperbolic lattice-grid. This happens because the difference between the divisors of the product xy always increases until it reaches $|x - y| = xy - 1$ where one divisor is 1.

4.2 Conclusion II:

For each right-isosceles triangle of side a, the largest possible xy product covered by the triangle at all times it occurs in the hyperbolic lattice-grid is $xy = a - 1$.

- If we want to count, collect, or organize all the xy products that appear in the hyperbolic lattice-grid up to a certain value z, we do a right-isosceles triangle of side $a = z + 1$. Then, we count, collect, or organize all the xy products covered by the triangle and despise all products $xy > z$.

5. The Multiplication Table in Hyperbolic Lattice-Grid

Note that there is no need to study both the products $z = xy$ and $z = yx$ formed by Hyperbolic Lattice-Grid. There is a total symmetry in all hyperbolas whose axis of symmetry representing the Squares is the diagonal $x = y$.

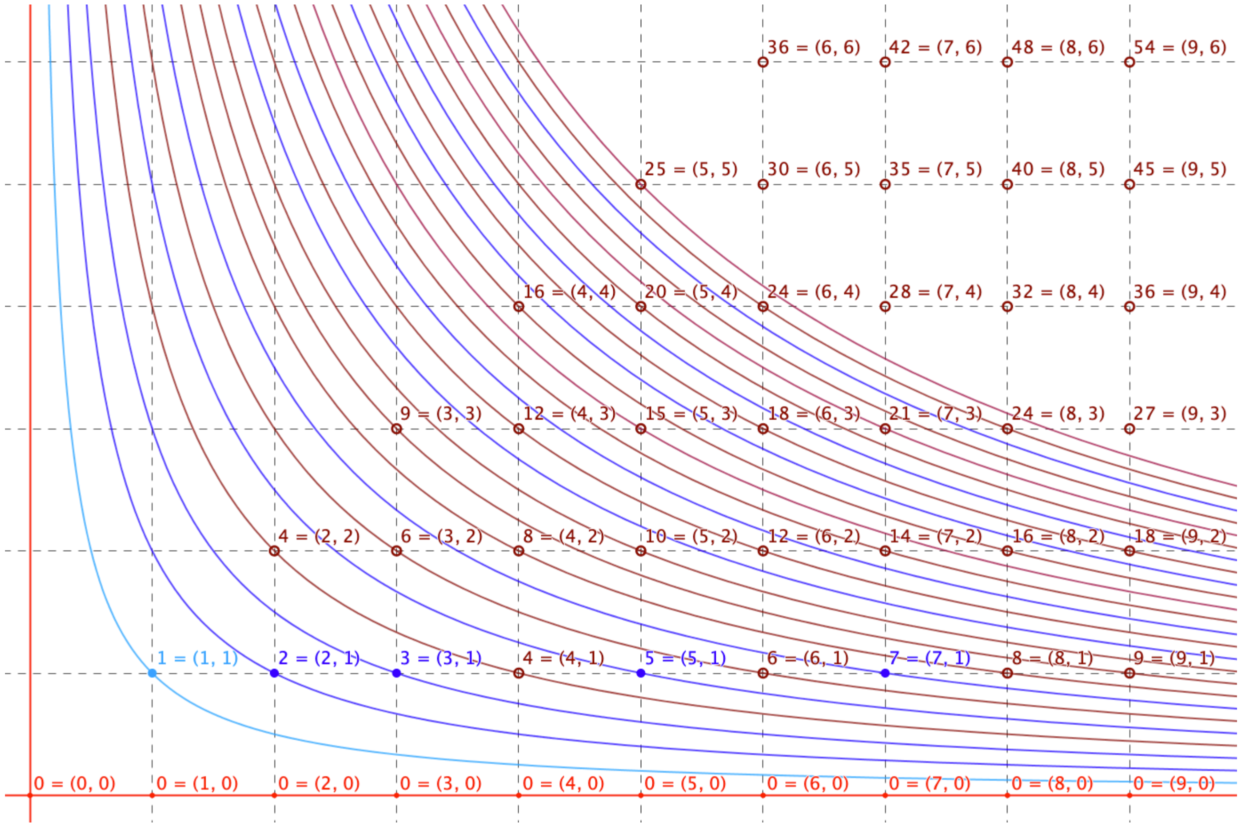

Figure 6. The Multiplication Table is made by the product of two Integers xy.

Then, to avoid repeated divisors ($multiplier * multiplicand$), in the multiplication table, all rows will start on a square number.

- Row 0 starts on $z = 0^2$ and all the dots in this row will be $z = 0, x \geq 0$.
- Row 1 starts on $z = 1^2$ and all the dots in this row will be $z = x, x \geq 1$.
- Row 2 starts on $z = 2^2$ and all the dots in this row will be $z = 2x, x \geq 2$.
- Row 3 starts on $z = 3^2$ and all the dots in this row will be $z = 3x, x \geq 3$.
- Row 4 starts on $z = 4^2$ and all the dots in this row will be $z = 4x, x \geq 4$.
- ...
- Row y starts on $z = y^2$ and all the dots in this row will be $z = yx, x \geq y$.

In this way, considering the results of Hyperbolic Sieve of Products $z = xy$ where x and y are Integers in Hyperbolic Lattice-Grid, we will have all possible multiplications of the form $z = xy = multiplier * multiplicand = product$ where:

1. All possible divisors of the products are used; and
2. There is no divisor repeated. There is no pair $(multiplier, multiplicand)$ repeated.

Consequently, besides the Zero and One, we have all the Primes results once and all the Composites results repeated at accordingly with their number of divisors in such a way there is no divisor repeated.

The divisors will be following exactly the sequence A038548 Number of divisors of $z > 0$ that are at most $\sqrt{z}$. The sequence is: {1, 1, 1, 2, 1, 2, 1, 2, 2, 2, 1, 3, 1, 2, 2, 3, 1, 3, 1, 3, 2, 2, 1, 4, 2, 2, 2, 3, 1, 4, 1, 3, 2, 2, 2, 5, 1, 2, 2, 4, 1, 4, 1, 3, 3, 2, 1, 5, 2, 3, 2, 3, 1, 4, 2, 4, 2, 2, 1, 6, 1, 2, 3, 4, 2, 4, 1, 3, 2, 4, 1, 6, 1, 2, 3, 3, 2, 4, 1, 5, 3, 2, 1, 6, 2, 2, 2, 4, 1, 6, 2, 3, 2, 2, 2, 6, 1, 3, 3, 5, 1, 4, 1, 4, 4, …}

6. Organizing all xy Products using all possible divisors only once

Now we can use all the conclusions made so far in the multiplication table to organize all possible Integer multiplications in the form of $z = xy$. We have to order by-products the $divisors \leq \sqrt{z}$ in ascending order and the $divisors \geq \sqrt{z}$ in descending order, like the table below:

A161908 Array read by rows in which row z lists the divisors of z that are >= sqrt(z).		Array read by rows in which row z lists the divisors of z that are <= sqrt(z).		Array read by rows in which row z lists z A038548(z) times.
1	*	1	=	1
2	*	1	=	2
3	*	1	=	3
2	*	2	=	4
4	*	1	=	4
5	*	1	=	5
3	*	2	=	6
6	*	1	=	6
7	*	1	=	7
4	*	2	=	8
8	*	1	=	8
3	*	3	=	9
9	*	1	=	9
5	*	2	=	10
10	*	1	=	10
11	*	1	=	11
4	*	3	=	12
6	*	2	=	12
12	*	1	=	12
13	*	1	=	13
7	*	2	=	14
14	*	1	=	14
5	*	3	=	15
15	*	1	=	15
4	*	4	=	16
8	*	2	=	16
16	*	1	=	16
17	*	1	=	17
6	*	3	=	18
9	*	2	=	18
18	*	1	=	18
19	*	1	=	19
5	*	4	=	20
10	*	2	=	20
20	*	1	=	20
7	*	3	=	21
21	*	1	=	21
11	*	2	=	22
22	*	1	=	22
23	*	1	=	23

Table 1. All the unique products $z = xy$ until $z = 23$.

1. The multiplier column sorted by the products will form the sequence A161908 Array read by rows in which row z lists the divisors of z that are $\geq \sqrt{z}$.
{1, 2, 3, 2, 4, 5, 3, 6, 7, 4, 8, 3, 9, 5, 10, 11, 4, 6, 12, 13, 7, 14, 5, 15, 4, 8, 16, 17, 6, 9, 18, 19, 5, 10, 20, 7, 21, 11, 22, 23, 6, 8, 12, 24, 5, 25, 13, 26, 9, 27, 7, 14, 28, 29, 6, 10, 15, 30, 31, 8, 16, 32, 11, 33, 17, 34, 7, 35, 6, 9, 12, 18, 36, 37, 19, 38, 13, 39, 8, 10, 20, 40, 41, 7, 14, 21, 42, 43, 11, 22, 44, 9, 15, 45, 23, 46, 47, 8, 12, 16, …}.

Array begins:

A161908 Array read by rows in which row z lists the divisors of z that are >= sqrt(z).		
1		
2		
3		
2	4	
5		
3	6	
7		
4	8	
3	9	
5	10	
11		
4	6	12
13		
7	14	
5	15	
4	8	16

Table 2. The divisors $\geq \sqrt{z}$ of the products $z = xy$.

The array's row lengths are the number of divisors of z that are at most $\sqrt{z}$. It is given by the sequence A038548. {1, 1, 1, 2, 1, 2, 1, 2, 2, 2, 1, 3, 1, 2, 2, 3, 1, 3, 1, 3, 2, 2, 1, 4, 2, 2, 2, 3, 1, 4, 1, 3, 2, 2, 2, 5, 1, 2, 2, 4, 1, 4, 1, 3, 3, 2, 1, 5, 2, 3, 2, 3, 1, 4, 2, 4, 2, 2, 1, 6, 1, 2, 3, 4, 2, 4, 1, 3, 2, 4, 1, 6, 1, 2, 3, 3, 2, 4, 1, 5, 3, 2, 1, 6, 2, 2, 2, 4, 1, 6, 2, 3, 2, 2, 2, 6, 1, 3, 3, 5, 1, 4, 1, 4, 4, …}.

2. The multiplicand column sorted by the products form the sequence Array read by rows in which row z lists the divisors of z that are $\leq \sqrt{z}$. {1, 1, 1, 2, 1, 1, 2, 1, 1, 2, 1, 3, 1, 2, 1, 1, 3, 2, 1, 1, 2, 1, 3, 1, 4, 2, 1, 1, 3, 2, 1, 1, 4, 2, 1, 3, 1, 2, 1, 1, 4, 3, 2, 1, 5, 1, 2, 1, 3, 1, 4, 2, 1, 1, 5, 3, 2, 1, 1, 4, 2, 1, 3, 1, 2, 1, …}

Array begins:

Array read by rows in which row z lists the divisors of z that are <= sqrt(z).		
1		
1		
1		
2	1	
1		
2	1	
1		
2	1	
3	1	
2	1	
1		
3	2	1
1		
2	1	
3	1	
4	2	1

Table 3. The divisors $\leq \sqrt{z}$ of the products $z = xy$.

Also, the array's row lengths are the number of divisors of z that are at most $\sqrt{z}$. It is given by the sequence A038548.

3. Finally, the products of multiplier as divisors $\leq \sqrt{z}$ times multiplicand as divisors $\geq \sqrt{z}$, sorted by the products, form the sequence Array read by rows in which row z lists z [A038548(z)](https://oeis.org/A038548) times. {1, 2, 3, 4, 4, 5, 6, 6, 7, 8, 8, 9, 9, 10, 10, 11, 12, 12, 12, 13, 14, 14, 15, 15, 16, 16, 16, 17, 18, 18, 18, 19, 20, 20, 20, 21, 21, 22, 22, 23, 24, 24, 24, 24, 25, 25, 26, 26, 27, 27, 28, 28, 28, 29, 30, 30, 30, 30, 31, 32, 32, 32, 33, 33, 34, 34, ...}.

Arrays product:

A161908 Array read by rows in which row z lists the divisors of z that are >= sqrt(z).				Array read by rows in which row z lists the divisors of z that are <= sqrt(z).				Array read by rows in which row z lists z A038548(z) times.		
1				1				1		
2				1				2		
3				1				3		
2	4		x	2	1		=	4	4	
5				1				5		
3	6			2	1			6	6	
7				1				7		
4	8			2	1			8	8	
3	9			3	1			9	9	
5	10			2	1			10	10	
11				1				11		
4	6	12		3	2	1		12	12	12
13				1				13		
7	14			2	1			14	14	
5	15			3	1			15	15	
4	8	16		4	2	1		16	16	16

Table 3. The mechanism of the divisors and the products $z = xy$.

The number of divisors for each product is given by the sequence [A000005](https://oeis.org/A000005) d(z) (also called tau(z) or sigma_0(z)), the number of divisors of z. {1, 2, 2, 3, 2, 4, 2, 4, 3, 4, 2, 6, 2, 4, 4, 5, 2, 6, 2, 6, 4, 4, 2, 8, 3, 4, 4, 6, 2, 8, 2, 6, 4, 4, 4, 9, 2, 4, 4, 8, 2, 8, 2, 6, 6, 4, 2, 10, 3, 6, 4, 6, 2, 8, …}.

7. Acknowledgments

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8. References

[1] *The On-Line Encyclopedia of Integer Sequences*, Available online at http://oeis.org.