Self-Extensionality of Finitely-Valued Logics: Advances

Alexej Pynko

September 13, 2021

SELF-EXTENSIONALITY OF FINITELY-VALUED LOGICS: ADVANCES

ALEXEJ P. PYNKO

Abstract. We start from proving a general characterization of the self-extensionality of sentential logics implying the decidability of this problem for (not necessarily uniform) finitely-valued logics. And what is more, in case of logics defined by finitely many either implicative or both disjunctive and conjunctive finite *hereditarily* simple (viz., having no non-simple submatrix) matrices, we then derive a characterization yielding a quite effective algebraic criterion of checking their self-extensionality via analyzing homomorphisms between (viz., in the uniform case, endomorphisms of) the underlying algebras of their defining matrices and equally being a quite useful heuristic tool, manual applications of which are demonstrated within the framework of Łukasiewicz' finitely-valued logics, logics of three-valued super-classical matrices, four-valued expansions of Belnap's "useful" four-valued logic as well as their (not necessarily uniform) no-more-than-three-valued extensions, [uniform inferentially consistent non-]classical [three-valued] ones proving to be [non-]self-extensional.

1. Introduction

Perhaps, the principal value of universal logical investigations consists in discovering uniform points behind particular results originally proved *ad hoc*. This thesis is the main paradigm of the present universal logical study.

Recall that a sentential logic (cf., e.g., [5]) is said to be *self-extensional*, whenever its inter-derivability relation is a congruence of the formula algebra. This feature is typical of both two-valued (in particular, classical) and super-intuitionistic logics as well as some interesting many-valued ones (like Belnap's "useful" four-valued one [1]). Here, we explore self-extensionality laying a special emphasis onto the general framework of finitely-valued logics and the decidability issue with reducing the complexity of effective procedures of verifying self-extensionality, when restricting our consideration to finitely-valued logics of special kind — namely, those defined by finitely many either implicative or both conjunctive and disjunctive (and so having either classical implication or both classical conjunction and classical disjunction in Tarski's conventional sense) *hereditarily simple* (viz., having no non-simple submatrix; in particular, having an *equality determinant* in the sense of [13] — cf. Lemmas 3.2 and 3.3 of [17]) finite matrices. We then exemplify our universal elaboration by discussing four (perhaps, most representative) generic classes of logics of the kind involved: Łukasiewicz' finitely-valued logics [6]); logics of three-valued super-classical matrices; four-valued expansions of Belnap's logic [1] (cf. [12, 17]) and their (not necessarily uniform) no-more-than-three-valued extensions, [uniform inferentially consistent non-]classical [three-valued] ones proving to be [non-]self-extensional.

The rest of the paper is as follows. The exposition of the material of the paper is entirely self-contained (of course, modulo very basic issues concerning Set and

---

2020 *Mathematics Subject Classification.* 03B20, 03B22, 03B50, 03B53, 03G10.
Key words and phrases. logic; matrix; model; congruence; [semi-]lattice.

Lattice Theory, Universal Algebra and Logic to be found, if necessary, in standard mathematical handbooks like [7]). Section 2 is a concise summary of particular basic issues underlying the paper, most of which, though having become a part of algebraic and logical folklore, are still recalled just for the exposition to be properly self-contained. Likewise, in Section 3, we then summarize certain advanced generic issues concerning simple matrices, equality determinants, intrinsic varieties as well as both disjunctivity and implicativity. Section 4 is a collection of main *general* results of the paper that are then exemplified in Section 5 (aside from Łukasiewicz' finitely-valued logics, whose non-self-extensionality has actually been due [14], as we briefly discuss within Example 4.16 — this equally concerns certain particular instances discussed in Section 5 and summarized in Example 4.17). Finally, Section 6 is a brief summary of principal contributions of the paper.

2. BASIC ISSUES

2.1. **Set-theoretical background.** We follow the standard set-theoretical convention, according to which natural numbers (including 0) are treated as finite ordinals (viz., sets of lesser natural numbers), the ordinal of all them being denoted by ω. In this way, when dealing with n-tuples to be viewed as either sequences of length n or functions with domain n, where $n \in \omega$, π_i, where $i \in n$, denotes the i-th projection operator under enumeration started from rather 0 than 1. (In particular, when $n = 2$, $\pi_{0/1}$ denotes the left/right projection operator, respectively.) The proper class of all ordinals is denoted by ∞. Also, functions are viewed as binary relations, while singletons are identified with their unique elements, unless any confusion is possible. A function f is said to be *singular*, provided $|\operatorname{img} f| \in 2$, that is, $(\ker f) = (\operatorname{dom} f)^2$.

Given a set S, the set of all subsets of S [of cardinality $\in K \subseteq \infty$] is denoted by $\wp_{[K]}(S)$, respectively. Then, given any equivalence relation θ on S, by ν_θ we denote the function with domain S defined by $\nu_\theta(a) \triangleq \theta[\{a\}]$, for all $a \in S$, whereas we set $(T/\theta) \triangleq \nu_\theta[T]$, for every $T \subseteq S$. Next, S-tuples (viz., functions with domain S) are often written in the sequence $\bar{t}$ form, its s-th component (viz., the value under argument s), where $s \in S$, being written as t_s. Given two more sets A and B, any relation $R \subseteq (A \times B)$ (in particular, a mapping $R : A \to B$) determines the equally-denoted relation $R \subseteq (A^S \times B^S)$ (resp., mapping $R : A^S \to B^S$) point-wise. Further, set $\Delta_S \triangleq \{\langle a, a\rangle \mid a \in S\}$, functions of such a kind being referred to as *diagonal*. Furthermore, any $f : S^n \to S$, where $n \in \omega$, is said to be *R-monotonic*, where $R \subseteq S^2$, provided, for all $\bar{a} \in R^n$, it holds that $\langle f(\bar{a} \circ \pi_0), f(\bar{a} \circ \pi_1)\rangle \in R$. Finally, given any $T \subseteq S$, we have the *characteristic function* $\chi^T_S \triangleq ((T \times \{1\}) \cup ((S \setminus T) \times \{0\}))$ *of T in S*.

Let A be a set. Then, an $X \in S \subseteq \wp(A)$ is said to be *meet-irreducible in/of* S, provided, for each $T \in \wp(S)$, $X \in T$, whenever $T = (A \cap \bigcap T)$, the set of all them being denoted by $\mathrm{MI}(S)$. Next, a $U \subseteq \wp(A)$ is said to be *upward-directed*, provided, for every $S \in \wp_\omega(U)$, there is some $T \in U$ such that $(\bigcup S) \subseteq T$, in which case $U \neq \varnothing$, when taking $S = \varnothing$. Further, a subset of $\wp(A)$ is said to be *inductive*, whenever it is closed under unions of upward-directed subsets. Further, a *closure system over* A is any $\mathcal{C} \subseteq \wp(A)$ such that, for every $S \subseteq \mathcal{C}$, it holds that $(A \cap \bigcap S) \in \mathcal{C}$. In that case, any $\mathcal{B} \subseteq \mathcal{C}$ is called a *(closure) basis of* $\mathcal{C}$, provided $\mathcal{C} = \{A \cap \bigcap S | S \subseteq \mathcal{B}\}$. Furthermore, an *operator over* A is any unary operation O on $\wp(A)$. This is said to be *monotonic*, whenever it is $(\subseteq \cap \wp(A)^2)$-monotonic. Likewise, it is said to be *idempotent*|*transitive*, provided, for all $X \subseteq A$, it holds that $(X|O(O(X))) \subseteq O(X)$, respectively. Finally, it is said to be *inductive/finitary*, provided, for any upward-directed $U \subseteq \wp(A)$, it holds that $O(\bigcup U) \subseteq (\bigcup O[U])$. Then, a *closure operator over* A is any monotonic idempotent transitive operator over A, in which case $\operatorname{img} C$ is a[n inductive] closure system over A [iff C is inductive], determining C uniquely, because, for every closure basis $\mathcal{B}$ of $\operatorname{img} C$ (including $\operatorname{img} C$ itself) and each $X \subseteq A$, it holds that $C(X) = (A \cap \bigcap\{Y \in \mathcal{B} | X \subseteq Y\})$, C and $\operatorname{img} C$ being said to be *dual to* one another.

Remark 2.1. By Zorn Lemma, due to which any non-empty inductive subset of $\wp(A)$ has a maximal element, $\mathrm{MI}(\mathcal{C})$ is a basis of any inductive closure system $\mathcal{C}$ over A. $\square$

2.2. **Algebraic background.** Unless otherwise specified, abstract algebras are denoted by Fraktur letters [possibly, with indices], their carriers (viz., underlying sets) being denoted by corresponding Italic letters [with same indices, if any].

A *(propositional/sentential) language/signature* is any algebraic (viz., functional) signature Σ (to be dealt with throughout the paper by default) constituted by function (viz., operation) symbols of finite arity to be treated as *(propositional/sentential) [primary] connectives*, the set of all nullary ones being denoted by $\Sigma{\upharpoonright}0$.

Given a Σ-algebra $\mathfrak{A}$, $\mathrm{Con}(\mathfrak{A})$ is an inductive closure system over A^2, the dual closure operator (of congruence generation) being denoted by $\mathrm{Cg}^{\mathfrak{A}}$. Then, given a class K of Σ-algebras, set $\hom(\mathfrak{A}, \mathsf{K}) \triangleq (\bigcup\{\hom(\mathfrak{A}, \mathfrak{B}) \mid \mathfrak{B} \in \mathsf{K}\})$, in which case $\ker[\hom(\mathfrak{A}, \mathsf{K})] \subseteq \mathrm{Con}(\mathfrak{A})$, so $(A^2 \cap \bigcap \ker[\hom(\mathfrak{A}, \mathsf{K})]) \in \mathrm{Con}(\mathfrak{A})$.

Given any $\alpha \subseteq \omega$, put $\bar{x}_\alpha \triangleq \langle x_i \rangle_{i \in \alpha}$ and $\mathrm{Var}_\alpha \triangleq (\operatorname{img} \bar{x}_\alpha)$, elements of which being viewed as *(propositional/sentential) variables of rank* α. (In general, any mention of α within any context is normally omitted, whenever $\alpha = \omega$.) Then, providing either $\alpha \neq \varnothing$ or Σ has a nullary connective, we have the absolutely-free Σ-algebra $\mathfrak{Fm}^\alpha_\Sigma$ freely-generated by the set Var_α, "its endomorphisms"/"elements of its carrier $\mathrm{Fm}^\alpha_\Sigma$ (viz., Σ-terms of rank α)" being called *(propositional|sentential)* Σ-*substitutions/-formulas of rank* α. Any homomorphism h from $\mathfrak{Fm}^\alpha_\Sigma$ to a Σ-algebra $\mathfrak{A}(= \mathfrak{Fm}^\alpha_\Sigma)$ is uniquely determined by {and so identified with} $h' = (h{\upharpoonright}(\mathrm{Var}_\alpha(\setminus V)))$ (where $V \subseteq \mathrm{Var}_\alpha$ such that $h{\upharpoonright}V$ is diagonal) as well as often written in the standard assignment (resp., substitution) form $[v/h(v)]_{v \in (\operatorname{dom} h')}$, $\varphi^{\mathfrak{A}}\langle[\rangle h \langle]\rangle$, where $\varphi \in \mathrm{Fm}^\alpha_\Sigma$, standing for $h(\varphi)$ (the algebra superscript being normally omitted just like in denoting primary operations of $\mathfrak{A}$). Then, given any $n \in \omega$, a *secondary/"(term-wise) definable" n-ary connective of* Σ is any Σ-formula φ of rank $m = (n + (1 - \min(1, \max(n, |\Sigma{\upharpoonright}0|))))$, in which case, given any Σ-algebra $\mathfrak{A}$, an $f : A^n \to A$ is said to be *(term-wise) definable* $\{$*by* $\varphi\}$ *in* $\mathfrak{A}$, provided, for all $\bar{a} \in A^m$, it holds that $f(\bar{a}{\upharpoonright}n) = \varphi^{\mathfrak{A}}[x_i/a_i]_{i \in m}$. For the sake of formal unification, any primary n-ary connective $\varsigma \in \Sigma$ is identified with the secondary one $\varsigma(\bar{x}_n)$. A $\theta \in \mathrm{Con}(\mathfrak{Fm}^\alpha_\Sigma)$ is said to be *fully-invariant*, if, for every Σ-substitution σ of rank α, it holds that $\sigma[\theta] \subseteq \theta$. Recall that, for any [surjective] $h \in \hom(\mathfrak{A}, \mathfrak{B})$, where $\mathfrak{A}$ and $\mathfrak{B}$ are Σ-algebras, it holds that:

$$(\hom(\mathfrak{Fm}^\alpha_\Sigma, \mathfrak{B}) \supseteq [=]\{h \circ g \mid g \in \hom(\mathfrak{Fm}^\alpha_\Sigma, \mathfrak{A})\}). \tag{2.1}$$

Any $\langle \phi, \psi \rangle \in \mathrm{Eq}^\alpha_\Sigma \triangleq (\mathrm{Fm}^\alpha_\Sigma)^2$ is referred to as a Σ-*equation/-indentity of rank* α and normally written in the standard equational form $\phi \approx \psi$. In this way, given any $h \in \hom(\mathfrak{Fm}^\alpha_\Sigma, \mathfrak{A})$, $\ker h$ is the set of all Σ-identities of rank α *true/satisfied in* $\mathfrak{A}$ *under* h. Likewise, given a class K of Σ-algebras, $\theta^\alpha_{\mathsf{K}} \triangleq (\mathrm{Eq}^\alpha_\Sigma \cap \bigcap \ker[\hom(\mathfrak{Fm}^\alpha_\Sigma, \mathsf{K})]) \in \mathrm{Con}(\mathfrak{Fm}^\alpha_\Sigma)$, being fully invariant, in view of (2.1), is the set of all all Σ-identities of rank α *true/satisfied in* K, in which case we set $\mathfrak{F}^\alpha_{\mathsf{K}} \triangleq (\mathfrak{Fm}^\alpha_\Sigma / \theta^\alpha_{\mathsf{K}})$. (In case α as well as both K and all elements of it are finite, the class $I \triangleq \{\langle \mathfrak{A}, h \rangle \mid \mathfrak{A} \in \mathsf{K}, h \in \hom(\mathfrak{Fm}^\alpha_\Sigma, \mathfrak{A})\}$ is a finite set — more precisely, $|I| = \sum_{\mathfrak{A} \in \mathsf{K}} |A|^\alpha$, in which case, putting, for each $i \in I$, $\mathfrak{A}_i \triangleq \pi_0(i) \in \mathsf{K}$, $h_i \triangleq \pi_1(i) \in \hom(\mathfrak{Fm}^\alpha_\Sigma, \mathfrak{A})$ and $\mathfrak{B}_i \triangleq (\mathfrak{A}_i{\upharpoonright}(\operatorname{img} h_i))$, we have $\hom(\mathfrak{Fm}^\alpha_\Sigma, \prod_{i \in I} \mathfrak{B}_i) \ni g : \mathrm{Fm}^\alpha_\Sigma \to (\prod_{i \in I} B_i), \varphi \mapsto \langle h_i(\varphi) \rangle_{i \in I}$ with $(\ker g) = \theta \triangleq \theta^\alpha_{\mathsf{K}}$, and so, by the Homomorphism Theorem, $e \triangleq (\nu_\theta^{-1} \circ g)$ is an isomorphism from $\mathfrak{F}^{\alpha}_{\mathsf{K}}$ onto the subdirect product $(\prod_{i\in I}\mathfrak{B}_i){\restriction}(\operatorname{img} g)$ of $\langle\mathfrak{B}_i\rangle_{i\in I}$. In this way, the former is finite, for the latter is so — more precisely, $|F^{\alpha}_{\mathsf{K}}| \leqslant (\max\{|A| \mid \mathfrak{A}\in\mathsf{K}\}^{|I|}$.)

The class of all Σ-algebras satisfying every element of an $\mathcal{E}\subseteq\mathrm{Eq}^{\omega}_{\Sigma}$ is called the *variety axiomatized by* $\mathcal{E}$. Then, the variety $\mathbf{V}(\mathsf{K})$ axiomatized by $\theta^{\omega}_{\mathsf{K}}$ is the least variety including K and is said to be *generated by* K, in which case $\theta^{\alpha}_{\mathbf{V}(\mathsf{K})}=\theta^{\alpha}_{\mathsf{K}}$, and so $\mathfrak{F}^{\alpha}_{\mathbf{V}(\mathsf{K})}=\mathfrak{F}^{\alpha}_{\mathsf{K}}$.

Given a fully invariant $\theta\in\mathrm{Con}(\mathfrak{Fm}^{\omega}_{\Sigma})$, by (2.1), $\mathfrak{Fm}^{\omega}_{\Sigma}/\theta$ belongs to the variety V axiomatized by θ, in which case any Σ-identity satisfied in V belongs to θ, and so $\theta^{\omega}_{\mathsf{V}}=\theta$. In particular, given a variety V of Σ-algebras, we have $\mathfrak{F}^{\alpha}_{\mathsf{V}}\in\mathsf{V}$.

The mapping $\mathrm{Var}:\mathrm{Fm}^{\omega}_{\Sigma}\to\wp_{\omega}(\mathrm{Var}_{\omega})$ assigning the set of all *actually* occurring variables is defined in the standard way.

2.2.1. *Lattice-theoretic background.*

2.2.1.1. Semi-lattices. Let $\diamond$ be a (possibly, secondary) binary connective of Σ.

A Σ-algebra $\mathfrak{A}$ is called a $\diamond$*-semi-lattice*, provided it satisfies semi-lattice (viz., idempotence, commutativity and associativity) identities for $\diamond$, in which case we have the partial ordering $\leq^{\mathfrak{A}}_{\diamond}$ on A, given by $(a\leq^{\mathfrak{A}}_{\diamond}b)\overset{\mathrm{def}}{\Longleftrightarrow}(a=(a\diamond^{\mathfrak{A}}b))$, for all $a,b\in A$. Then, in case the [dual] poset $\langle A,(\leq^{\mathfrak{A}}_{\diamond})^{[-1]}\rangle$ has the least element (viz., lower *bound*), this is called the *[dual]* $\langle\diamond-\rangle$*bound of* $\mathfrak{A}$ and denoted by $[\delta]\beta^{\mathfrak{A}}_{\diamond}$, while $\mathfrak{A}$ is referred to as a $\diamond$*-semi-lattice with [dual] bound* $\{a$, whenever $a=[\delta]\beta^{\mathfrak{A}}_{\diamond}\}$.

Lemma 2.2. *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be* $\diamond$*-semi-lattices with bound and* $h\in\hom(\mathfrak{A},\mathfrak{B})$. *Suppose* $h[A]=B$. *Then,* $h(\beta^{\mathfrak{A}}_{\diamond})=\beta^{\mathfrak{B}}_{\diamond}$.

Proof. There is some $a\in A$ such that $h(a)=\beta^{\mathfrak{B}}_{\diamond}$, in which case $(a\diamond^{\mathfrak{A}}\beta^{\mathfrak{A}}_{\diamond})=\beta^{\mathfrak{A}}_{\diamond}$, so $h(\beta^{\mathfrak{A}}_{\diamond})=(h(a)\diamond^{\mathfrak{B}}h(\beta^{\mathfrak{A}}_{\diamond}))=(\beta^{\mathfrak{B}}_{\diamond}\diamond^{\mathfrak{B}}h(\beta^{\mathfrak{A}}_{\diamond}))=\beta^{\mathfrak{B}}_{\diamond}$, as required. ☐

2.2.1.2. Lattices. Let $\barwedge$ and $\veebar$ be (possibly, secondary) binary connectives of Σ fixed throughout the paper by default.

A Σ-algebra $\mathfrak{A}$ is called a *[distributive]* $(\barwedge,\veebar)$*-lattice*, provided it satisfies [distributive] lattice identities for $\barwedge$ and $\veebar$ (viz., semilattice identities for both $\barwedge$ and $\veebar$ as well as mutual [both] absorption [and distributivity] identities for them), in which case $\leq^{\mathfrak{A}}_{\barwedge}$ and $\leq^{\mathfrak{A}}_{\veebar}$ are inverse/dual to one another, and so, in case $\mathfrak{A}$ is a $\veebar$-semi-lattice with bound (in particular, when A is finite), $\beta^{\mathfrak{A}}_{\veebar}$ is the dual $\barwedge$-bound of $\mathfrak{A}$ (viz., the greatest element of the poset $\langle A,\leq^{\mathfrak{A}}_{\barwedge}\rangle$). Then, in case $\mathfrak{A}$ is a {distributive} $(\barwedge,\veebar)$-lattice, it is said to be that *with zero|unit (a)*, whenever it is a $(\barwedge|\veebar)$-semilattice with bound (a).

2.2.1.2.1. Bounded lattices. Let $\Sigma_{+[,01]}\triangleq\{\wedge,\vee[,\bot,\top]\}$ be the [bounded] lattice signature with binary $\wedge$ (conjunction) and $\vee$ (disjunction) [as well as nullary $\bot$ and $\top$ (falsehood/zero and truth/unit constants, respectively)]. Then, a $\Sigma_{+[,01]}$-algebra $\mathfrak{A}$ is called a *[bounded] (distributive) lattice*, whenever it is a (distributive) $(\wedge,\vee)$-lattice [with zero $\bot^{\mathfrak{A}}$ and unit $\top^{\mathfrak{A}}$].

Given any $n\in(\omega\setminus 2)$, by $\mathfrak{D}_{n[,01]}$ we denote the [bounded] distributive lattice given by the chain $(n\div(n-1))\triangleq\{\frac{m}{n-1}\mid m\in n\}$ ordered by $\leqslant$.

2.3. **Logical background.**

2.3.1. *Propositional calculi and logics.* A *(propositional/sentential)* Σ*-rule* is any element of $\wp(\mathrm{Fm}^{\omega}_{\Sigma})\times\mathrm{Fm}^{\omega}_{\Sigma}$. Then, any Σ-rule $\langle\Gamma,\varphi\rangle$ is normally written in the standard sequent form $\Gamma\vdash\varphi$, $\varphi|(\psi\in\Gamma)$ being referred to as the|a *conclusion|premise of* it. In that case, we set $\sigma(\Gamma\vdash\varphi)\triangleq(\sigma[\Gamma]\vdash\sigma(\varphi))$, where σ is a Σ-substitution. As usual, Σ-rules without premises are called *(propositional/sentential)* Σ*-axioms* and are identified with their conclusions.

A *(propositional/sentential)* Σ*-logic* is any closure operator C over $\mathrm{Fm}^{\omega}_{\Sigma}$ that is *structural* in the sense that $\sigma[C(X)] \subseteq C(\sigma[X])$, for all $X \subseteq \mathrm{Fm}^{\omega}_{\Sigma}$ and all $\sigma \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{Fm}^{\omega}_{\Sigma})$, that is, $\mathrm{img}\, C$ is closed under inverse Σ-substitutions. Then, we have the equivalence relation $\equiv^{\alpha}_{C} \triangleq \{\langle\phi,\psi\rangle \in \mathrm{Eq}^{\alpha}_{\Sigma} \mid C(\phi) = C(\psi)\}$ on $\mathrm{Fm}^{\alpha}_{\Sigma}$, where [unless Σ has a nullary connective] $\alpha \in \wp_{\infty[\setminus 1]}(\omega)$, called the *inter-derivablity relation of* C, whenever $\alpha = \omega$. A *congruence of* C is any $\theta \in \mathrm{Con}(\mathfrak{Fm}^{\omega}_{\Sigma})$ such that $\theta \subseteq \equiv^{\omega}_{C}$, the set of all them being denoted by $\mathrm{Con}(C)$. Then, given any $\theta, \vartheta \in \mathrm{Con}(C)$, the transitive closure of $\theta \cup \vartheta$, being well-known to be a congruence of $\mathfrak{Fm}^{\omega}_{\Sigma}$, is then that of C, for θ^{ω}_{C}, being an equivalence relation, is transitive. In particular, any maximal congruence of C (that exists, by Zorn Lemma, because $\mathrm{Con}(C) \ni \Delta_{\mathrm{Fm}^{\omega}_{\Sigma}}$ is both non-empty and inductive, for $\mathrm{Con}(\mathfrak{Fm}^{\omega}_{\Sigma})$ is so) is the greatest one to be denoted by $\Game(C)$. Then, C is said to be *self-extensional*, whenever $\equiv^{\omega}_{C} \in \mathrm{Con}(\mathfrak{Fm}^{\omega}_{\Sigma})$, that is, $\Game(C) = \equiv^{\omega}_{C}$.

Definition 2.3. Given a Σ-logic C, the variety $\mathrm{IV}(C)$ axiomatized by $\Game(C)$ is called the *intrinsic variety of* C (cf. [11]). ☐

Next, a Σ-logic C is said to be *[inferentially] (in)consistent*, provided $x_1 \notin (\in)C(\varnothing[\cup\{x_0\}])$ [(in which case $\equiv^{\omega}_{C} = \mathrm{Eq}^{\omega}_{\Sigma} \in \mathrm{Con}(\mathfrak{Fm}^{\omega}_{\Sigma})$, and so C is self-extensional)]. Further, a Σ-rule $\Gamma \to \Phi$ is said to be *satisfied/derivable in* C, provided $\Phi \in C(\Gamma)$, Σ-axioms satisfied in C being referred to as *theorems of* C.

Definition 2.4. A Σ-logic C' is said to be a *(proper) [K-]extension of* a Σ-logic C [where $K \subseteq \infty$], whenever ($C' \neq C$ and) $C(X) \subseteq C'(X)$, for all $X \in \wp_{[K]}(\mathrm{Fm}^{\omega}_{\Sigma})$, in which case C is said to be a *(proper) [K-]sublogic of* C'. ☐

Next, a Σ-logic C is said to be *(strongly)/weakly* $\barwedge$*-conjunctive*, provided $C(\{x_0, x_1\}) = / \subseteq C(x_0 \barwedge x_1)$. Likewise, C is said to be *(strongly)/weakly* $\veebar$*-disjunctive*, if $C(X \cup \{\phi \veebar \psi\}) = / \subseteq (C(X \cup \{\phi\}) \cap C(X \cup \{\psi\}))$, where $(X \cup \{\phi,\psi\}) \subseteq \mathrm{Fm}^{\omega}_{\Sigma}$, "in which case"/"that is, the first two of" the following rules:

$$x_0 \vdash (x_0 \veebar x_1), \tag{2.2}$$

$$x_1 \vdash (x_0 \veebar x_1), \tag{2.3}$$

$$(x_0 \veebar x_1) \vdash (x_1 \veebar x_0), \tag{2.4}$$

$$(x_0 \veebar x_0) \vdash x_0, \tag{2.5}$$

are satisfied in C. Further, C is said to *have/satisfy Deduction Theorem (DT) with respect to* a (possibly, secondary) binary connective $\sqsupset$ of Σ (fixed throughout the paper by default), provided, for all $\phi \in X \subseteq \mathrm{Fm}^{\omega}_{\Sigma}$ and all $\psi \in C(X)$, it holds that $(\phi \sqsupset \psi) \in C(X \setminus \{\phi\})$, in which case the following axioms:

$$x_0 \sqsupset x_0, \tag{2.6}$$

$$x_0 \sqsupset (x_1 \sqsupset x_0) \tag{2.7}$$

are satisfied in C. Then, C is said to be *weakly* $\sqsupset$*-implicative*, if it has DT w.r.t. $\sqsupset$ as well as satisfies the *Modus Ponens* rule:

$$\{x_0, x_0 \sqsupset x_1\} \vdash x_1, \tag{2.8}$$

in which case the following axiom:

$$(x_0 \uplus_{\sqsupset} (x_0 \sqsupset x_1)), \tag{2.9}$$

where $(x_0 \uplus_{\sqsupset} x_1) \triangleq ((x_0 \sqsupset x_1) \sqsupset x_1)$ is the *intrinsic disjunction of (implication)* $\sqsupset$, is satisfied in C. Likewise, C is said to be *(strongly)* $\sqsupset$*-implicative*, whenever it is weakly so and satisfies the *Peirce Law* axiom:

$$((x_0 \sqsupset x_1) \uplus_{\sqsupset} x_0). \tag{2.10}$$

Furthermore, C is said to be *$\wr$-paraconsistent*, where $\wr$ is a (possibly, secondary) unary connective of Σ (tacitly fixed throughout the paper by default), provided it does not satisfy the *Ex Contradictione Quodlibet* rule:

$$\{x_0, \wr x_0\} \vdash x_1. \tag{2.11}$$

Likewise, C is said to be *$(\veebar, \wr)$-paracomplete*, whenever it does not satisfy the *Excluded Middle Law* axiom:

$$x_0 \veebar \wr x_0. \tag{2.12}$$

Given any $\Sigma' \subseteq \Sigma$, the Σ'-logic C', defined by $C'(X) \triangleq (\mathrm{Fm}^{\omega}_{\Sigma'} \cap C(X))$, for all $X \subseteq \mathrm{Fm}^{\omega}_{\Sigma'}$, is called the *$\Sigma'$-fragment of C*, C being referred to as a *(Σ-)expansion of C'*, in which case $\equiv^{\omega}_{C'} = (\equiv^{\omega}_{C} \cap \mathrm{Eq}^{\omega}_{\Sigma'})$, and so C' is self-extensional, whenever C is so. Finally, C is said to be *theorem-less/purely-inferential*, whenever it has no theorem, that is, $\varnothing \in (\mathrm{img}\, C)$. In general, $(\mathrm{img}\, C) \cup \{\varnothing\}$ is closed under inverse Σ-substitutions, for $\mathrm{img}\, C$ is so, in which case the dual closure operator C_{+0} is the greatest purely-inferential sublogic of C, called the *purely-inferential version of C* and being an $(\infty \setminus 1)$-extension of C (cf. Definition 2.4), so

$$\equiv^{\omega}_{C} = \equiv^{\omega}_{C_{+0}} \tag{2.13}$$

(in particular, C_{+0} is self-extensional iff C is so).

2.3.2. *Logical matrices.* A *(logical) Σ-matrix* is any pair of the form $\mathcal{A} = \langle \mathfrak{A}, D^{\mathcal{A}} \rangle$, where $\mathfrak{A}$ is a Σ-algebra, called the *underlying algebra of $\mathcal{A}$*, while A is called the *carrier/"underlying set" of $\mathcal{A}$*, whereas $D^{\mathcal{A}} \subseteq A$ is called the *truth predicate of $\mathcal{A}$*, elements of $A[\cap D^{\mathcal{A}}]$ being referred to as *[distinguished] values of $\mathcal{A}$*. (In general, matrices are denoted by Calligraphic letters [possibly, with indices], their underlying algebras being denoted by corresponding capital Fraktur letters [with same indices, if any].) This is said to be *[no-more-than-]n-valued*, where $n \in (\omega \setminus 1)$, provided $|A| = [\leqslant]n$. Next, it is said to be *[in]consistent*, whenever $D^{\mathcal{A}} \neq [=]A$, respectively. Likewise, it is is said to be *truth[-non]-empty*, whenever $D^{\mathcal{A}} = [\neq]\varnothing$. Further, it is said to be *truth-/false-singular*, provided $|((D^{\mathcal{A}}/(A \setminus D^{\mathcal{A}}))| \in 2$, respectively. Finally, $\mathcal{A}$ is said to be *finite[ly generated]/"generated by* a $B \subseteq A$", whenever $\mathfrak{A}$ is so.

Given any $\alpha \in \wp_{\infty[\setminus 1]}(\omega)$ [unless Σ contains a nullary connective] and any class M of Σ-matrices, we have the closure operator $\mathrm{Cn}^{\alpha}_{\mathsf{M}}$ over $\mathrm{Fm}^{\alpha}_{\Sigma}$ dual to the closure system with basis $\{h^{-1}[D^{\mathcal{A}}] \mid \mathcal{A} \in \mathsf{M}, h \in \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{A})\}$, in which case:

$$\mathrm{Cn}^{\alpha}_{\mathsf{M}}(X) = (\mathrm{Fm}^{\alpha}_{\Sigma} \cap \mathrm{Cn}^{\omega}_{\mathsf{M}}(X)), \tag{2.14}$$

for all $X \subseteq \mathrm{Fm}^{\alpha}_{\Sigma}$. Then, by (2.1), $\mathrm{Cn}^{\omega}_{\mathsf{M}}$ is a Σ-logic, called the *logic of/"defined by"* M. A Σ-logic is said to be *("unitary/uniform[ly]"|double|finitely) {no-more-than-}n-valued*, where $n \in (\omega \setminus 1)$, whenever it is defined by a (one-element|two-element|finite) class of {no-more-than-}n-valued Σ-matrices (in which case it is finitary, and so is the logic of any finite class of finite Σ-matrices; cf. [5]).

As usual, Σ-matrices are treated as first-order model structures (viz., algebraic systems; cf. [7]) of the first-order signature $\Sigma \cup \{D\}$ with unary predicate D, any [in]finitary Σ-rule $\Gamma \vdash \phi$ being viewed as the [in]finitary equality-free basic strict Horn formula $(\bigwedge \Gamma) \to \phi$ under the standard identification of any propositional Σ-formula ψ with the first-order atomic formula $D(\psi)$, as well as being *true/satisfied in* a class M of Σ-matrices (in the conventional model-theoretic sense; cf., e.g., [7]) iff it being satisfied in the logic of M.

Remark 2.5. Since any rule with[out] premises is [not] true in any truth-empty matrix, given any class M of Σ-matrices, the purely-inferential version of the logic of M is defined by $\mathsf{M} \cup \mathsf{S}$, where S is a non-empty class of truth-empty Σ-matrices. $\square$

Let $\mathcal{A}$ and $\mathcal{B}$ be two Σ-matrices. A *(strict) [surjective]* $\{$*injective*$\}$ *homomorphism from* $\mathcal{A}$ *[on]to* $\mathcal{B}$ is any $\{$injective$\}$ $h \in \mathrm{hom}(\mathfrak{A}, \mathfrak{B})$ such that $[h[A] = B$ and$]$ $D^{\mathcal{A}} \subseteq h^{-1}[D^{\mathcal{B}}](\subseteq D^{\mathcal{A}})$, the set of all them being denoted by $\mathrm{hom}^{[\mathrm{S}]}_{(\mathrm{S})}(\mathcal{A}, \mathcal{B})$, in which case $\mathcal{B}/\mathcal{A}$ is said to be a *(strictly) [surjectively]* $\{$*injectively*$\}$ *homomorphic image/counter-image* ([$\{$as well as called an *isomorphic copy*$\}$]) *of* $\mathcal{A}/\mathcal{B}$, respectively. Then, by (2.1), we have:

$$(\mathrm{hom}^{(\mathrm{S})}_{\mathrm{S}}(\mathcal{A}, \mathcal{B}) \neq \varnothing) \Rightarrow (\mathrm{Cn}^{\alpha}_{\mathcal{B}}(X) \subseteq (=) \, \mathrm{Cn}^{\alpha}_{\mathcal{A}}(X)), \tag{2.15}$$

for all $\alpha \in \wp_{\infty[\setminus 1]}(\omega)$ [unless Σ has a nullary connective] and all $X \subseteq \mathrm{Fm}^{\alpha}_{\Sigma}$. Further, $\mathcal{A}[\neq \mathcal{B}]$ is said to be a *[proper] submatrix of* $\mathcal{B}$, whenever $\Delta_A \in \mathrm{hom}_{\mathrm{S}}(\mathcal{A}, \mathcal{B})$, in which case we set $(\mathcal{B} \restriction A) \triangleq \mathcal{A}$. Injective/bijective strict homomorphisms from $\mathcal{A}$ to $\mathcal{B}$ are called *embeddings/isomorphisms of/from* $\mathcal{A}$ *into/onto* $\mathcal{B}$, in case of existence of which $\mathcal{A}$ is said to be *embeddable/isomorphic into/to* $\mathcal{B}$.

Given a Σ-matrix $\mathcal{A}$, $(\chi^{\mathcal{A}}/\theta^{\mathcal{A}}) \triangleq (\chi^{D^{\mathcal{A}}}_{A})/(\ker \chi^{\mathcal{A}}))$ is referred to as the *characteristic function/relation of* $\mathcal{A}$. Then, any $\theta \in \mathrm{Con}(\mathfrak{A})$ such that $\theta \subseteq \theta^{\mathcal{A}}$, in which case ν_θ is a strict surjective homomorphism from $\mathcal{A}$ onto $(\mathcal{A}/\theta) \triangleq \langle \mathfrak{A}/\theta, D^{\mathcal{A}}/\theta \rangle$, is called a *congruence of* $\mathcal{A}$, the set of all them being denoted by $\mathrm{Con}(\mathcal{A})$. Given any $\theta, \vartheta \in \mathrm{Con}(\mathcal{A})$, the transitive closure of $\theta \cup \vartheta$, being well-known to be a congruence of $\mathfrak{A}$, is then that of $\mathcal{A}$, for $\theta^{\mathcal{A}}$, being an equivalence relation, is transitive. In particular, any maximal congruence of $\mathcal{A}$ (that exists, by Zorn Lemma, because $\mathrm{Con}(\mathcal{A}) \ni \Delta_A$ is both non-empty and inductive, for $\mathrm{Con}(\mathfrak{A})$ is so) is the greatest one to be denoted by $\backepsilon(\mathcal{A})$ (this is traditionally called the *Leibniz* congruence of $\mathcal{A}$ and denoted, for unclear reasons, by $\Omega(\mathcal{A})$, though here we naturally adapt conventions adopted in [17] to use its results *immediately*). Finally, $\mathcal{A}$ is said to be *[(finitely) hereditarily] simple*, whenever it has no non-diagonal congruence [and no non-simple (finitely-generated) submatrix].

Remark 2.6. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be two* Σ*-matrices and* $h \in \mathrm{hom}(\mathcal{A}, \mathcal{B})$ *strict [and surjective]. Then,* $\theta^{\mathcal{A}} = h^{-1}[\theta^{\mathcal{B}}]$ and, for every $\theta \in \mathrm{Con}(\mathfrak{B})$, $(\ker h) \subseteq h^{-1}[\theta] \in \mathrm{Con}(\mathfrak{A})$ [while $h[\theta^{\mathcal{A}}] = \theta^{\mathcal{B}}$ as well as $h[h^{-1}[\theta]] = \theta$, whereas, for every $\vartheta \in \mathrm{Con}(\mathfrak{A})$ including $(\ker h)$, both $h[\vartheta] \in \mathrm{Con}(\mathfrak{B})$ and $h^{-1}[h[\vartheta]] = \vartheta$]. Therefore,

(i) *for every* $\theta \in \mathrm{Con}(\mathcal{B})$, $(\ker h) \subseteq h^{-1}[\theta] \in \mathrm{Con}(\mathcal{A})$ *[while* $h[h^{-1}[\theta]] = \theta$*, whereas, for every* $\vartheta \in \mathrm{Con}(\mathcal{A})$ *including* $(\ker h)$*, both* $h[\vartheta] \in \mathrm{Con}(\mathcal{B})$ *and* $h^{-1}[h[\vartheta]] = \vartheta$*].*

In particular (when $\theta = \Delta_B$), we have $(\ker h) = h^{-1}[\Delta_B] \in \mathrm{Con}(\mathcal{A})$, in which case we get $(\ker h) \subseteq \backepsilon(\mathcal{A})$, and so

(ii) h *is injective, whenever* $\mathcal{A}$ *is simple.*

[Likewise, when $\vartheta = \backepsilon(\mathcal{A}) \supseteq (\ker h)$ and $\theta = \backepsilon(\mathcal{B})$, we have $h[\vartheta] \in \mathrm{Con}(\mathcal{B})$ and $h^{-1}[\theta] \in \mathrm{Con}(\mathcal{A})$, in which case we get $h[\vartheta] \subseteq \theta$ and $h^{-1}[\theta] \subseteq \vartheta$, and so:

(iii) $\backepsilon(\mathcal{A}) = h^{-1}[\backepsilon(\mathcal{B})]$ *and* $\backepsilon(\mathcal{B}) = h[\backepsilon(\mathcal{A})]$.

In particular, when $\mathcal{B} = (\mathcal{A}/\vartheta)$ and $h = \nu_\vartheta$, we have $\theta = h[\vartheta] = \Delta_B$, and so

(iv) $\mathcal{A}/\backepsilon(\mathcal{A})$ *is simple.*] □

Definition 2.7. A Σ-matrix $\mathcal{A}$ is said to be a *[K-]model of* a Σ-logic C $\{$*over* $\mathfrak{A}\}$ [where $K \subseteq \infty$], provided C is a [K-]sublogic of the logic of $\mathcal{A}$ $\langle$cf. Definition 2.4$\rangle$, the class of all (simple of) them being denoted by $\mathrm{Mod}^{(*)}_{[K]}(C\{, \mathfrak{A}\})$, respectively. Then, $\mathrm{Fi}_C(\mathfrak{A}) \triangleq \pi_1[\mathrm{Mod}(C, \mathfrak{A})]$, elements of which are called *filters of* C *over* $\mathfrak{A}$, is a closure system over A, the dual closure operator — of filter generation — being denoted by $\mathrm{Fg}^{\mathfrak{A}}_{C}$. □

A Σ-matrix $\mathcal{A}$ is said to be $\wr$*-paraconsistent/*$(\veebar, \wr)$*-paracomplete*, whenever its logic is so. Next, $\mathcal{A}$ is said to be *(strongly)/weakly* $\diamond$*-conjunctive*, provided $(\{a, b\} \subseteq$

$D^{\mathcal{A}}) \Leftrightarrow / \Leftarrow ((a \diamond^{\mathfrak{A}} b) \in D^{\mathcal{A}})$, for all $a, b \in A$, that is, the logic of $\mathcal{A}$ is strongly/weakly $\diamond$-conjunctive. Then, $\mathcal{A}$ is said to be *(strongly)/weakly $\diamond$-disjunctive*, whenever $\langle \mathfrak{A}, A \setminus D^{\mathcal{A}} \rangle$ is strongly/weakly $\diamond$-conjunctive, "in which case"/"that is," the logic of $\mathcal{A}$ is strongly/weakly $\diamond$-disjunctive, and so is the logic of any class of strongly/weakly $\diamond$-disjunctive Σ-matrices. Likewise, $\mathcal{A}$ is said to be *(strongly) $\sqsupset$-implicative*, whenever $((a \in D^{\mathcal{A}}) \Rightarrow (b \in D^{\mathcal{A}})) \Leftrightarrow ((a \sqsupset^{\mathfrak{A}} b) \in D^{\mathcal{A}})$, for all $a, b \in A$, in which case it is $\uplus_{\sqsupset}$-disjunctive, while the logic of $\mathcal{A}$ is $\sqsupset$-implicative, for both (2.8) and (2.10) are true in any $\sqsupset$-implicative (and so $\uplus_{\sqsupset}$-disjunctive) Σ-matrix, while DT is immediate, and so is the logic of any class of $\sqsupset$-implicative Σ-matrices. Furthermore, given any $\Sigma' \subseteq \Sigma$, $\mathcal{A}$ is said to be a *(Σ-)expansion of* its Σ'*-reduct* $(\mathcal{A} \upharpoonright \Sigma') \triangleq \langle \mathfrak{A} \upharpoonright \Sigma', D^{\mathcal{A}} \rangle$, clearly defining the Σ'-fragment of the logic of $\mathcal{A}$. Finally, $\mathcal{A}$ is said to be *(classically) $\wr$-negative*, provided, for all $a \in A$, $(a \in D^{\mathcal{A}}) \Leftrightarrow (\wr^{\mathfrak{A}} a \notin D^{\mathcal{A}})$, in which case it is truth-non-empty, and so consistent.

Remark 2.8. The following hold:

(i) any $\wr$-negative Σ-matrix $\mathcal{A}$:
  (a) is [weakly] $\diamond$-disjunctive/-conjunctive iff it is [weakly] $\diamond^{\wr}$-conjunctive/-disjunctive, respectively, where $(x_0 \diamond^{\wr} x_1) \triangleq \wr(\wr x_0 \diamond \wr x_1)$ is the *$\wr$-dual counterpart of* $\diamond$;
  (b) is $\sqsupset^{\wr}_{\diamond}$-implicative, whenever it is $\diamond$-disjunctive, where $(x_0 \sqsupset^{\wr}_{\diamond} x_1) \triangleq (\wr x_0 \diamond x_1)$ is the *material implication of/"defined|given by" (negation) $\wr$ and (disjunction) $\diamond$.*
  (c) is not $\wr$-paraconsistent/"$(\diamond, \wr)$-paracomplete, whenever it is weakly $\diamond$-disjunctive";
(ii) given any Σ-matrices $\mathcal{A}$ and $\mathcal{B}$ as well as any strict [surjective] $h \in \hom(\mathcal{A}, \mathcal{B})$, the following hold:
  (a) $\mathcal{A}$ is (weakly, if applicable) $\wr$-negative|$\diamond$-conjunctive/-disjunctive/-implicative if[f] $\mathcal{B}$ is so;
  (b) $\mathcal{B}$ is consistent/truth-non-empty if[f] $\mathcal{A}$ is so;
  (c) providing h is injective, $\mathcal{A}$ is false-/truth-singular if[f] $\mathcal{B}$ is so. □

Given a set I and an I-tuple $\overline{\mathcal{A}}$ of Σ-matrices, [any submatrix $\mathcal{B}$ of] the Σ-matrix $(\prod_{i \in I} \mathcal{A}_i) \triangleq \langle \prod_{i \in I} \mathfrak{A}_i, \prod_{i \in I} D^{\mathcal{A}_i} \rangle$ is called the [a] *[sub]direct product of* $\overline{\mathcal{A}}$ [whenever, for each $i \in I$, $\pi_i[B] = A_i$].

Given a class M of Σ-matrices, the class of all "strictly surjectively homomorphic [counter-]images"/"isomorphic copies"/"(consistent) submatrices" of elements of M is denoted, respectively, by $(\mathbf{H}^{[-1]}/\mathbf{I}/\mathbf{S}_{(*)})(\mathsf{M})$. Likewise, the class of all [sub]direct products of tuples (of cardinality $\in K \subseteq \infty$) constituted by elements of M is denoted by $\mathbf{P}^{[\mathrm{SD}]}_{(K)}(\mathsf{M})$.

2.3.2.1. Classical matrices and logics. Σ-matrices with diagonal characteristic function (and so relation) are said to be *classically-canonical*, isomorphisms between them being diagonal, in which case isomorphic ones being equal. Then, the characteristic function of any Σ-matrix $\mathcal{A}$ with diagonal characteristic relation — viz., injective characteristic function — (and so no-more-than-two-valued) is an isomorphism from it onto the classically-canonical Σ-matrix $\mathsf{C}(\mathcal{A}) \triangleq \langle \chi^{\mathcal{A}}[\mathfrak{A}], \{1\} \rangle$, called the *[classical] canonization of* $\mathcal{A}$.

A (classically-canonical) two-valued Σ-matrix $\mathcal{A}$ is said to be *(canonical[ly]) $\wr$-classical*, whenever it is $\wr$-negative, in which case it is both false- and truth-singular (and so its characteristic relation is diagonal) but is not $\wr$-paraconsistent, by Remark 2.8(i)(c).

A Σ-logic is said to be *$\wr$-[sub]classical*, whenever it is [a sublogic of] the logic of a $\wr$-classical Σ-matrix, in which case it is inferentially consistent.

2.3.2.2. Equality determinants. According to [13], an *equality determinant for* a Σ-matrix $\mathcal{A}$ is any $\Upsilon \subseteq \mathrm{Fm}^1_\Sigma$ such that, any $a, b \in A$ are equal, whenever, for all $\upsilon \in \Upsilon$, it holds that $(\upsilon^{\mathfrak{A}}(a) \in D^{\mathcal{A}}) \Leftrightarrow (\upsilon^{\mathfrak{A}}(b) \in D^{\mathcal{A}})$, in which case Υ is an equality determinant for every Σ-matrix embeddable into $\mathcal{A}$ (cf. Lemma 3.3 of [17]), while $\mathcal{A}$ is simple (cf. Lemma 3.2 therein), and so hereditarily simple. Clearly, any consistent truth-non-empty two-valued (in particular, classical) Σ-matrix $\mathcal{A}$ is both false- and truth-singular, in which case its characteristic relation is diagonal, and so $\{x_0\}$ is an equality determinant for $\mathcal{A}$ (cf. Example 1/3.1 of [13]/[17]).

3. Preliminary key advanced generic issues

3.1. **Disjunctivity.**

3.1.1. *Disjunctivity versus multiplicativity.* To unify further notations, set $(X \veebar Y) \triangleq \veebar[X \times Y]$, where $X, Y \subseteq \mathrm{Fm}^\omega_\Sigma$.

Then, a Σ-logic C is said to be $\veebar$*-(singularly-)multiplicative*, provided, for all $X \subseteq \mathrm{Fm}^\omega_\Sigma$ and all $\phi, \psi \in \mathrm{Fm}^\omega_\Sigma$, it holds that $(C(X \cup \{\phi\}) \veebar \psi) \subseteq C(X \cup \{\phi \veebar \psi\})$.

Lemma 3.1. *Any Σ-logic C is $\veebar$-disjunctive iff it is both weakly $\veebar$-disjunctive and $\veebar$-multiplicative as well as satisfies both* (2.4) *and* (2.5).

Proof. The "only if" part is immediate. Conversely, assume C is both weakly $\veebar$-disjunctive and $\veebar$-multiplicative as well as satisfies both (2.4) and (2.5). Consider any $X \subseteq \mathrm{Fm}^\omega_\Sigma$, any $\phi, \psi \in \mathrm{Fm}^\omega_\Sigma$ and any $\varphi \in (C(X \cup \{\phi\}) \cap C(X \cup \{\psi\}))$. Then, by the $\veebar$-multiplicativity of C and (2.4), we have $(\psi \veebar \varphi) \in C(\varphi \veebar \psi) \subseteq C(X \cup \{\phi \veebar \psi\})$. Likewise, by the $\veebar$-multiplicativity of C and (2.5), we have $\varphi \in C(\varphi \veebar \varphi) \subseteq C(X \cup \{\psi \veebar \varphi\})$. In this way, we eventually get $\varphi \in C(X \cup \{\phi \veebar \psi\})$. ☐

3.1.2. *Disjunctive consistent finitely-generated models of finitely-valued weakly disjunctive logics.*

Lemma 3.2. $\mathbf{H}(\mathbf{H}^{-1}(\mathsf{M})) \subseteq \mathbf{H}^{-1}(\mathbf{H}(\mathsf{M}))$*, for any class of Σ-matrices* M*.*

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be Σ-matrices, $\mathcal{C} \in \mathsf{M}$ and $(h|g) \in \mathrm{hom}^{\mathrm{S}}_{\mathrm{S}}(\mathcal{B}, \mathcal{C}|\mathcal{A})$. Then, by Remark 2.6(i), $(\ker(h|g)) \in \mathrm{Con}(\mathcal{B})$, in which case $(\ker(h|g)) \subseteq \theta \triangleq \partial(\mathcal{B}) \in \mathrm{Con}(\mathcal{B})$, and so, by the Homomorphism Theorem, $(\nu_\theta \circ (h|g)^{-1}) \in \mathrm{hom}^{\mathrm{S}}_{\mathrm{S}}(\mathcal{C}|\mathcal{A}, \mathcal{B}/\theta)$. ☐

Lemma 3.3 (Finite Subdirect Product Lemma; cf. Lemma 2.7 of [17])**.** *Let* M *be a finite class of finite Σ-matrices and $\mathcal{A}$ a [non-]simple finite(ly-generated) model of the logic of* M*. Then,* $(\mathcal{A}[/\partial(\mathcal{A})]) \in \mathbf{HP}^{\mathrm{SD}}_\omega \mathbf{S}_* \mathsf{M}$*.*

Lemma 3.4. *Let* M *be a class of weakly $\veebar$-disjunctive Σ-matrices, I a finite set, $\overline{\mathcal{C}} \in \mathsf{M}^I$, and $\mathcal{D}$ a consistent $\veebar$-disjunctive submatrix of $\prod \overline{\mathcal{C}}$. Then, there is some $i \in I$ such that $(\pi_i \restriction D) \in \mathrm{hom}^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D}, \mathcal{C}_i)$.*

Proof. By contradiction. For suppose that, for every $i \in I$, $(\pi_i \restriction D) \notin \mathrm{hom}^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D}, \mathcal{C}_i)$, in which case $D^{\mathcal{D}} \subsetneq (\pi_i \restriction D)^{-1}[D^{\mathcal{C}_i}] = (D \cap \pi_i^{-1}[D^{\mathcal{C}_i}])$, for $(\pi_i \restriction D) \in \mathrm{hom}(\mathcal{D}, \mathcal{C}_i)$ is surjective, and so there is some $a_i \in (D \setminus D^{\mathcal{D}})$ such that $\pi_i(a_i) \in D^{\mathcal{C}_i}$. By induction on the cardinality of any $J \subseteq I$, let us prove that there is some $b \in (D \setminus D^{\mathcal{D}})$ such that $\pi_j(b) \in D^{\mathcal{C}_j}$, for all $j \in J$, as follows. In case $J = \varnothing$, take any $b \in (D \setminus D^{\mathcal{D}}) \neq \varnothing$, for $\mathcal{D}$ is consistent. Otherwise, take any $j \in J$, in which case $K \triangleq (J \setminus \{j\}) \subseteq I$, while $|K| < |J|$, so, by the induction hypothesis, there is some $c \in (D \setminus D^{\mathcal{D}})$ such that $\pi_k(c) \in D^{\mathcal{C}_k}$, for all $k \in K$. Then, by the $\veebar$-disjunctivity of $\mathcal{D}$, $b \triangleq (c \veebar^{\mathfrak{D}} a_j) \in (D \setminus D^{\mathcal{D}})$, while $\pi_i(b) \in D^{\mathcal{C}_i}$, for all $i \in J = (K \cup \{j\})$, because $(\pi_i \restriction D) \in \mathrm{hom}(\mathfrak{D}, \mathfrak{C}_i)$, while $\mathcal{C}_i$ is weakly $\veebar$-disjunctive. In particular, when $J = I$, there is some $b \in (D \setminus D^{\mathcal{D}})$ such that $\pi_i(b) \in D^{\mathcal{C}_i}$, for all $i \in I$. This contradicts to the fact that $D^{\mathcal{D}} = (D \cap \bigcap_{i \in I} \pi_i^{-1}[D^{\mathcal{C}_i}])$, as required. ☐

By Lemmas 3.2, 3.3, 3.4 and Remark 2.8(ii), we immediately have:

Theorem 3.5. *Let* M *be a finite class of finite weakly* $\underline{\vee}$*-disjunctive* Σ*-matrices,* C *the logic of* M *and* $\mathcal{A}$ *a finite[ly-generated] consistent* $\underline{\vee}$*-disjunctive model of* C*. Then,* $\mathcal{A} \in \mathbf{H}^{-1}(\mathbf{H}(\mathbf{S}_*(\mathsf{M})))$.

3.1.2.1. Theorems of weakly disjunctive finitely-valued logics versus truth-empty submatrices of defining matrices.

Corollary 3.6. *Let* C *be a* Σ*-logic. (Suppose it is defined by a finite class* M *of finite [weakly* $\underline{\vee}$*-disjunctive]* Σ*-matrices.) Then, (i)⇔(ii)⇔(iii)(⇔(iv)), where:*

(i) *C is purely-inferential;*
(ii) *C has a truth-empty model;*
(iii) *C has a one-valued truth-empty model;*
(iv) $\mathbf{P}^{\mathrm{SD}}_{\omega[\cap 0]}(\mathbf{S}_*(\mathsf{M}))[\cup \mathbf{S}_*(\mathsf{M})]$ *has a truth-empty element.*

Proof. First, (ii)⇒(i) is immediate. The converse is by the fact that, by the structurality of C, $\langle \mathfrak{Fm}^{\omega}_{\Sigma}, C(\varnothing)\rangle$ is a model of C.

Next, (ii) is a particular case of (iii). Conversely, let $\mathcal{A} \in \mathrm{Mod}(C)$ be truth-empty. Then, $\chi^{\mathcal{A}}$ is singular, in which case $\theta^{\mathcal{A}} = A^2 \in \mathrm{Con}(\mathfrak{A})$, and so, by (2.15), $(\mathcal{A}/\theta^{\mathcal{A}}) \in \mathrm{Mod}(C)$ is both one-valued and truth-empty.

(Finally, (iv)⇒(ii) is by (2.15). Conversely, (iii)⇒(iv) is by Lemma 3.3 [resp., Theorem 3.5 as well as the consistency and $\underline{\vee}$-disjunctivity of truth-empty Σ-matrices].) ☐

3.2. **Implicativity versus weak implicativity.**

3.2.1. *Implicativity versus intrinsic disjunctivity.*

Theorem 3.7. *Let* C *be a weakly* $\sqsupset$*-implicative* Σ*-logic and* $\underline{\vee} \triangleq \uplus_{\sqsupset}$*. Then, the following hold:*

(i) *C is both weakly $\underline{\vee}$-disjunctive and $\underline{\vee}$-multiplicative;*
(ii) *C is $\sqsupset$-implicative iff it is $\underline{\vee}$-disjunctive iff it satisfies* (2.4).

Proof. (i) First, (2.2) is by DT and (2.8). Likewise, (2.3) is by (2.7) and (2.8). Now, consider any $X \subseteq \mathrm{Fm}^{\omega}_{\Sigma}$ and any $\phi, \psi, \varphi \in \mathrm{Fm}^{\omega}_{\Sigma}$. Then, by DT and (2.8), we have $((\psi \in C(X \cup \{\phi\}) \Rightarrow ((\phi \sqsupset \varphi) \in C(X \cup \{\psi \sqsupset \varphi\})$, applying which twice, the second time being with $(\psi \sqsupset \varphi)|(\phi \sqsupset \varphi)$ instead of $\phi|\psi$, respectively, we conclude that C is $\underline{\vee}$-multiplicative.

(ii) Assume C is $\sqsupset$-implicative. Then, $((x_0 \underline{\vee} x_0) \sqsupset x_0) = ((2.10)[x_1/x_0])$ is satisfied in C, for this is structural, and so is (2.5), in view of (2.8). Furthermore, by (2.8), we have $x_0 \in C(\{x_0 \underline{\vee} x_1, x_0 \sqsupset x_1, x_1 \sqsupset x_0\})$, in which case, by DT, we get $((x_0 \sqsupset x_1) \sqsupset x_0) \in C(\{x_0 \underline{\vee} x_1, x_1 \sqsupset x_0\})$, and so, by (2.8) and (2.10), we eventually get $x_0 \in C(\{x_0 \underline{\vee} x_1, x_1 \sqsupset x_0\})$ (in particular, by DT, (2.4) is satisfied in C). In this way, Lemma 3.1, (i) and (2.9) complete the argument. ☐

3.2.2. *False-singular models of weakly implicative logics.*

Lemma 3.8. *Let* $\mathcal{A}$ *be a false-singular* Σ*-matrix. Suppose* (2.6), (2.7) *and* (2.8) *are true in* $\mathcal{A}$*. Then,* $\mathcal{A}$ *is* $\sqsupset$*-implicative. In particular, any false-singular* Σ*-matrix is* $\sqsupset$*-implicative iff its logic is [weakly] so.*

Proof. Then, for all $a, b \in (A \setminus D^{\mathcal{A}})$, we have $a = b$, in which case, by (2.6), we get $(a \sqsupset^{\mathfrak{A}} b) = (a \sqsupset^{\mathfrak{A}} a) \in D^{\mathcal{A}}$, and so (2.7) and (2.8) complete the argument. ☐

3.3. **Logic versus model congruences.**

Lemma 3.9. *Let C be a Σ-logic, $\theta \in \mathrm{Con}(C)$, $\mathcal{A} \in \mathrm{Mod}(C)$ and $h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$. Then, $h[\theta] \subseteq \reflectbox{\ensuremath{\partial}}(\mathcal{A})$.*

Proof. Then, $\vartheta \triangleq (\bigcup\{g[\theta] \mid g \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})\})$ is symmetric, for θ is so. And what is more, since $\theta \subseteq \equiv^{\omega}_{C}$, while $\mathcal{A} \in \mathrm{Mod}(C)$, $\vartheta \subseteq \theta^{\mathcal{A}}$. Next, consider any $a \in A$. Let $g \triangleq [x_k/a]_{k\in\omega} \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$. Then, since $\langle x_0, x_0\rangle \in \theta$, $\langle a, a\rangle = g(\langle x_0, x_0\rangle) \in g[\theta] \subseteq \vartheta$, and so $\Delta_A \subseteq \vartheta$. Now, consider any $\varsigma \in \Sigma$ of arity $n \in \omega$, any $i \in n$, any $\langle a, b\rangle \in \vartheta$ and any $\bar{c} \in A^{n-1}$. Then, there are some $\langle \phi, \psi\rangle \in \theta$ and some $f \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$ such that $a = f(\phi)$ and $b = f(\psi)$. Let $V \triangleq (\mathrm{Var}(\phi) \cup \mathrm{Var}(\psi) \cup \{x_i\}) \in \wp_{\omega}(\mathrm{Var}_{\omega})$, in which case $|\mathrm{Var}_{\omega} \setminus V| = \omega \geqslant (n-1)$, for $|\mathrm{Var}_{\omega}| = \omega$ is infinite, and so there is some injective $\bar{v} \in (\mathrm{Var}_{\omega} \setminus V)^{n-1}$. Let $\varphi \triangleq (\varsigma(\bar{x}_n)[x_j/v_j; x_k/v_{k-1}]_{j\in i; k\in(n\setminus(i+1))}) \in \mathrm{Fm}^{\omega}_{\Sigma}$ and $g \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$ extend $(f{\restriction}(\mathrm{Var}_{\omega} \setminus (\mathrm{img}\,\bar{v}))) \cup (\bar{c} \circ \bar{v}^{-1})$, in which case $\langle \varphi[x_i/\phi], \varphi[x_i/\psi]\rangle \in \theta$, so $\langle \varphi^{\mathfrak{A}}[x_i/a; v_l/c_l]_{l\in(n-1)}, \varphi^{\mathfrak{A}}[x_i/b; v_l/c_l]_{l\in(n-1)}\rangle = g(\langle \varphi[x_i/\phi], \varphi[x_i/\psi]\rangle) \in g[\theta] \subseteq \vartheta$. Thus, unary algebraic operations of $\mathfrak{A}$ are ϑ-monotonic. Therefore, the transitive closure η of ϑ is a congruence of $\mathfrak{A}$. And what is more, $\theta^{\mathcal{A}} \supseteq \vartheta$, being transitive, includes η, in which case $\eta \in \mathrm{Con}(\mathcal{A})$, and so $h[\theta] \subseteq \vartheta \subseteq \eta \subseteq \reflectbox{\ensuremath{\partial}}(\mathcal{A})$. ☐

3.3.1. *Simple models versus intrinsic varieties.* As a particular case of Lemma 3.9, we first have (from now on, we follow Definition 2.3 tacitly):

Corollary 3.10. *Let C be a Σ-logic. Then, $\pi_0[\mathrm{Mod}^*(C)] \subseteq \mathrm{IV}(C)$.*

Corollary 3.11. *Let C be a Σ-logic. Then, $\reflectbox{\ensuremath{\partial}}(C)$ is fully-invariant. In particular, $\reflectbox{\ensuremath{\partial}}(C) = \theta^{\omega}_{\mathrm{IV}(C)}$.*

Proof. Consider any $\sigma \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{Fm}^{\omega}_{\Sigma})$ and any $T \in (\mathrm{img}\,C)$, in which case, by the structurality of C, $\mathcal{A}_T \triangleq \langle \mathfrak{Fm}^{\omega}_{\Sigma}, T\rangle \in \mathrm{Mod}(C)$, so, by Lemma 3.9, $\sigma[\reflectbox{\ensuremath{\partial}}(C)] \subseteq \reflectbox{\ensuremath{\partial}}(\mathcal{A}_T)$. Then, $\sigma[\reflectbox{\ensuremath{\partial}}(C)] \subseteq \theta \triangleq (\mathrm{Eq}^{\omega}_{\Sigma} \cap \bigcap\{\reflectbox{\ensuremath{\partial}}(\mathcal{A}_T) \mid T \in (\mathrm{img}\,C)\}) \subseteq (\mathrm{Eq}^{\omega}_{\Sigma} \cap \bigcap\{\theta^{\mathcal{A}_T} \mid T \in (\mathrm{img}\,C)\} = \equiv^{\omega}_{C}$. Moreover, for each $T \in (\mathrm{img}\,C)$, $\reflectbox{\ensuremath{\partial}}(\mathcal{A}_T) \in \mathrm{Con}(\mathfrak{Fm}^{\omega}_{\Sigma})$, in which case $\theta \in \mathrm{Con}(\mathfrak{Fm}^{\omega}_{\Sigma})$, and so $\sigma[\reflectbox{\ensuremath{\partial}}(C)] \subseteq \theta \subseteq \reflectbox{\ensuremath{\partial}}(C)$. ☐

Lemma 3.12. *Let M be a class of Σ-matrices, $\mathsf{K} \triangleq \pi_0[\mathsf{M}]$ and C the logic of M. Then, $\theta^{\omega}_{\mathsf{K}} \subseteq \equiv^{\omega}_{C}$, in which case $\theta^{\omega}_{\mathsf{K}} \subseteq \reflectbox{\ensuremath{\partial}}(C)$, and so $\mathrm{IV}(C) \subseteq \mathbf{V}(\mathsf{K})$.*

Proof. Then, for any $\langle \phi, \psi\rangle \in \theta^{\omega}_{\mathsf{K}}$, $\mathcal{A} \in \mathsf{M}$ and $h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$, $\mathfrak{A} \in \mathsf{K}$, in which case $\langle h(\phi), h(\psi)\rangle \in \Delta_A \subseteq \theta^{\mathcal{A}}$, and so $\phi \equiv^{\omega}_{C} \psi$. ☐

By Corollary 3.10 and Lemma 3.12, we immediately have:

Corollary 3.13. *Let M be a class of Σ-matrices, $\mathsf{K} \triangleq \pi_0[\mathsf{M}]$ and C the logic of M. Then, $\pi_0[\mathrm{Mod}^*(C)] \subseteq \mathbf{V}(\mathsf{K})$.*

Likewise, by Corollary 3.10 and Lemma 3.12, we also have:

Theorem 3.14. *Let M be a class of simple Σ-matrices, $\mathsf{K} \triangleq \pi_0[\mathsf{M}]$ and C the logic of M. Then, $\mathrm{IV}(C) = \mathbf{V}(\mathsf{K})$.*

4. Self-extensional logics versus simple matrices

Theorem 4.1. *Let C be a Σ-logic and $\mathsf{V} \triangleq \mathrm{IV}(C)$ (as well as M a class of simple Σ-matrices, $\mathsf{K} \triangleq \pi_0[\mathsf{M}]$ and $\alpha \triangleq ([1\cup](\omega \cap \bigcup\{|A| \mid \mathcal{A} \in \mathsf{M}\})) \in \wp_{\infty[\setminus 1]}(\omega)$ [unless Σ contains a nullary connective]). (Suppose C is defined by M.) Then, (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii)($\Rightarrow$(iv)$\Rightarrow$(v)$\Rightarrow$) (vi)$\Rightarrow$(i), where:*

- (i) *C is self-extensional;*
- (ii) *$\equiv^{\omega}_{C} \subseteq \theta^{\omega}_{\mathsf{V}}$;*
- (iii) *$\equiv^{\omega}_{C} = \theta^{\omega}_{\mathsf{V}}$;*

(iv) *for all distinct* $a, b \in F^{\alpha}_{\mathsf{V}}$*, there are some* $\mathcal{A} \in \mathsf{M}$ *and some* $h \in \hom(\mathfrak{F}^{\alpha}_{\mathsf{V}}, \mathfrak{A})$ *such that* $\chi^{\mathcal{A}}(h(a)) \neq \chi^{\mathcal{A}}(h(b))$*;*

(v) *there is some class* C *of* Σ*-algebras such that* $\mathsf{K} \subseteq \mathbf{V}(\mathsf{C})$ *and, for each* $\mathfrak{A} \in \mathsf{C}$ *and all distinct* $a, b \in A$*, there are some* $\mathcal{B} \in \mathsf{M}$ *and some* $h \in \hom(\mathfrak{A}, \mathfrak{B})$ *such that* $\chi^{\mathcal{B}}(h(a)) \neq \chi^{\mathcal{B}}(h(b))$*;*

(vi) *there is some* $\mathsf{S} \subseteq \mathrm{Mod}(C)$ *such that* $\mathsf{V} \subseteq \mathbf{V}(\pi_0[\mathsf{S}])$ *and, for each* $\mathcal{A} \in \mathsf{S}$*, it holds that* $(A^2 \cap \bigcap\{\theta^{\mathcal{B}} \mid \mathcal{B} \in \mathsf{S}, \mathfrak{B} = \mathfrak{A}\}) \subseteq \Delta_A$*.*

(In particular, (i–vi) are equivalent.)

Proof. In that case, by Corollary 3.11 (and Theorem 3.14), $\reflectbox{\ensuremath{\partial}}(C) = \theta^{\omega}_{\mathsf{V}}$ (as well as $\mathsf{V} = \mathbf{V}(\mathsf{K})$, and so $\theta^{\omega}_{\mathsf{V}} = \theta^{\omega}_{\mathsf{K}}$). Then, (i)⇔(iii) is immediate, while (ii) is a particular case of (iii), whereas the converse is by the inclusion $\reflectbox{\ensuremath{\partial}}(C) \subseteq \equiv^{\omega}_{C}$.

(Next, assume (iii) holds. Then, $\theta^{\alpha'} \triangleq \equiv^{\alpha'}_{C} = \theta^{\alpha'}_{\mathsf{K}} = \theta^{\alpha'}_{\mathsf{V}} \in \mathrm{Con}(\mathfrak{Fm}^{\alpha'}_{\Sigma})$, for all $\alpha' \in \wp_{\infty[\setminus 1]}(\omega)$. Furthermore, consider any distinct $a, b \in F^{\alpha}_{\mathsf{V}}$. Then, there are some $\phi, \psi \in \mathrm{Fm}^{\alpha}_{\Sigma}$ such that $\nu_{\theta^{\alpha}}(\phi) = a \neq b = \nu_{\theta^{\alpha}}(\phi)$, in which case, by (2.14), $\mathrm{Cn}^{\alpha}_{\mathsf{M}}(\phi) \neq \mathrm{Cn}^{\alpha}_{\mathsf{M}}(\psi)$, and so there are some $\mathcal{A} \in \mathsf{M}$ and some $g \in \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{A})$ such that $\chi^{\mathcal{A}}(g(\phi)) \neq \chi^{\mathcal{A}}(g(\phi))$. In that case, $\theta^{\alpha} \subseteq (\ker g)$, and so, by the Homomorphism Theorem, $h \triangleq (g \circ \nu^{-1}_{\theta^{\alpha}}) \in \hom(\mathfrak{F}^{\alpha}_{\mathsf{V}}, \mathfrak{A})$. Then, $h(a/b) = g(\phi/\psi)$, in which case $\chi^{\mathcal{A}}(h(a)) \neq \chi^{\mathcal{A}}(h(b))$, and so (iv) holds.

Further, assume (iv) holds. Let $\mathsf{C} \triangleq \{\mathfrak{F}^{\alpha}_{\mathsf{V}}\}$. Consider any $\mathfrak{A} \in \mathsf{K}$ and the following complementary cases:

- $|A| \leqslant \alpha$.
  Let $h \in \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{A})$ extend any surjection from Var_{α} onto A, in which case it is surjective, while $\theta \triangleq \theta^{\alpha}_{\mathsf{V}} = \theta^{\alpha}_{\mathsf{K}} \subseteq (\ker h)$, and so, by the Homomorphism Theorem, $g \triangleq (h \circ \nu^{-1}_{\theta}) \in \hom(\mathfrak{F}^{\alpha}_{\mathsf{V}}, \mathfrak{A})$ is surjective. In this way, $\mathfrak{A} \in \mathbf{V}(\mathfrak{F}^{\alpha}_{\mathsf{V}})$.
- $|A| \nleqslant \alpha$.
  Then, $\alpha = \omega$. Consider any Σ-identity $\phi \approx \psi$ true in $\mathfrak{F}^{\omega}_{\mathsf{V}}$ and any $h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$, in which case, we have $\theta \triangleq \theta^{\omega}_{\mathsf{V}} = \theta^{\omega}_{\mathsf{K}} \subseteq (\ker h)$, and so, since $\nu_{\theta} \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{F}^{\omega}_{\mathsf{V}})$, we get $\langle \phi, \psi \rangle \in (\ker \nu_{\theta}) \subseteq (\ker h)$. In this way, $\mathfrak{A} \in \mathbf{V}(\mathfrak{F}^{\alpha}_{\mathsf{V}})$.

Thus, $\mathsf{K} \subseteq \mathbf{V}(\mathsf{C})$, and so (v) holds.

Now, assume (v) holds. Let C' be the class of all non-one-element elements of C and $\mathsf{S} \triangleq \{\langle \mathfrak{A}, h^{-1}[D^{\mathcal{B}}] \rangle \mid \mathfrak{A} \in \mathsf{C}', \mathcal{B} \in \mathsf{M}, h \in \hom(\mathfrak{A}, \mathfrak{B})\}$. Then, for all $\mathfrak{A} \in \mathsf{C}'$, each $\mathcal{B} \in \mathsf{M}$ and every $h \in \hom(\mathfrak{A}, \mathfrak{B})$, h is a strict homomorphism from $\mathcal{C} \triangleq \langle \mathfrak{A}, h^{-1}[D^{\mathcal{B}}] \rangle$ to $\mathcal{B}$, in which case, by (2.15), $\mathcal{C} \in \mathrm{Mod}(C)$, and so $\mathsf{S} \subseteq \mathrm{Mod}(C)$, while $\chi^{\mathcal{C}} = (h \circ \chi^{\mathcal{B}})$, whereas $\pi_0[\mathsf{S}] = \mathsf{C}'$ generates the variety $\mathbf{V}(\mathsf{C})$. In this way, (vi) holds.)

Finally, assume (vi) holds. Consider any $\phi, \psi \in \mathrm{Fm}^{\omega}_{\Sigma}$ such that $\phi \equiv^{\omega}_{C} \psi$, any $\mathcal{A} \in \mathsf{S}$ and any $h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$. Then, for each $\mathcal{B} \in \mathsf{S}$ with $\mathfrak{B} = \mathfrak{A}$, $h(\phi)\ \theta^{\mathcal{B}}\ h(\psi)$, in which case $h(\phi) = h(\psi)$, so $\mathfrak{A} \models (\phi \approx \psi)$. Thus, $\mathsf{V} \subseteq \mathbf{V}(\pi_0[\mathsf{S}]) \models (\phi \approx \psi)$, so (ii) holds. ☐

When both M and all elements of it are finite, α is finite, in which case $\mathfrak{F}^{\alpha}_{\mathsf{V}}$ is finite and can be found effectively, and so, taking (2.15) and Remark 2.6(iv) into account, the item (iv) of Theorem 4.1 yields an effective procedure of checking the self-extensionality of any logic defined by a finite class of finite matrices. However, its computational complexity may be too large to count it *practically* applicable. For instance, in the unitary n-valued case, where $n \in (\omega \setminus 1)$, the upper limit $n^{n^{n}}$ of $|F^{\alpha}_{\mathsf{V}}|$ as well as the predetermined computational complexity $n^{n^{n^{n}}}$ of the procedure involved become too large even in the three-/four-valued case. And, though, in the two-valued case, this limit — 16 — as well as the respective complexity —

$2^{16} = 65536$ — are reasonably acceptable, this is no longer matter in view of the following universal observation:

Example 4.2. Let $\mathcal{A}$ be a Σ-matrix. Suppose it is both false- and truth-singular (in particular, two-valued as well as both consistent and truth-non-empty [in particular, classical]), in which case $\theta^{\mathcal{A}} = \Delta_A$, for $\chi^{\mathcal{A}}$ is injective, and so $\mathcal{A}$ is simple. Then, by Theorems 3.14 and 4.1(vi)⇒(i) with $\mathsf{S} = \{\mathcal{A}\}$, the logic of $\mathcal{A}$ is self-extensional, its intrinsic variety being generated by $\mathfrak{A}$. Thus, by the self-extensionality of inferentially inconsistent logics, *any two-valued (in particular, classical) logic is self-extensional.* □

Nevertheless, the procedure involved is simplified much under hereditary simplicity as well as either implicativity or both conjunctivity and disjunctivity of finitely many finite defining matrices upon the basis of the item (v) of Theorem 4.1.

4.1. **Self-extensionality of conjunctive disjunctive logics versus distributive lattices.**

Remark 4.3. Let C be a $\overline{\wedge}$-conjunctive or/and $\underline{\vee}$-disjunctive Σ-logic and $\phi \approx \psi$ a semi-lattice/"distributive lattice" identity for $\overline{\wedge}$ or/and $\underline{\vee}$, respectively. Then, $\phi \equiv^{\omega}_{C} \psi$. □

Theorem 4.4. *Let C be a $\diamond$-conjunctive/-disjunctive Σ-logic (defined by a class M of simple Σ-matrices) and $i = (0/1)$ (as well as $\mathsf{K} \triangleq \pi_0[\mathsf{M}]$). Then, C is self-extensional iff the following hold:*

(i) *each element of $\mathrm{IV}(C)(= \mathbf{V}(\mathsf{K}))$ is a $\diamond$-semi-lattice;*
(ii) *for all $\bar{\varphi} \in (\mathrm{Fm}^{\omega}_{\Sigma})^2$, it holds that $(\varphi_1 \in C(\varphi_0)) \Leftrightarrow | \Rightarrow (\mathrm{IV}(C) \models (\varphi_i \approx (\varphi_0 \diamond \varphi_1))$.*

Proof. The "if" part is by Theorem 4.1(ii)⇒(i) and semi-lattice identities (more specifically, the commutativity one) for $\diamond$. Conversely, if C is self-extensional, then, by Theorem 4.1(i)⇒(iii), we have $\equiv^{\omega}_{C} = \theta^{\omega}_{\mathrm{IV}(C)}$, in which case, since C is $\diamond$-conjunctive/-disjunctive, (i) is by Remark 4.3 (and Theorem 3.14), while, for all $\bar{\varphi} \in (\mathrm{Fm}^{\omega}_{\Sigma})^2$, $(\varphi_1 \in C(\varphi_0)) \Leftrightarrow (\varphi_i \equiv^{\omega}_{C} (\varphi_0 \diamond \varphi_1))$, so (ii) holds. □

Lemma 4.5. *Let C be a [finitary $\overline{\wedge}$-conjunctive] Σ-logic and $\mathcal{A}$ a [truth-non-empty $\overline{\wedge}$-conjunctive] Σ-matrix. Then, $\mathcal{A} \in \mathrm{Mod}_{2\setminus 1}(C)$ if[f] $\mathcal{A} \in \mathrm{Mod}(C)$ (cf. Definition 2.7).*

Proof. The "if" part is trivial. [Conversely, assume $\mathcal{A} \in \mathrm{Mod}_{2\setminus 1}(C)$. Consider any $\varphi \in C(\varnothing)$ and any $h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$, in which case $V \triangleq \mathrm{Var}(\varphi) \in \wp_{\omega}(\mathrm{Var}_{\omega})$, and so $(\mathrm{Var}_{\omega} \setminus V) \neq \varnothing$, for, otherwise, we would have $V = \mathrm{Var}_{\omega}$, and so would get $\omega = |\mathrm{Var}_{\omega}| = |V| \in \omega$. Take any $v \in (\mathrm{Var}_{\omega} \setminus V)$ and any $a \in D^{\mathcal{A}} \neq \varnothing$. Let $g \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$ extend $(h{\restriction}(V \setminus \{v\})) \cup [v/a]$. Then, $\varphi \in C(v)$, $\{v\} \in \wp_{2\setminus 1}(\mathrm{Fm}^{\omega}_{\Sigma})$ and $g(v) = a \in D^{\mathcal{A}}$, in which case $h(\varphi) = g(\varphi) \in D^{\mathcal{A}}$, for $\mathcal{A} \in \mathrm{Mod}_{2\setminus 1}(C)$, and so $\mathcal{A} \in \mathrm{Mod}_2(C)$. By induction on any $n \in \omega$, let us prove that $\mathcal{A} \in \mathrm{Mod}_n(C)$. For consider any $X \in \wp_n(\mathrm{Fm}^{\omega}_{\Sigma})$, in which case $n \neq 0$. In case $|X| \in 2$, $X \in \wp_2(\mathrm{Fm}^{\omega}_{\Sigma})$, and so $C(X) \subseteq \mathrm{Cn}^{\omega}_{\mathcal{A}}(X)$, for $\mathcal{A} \in \mathrm{Mod}_2(C)$. Otherwise, $|X| \geqslant 2$, in which case there are some distinct $\phi, \psi \in X$, and so $Y \triangleq ((X \setminus \{\phi, \psi\}) \cup \{\phi \overline{\wedge} \psi\}) \in \wp_{n-1}(\mathrm{Fm}^{\omega}_{\Sigma})$. Then, by the induction hypothesis and the $\overline{\wedge}$-conjunctivity of both C and $\mathcal{A}$, we get $C(X) = C(Y) \subseteq \mathrm{Cn}^{\omega}_{\mathcal{A}}(Y) = \mathrm{Cn}^{\omega}_{\mathcal{A}}(X)$. Thus, $\mathcal{A} \in \mathrm{Mod}_{\omega}(C)$, for $\omega = (\bigcup \omega)$, and so $\mathcal{A} \in \mathrm{Mod}(C)$, for C is finitary.] □

Theorem 4.6. *Let C be a $\overline{\wedge}$-conjunctive [$\underline{\vee}$-disjunctive] Σ-logic and $\mathsf{V} \triangleq \mathrm{IV}(C)$ (as well as M a class of simple Σ-matrices defining C, and $\mathsf{K} \triangleq \pi_0[\mathsf{M}]$). {Suppose C is finitary (in particular, both M and all elements of it are finite).} Then, (i)⇔(ii){⇒}(iii)(⇒(iv))⇒(i), where:*

(i) *C is self-extensional;*

(ii) *for all $\phi, \psi \in \mathrm{Fm}_\Sigma^\omega$, it holds that $(\psi \in C(\phi)) \Leftrightarrow | \Rightarrow (\mathsf{V} \models (\phi \approx (\phi \barwedge \psi)))$, while every element of V is a $\barwedge$-semi-lattice [resp., distributive $(\barwedge, \veebar)$-lattice];*

(iii) *every truth-non-empty $\barwedge$-conjunctive [consistent $\veebar$- disjunctive] Σ-matrix with underlying algebra in V is a model of C, while every element of V is a $\barwedge$-semi-lattice [resp., distributive $(\barwedge, \veebar)$-lattice];*

(iv) *any truth-non-empty $\barwedge$-conjunctive [consistent $\veebar$- disjunctive] Σ-matrix with underlying algebra in K is a model of C, while every element of K is a $\barwedge$-semi-lattice [resp., distributive $(\barwedge, \veebar)$-lattice].*

{*(In particular, (i–iv) are equivalent.)*}

Proof. First, (i)⇔(ii) is by Remark 4.3 and Theorem 4.4 with $i = 0$ and $\diamond = \barwedge$. {Next, (ii)⇒(iii) is by Lemma 4.5.} (Further, (iv) is a particular case of (iii), in view of Theorem 3.14.) Finally, assume (iii) (resp., (iv)) holds. Let S be the class of all truth-non-empty $\barwedge$-conjunctive [consistent $\veebar$- disjunctive] Σ-matrices with underlying algebra in V (resp., in K). Consider any $\mathcal{A} \in \mathsf{S}$ and any $\bar{a} \in (A^2 \setminus \Delta_A)$, in which case, by the semi-lattice identities ⟨more specifically, the commutativity one⟩ for $\barwedge$, $a_i \neq (a_i \barwedge^{\mathfrak{A}} a_{1-i})$, for some $i \in 2$, and so $\mathcal{B} \triangleq \langle \mathfrak{A}, \{b \in A \mid a_i = (a_i \barwedge^{\mathfrak{A}} b)\}\rangle \in \mathsf{S}$ [resp., by the Prime Ideal Theorem, there is some $\mathcal{B} \in \mathsf{S}$] such that $\mathfrak{B} = \mathfrak{A}$ and $a_i \in D^{\mathcal{B}} \not\ni a_{1-i}$. In this way, (i) is by Theorem(s) 4.1(vi)⇒(i) (and 3.14). □

Theorem 4.7. *Let M be a [finite] class of [finite hereditarily] simple [$\barwedge$-conjunctive $\veebar$-disjunctive] Σ-matrices, $\mathsf{K} \triangleq \pi_0[\mathsf{M}]$ and C the logic of M. Then, C is self-extensional if[f], for each $\mathfrak{A} \in \mathsf{K}$ and all distinct $a, b \in A$, there are some $\mathcal{B} \in \mathsf{M}$ and some $h \in \hom(\mathfrak{A}, \mathfrak{B})$ such that $\chi^{\mathcal{B}}(h(a)) \neq \chi^{\mathcal{B}}(h(b))$.*

Proof. The "if" part is by Theorem 4.1(v)⇒(i) with $\mathsf{C} = \mathsf{K}$. [Conversely, assume C is self-extensional. Consider any $\mathfrak{A} \in \mathsf{K}$ and any $\bar{a} \in (A^2 \setminus \Delta_A)$. Then, by Theorem 4.6(i)⇒(iv), $\mathfrak{A}$ is a distributive $(\barwedge, \veebar)$-lattice, in which case, by the commutativity identity for $\barwedge$, $a_i \neq (a_i \barwedge^{\mathfrak{A}} a_{1-i})$, for some $i \in 2$, and so, by the Prime Ideal Theorem, there is some $\barwedge$-conjunctive $\veebar$-disjunctive Σ-matrix $\mathcal{D}$ with $\mathfrak{D} = \mathfrak{A}$ such that $a_i \in D^{\mathcal{D}} \not\ni a_{1-i}$, in which case $\mathcal{D}$ is both consistent and truth-non-empty, and so is a model of C. Hence, by Theorem 3.5 and Remark 2.6(ii), there are some $\mathcal{B} \in \mathsf{M}$ and some strict $h \in \hom(\mathcal{D}, \mathcal{B}) \subseteq \hom(\mathfrak{A}, \mathfrak{B})$, in which case $h(a_i) \in D^{\mathcal{B}} \not\ni h(a_{1-i})$, so $\chi^{\mathcal{B}}(h(a_i)) = 1 \neq 0 = \chi^{\mathcal{B}}(h(a_{1-i}))$.] □

4.2. Self-extensionality of implicative logics versus implicative intrinsic semi-lattices.

A Σ-algebra $\mathfrak{A}$ is called an *$\sqsupset$-implicative intrinsic semi-lattice [with bound (a)]*, provided it is a $\uplus_{\sqsupset}$-semi-lattice [with bound (a)] and satisfies the Σ-identities:

$$(x_0 \sqsupset x_0) \approx (x_1 \sqsupset x_1), \tag{4.1}$$

$$((x_0 \sqsupset x_0) \sqsupset x_1) \approx x_1, \tag{4.2}$$

in which case it is that with bound $a \sqsupset^{\mathfrak{A}} a$, for any $a \in A$.

Remark 4.8. Let C be a [self-extensional] Σ-logic and $\phi, \psi \in C(\varnothing)$, in which case $\phi \equiv_C^\omega \psi$ [and so $\mathrm{IV}(C) \models (\phi \approx \psi)$]. □

Theorem 4.9. *Let M be an $\sqsupset$-implicative Σ-logic C (defined by a class M of simple Σ-matrices and $\mathsf{K} \triangleq \pi_0[\mathsf{M}]$). Then, C is self-extensional iff, for all $\phi, \psi \in \mathrm{Fm}_\Sigma^\omega$, it holds that $(\psi \in C(\phi)) \Leftrightarrow | \Rightarrow (\mathrm{IV}(C) \models (\psi \approx (\phi \uplus_{\sqsupset} \psi)))$, while each element of $\mathrm{IV}(C) (= \mathbf{V}(\mathsf{K}))$ is an $\sqsupset$-implicative intrinsic semi-lattice.*

Proof. First, by (2.6), Remark 4.8 and the strucuruality of C, $(4.1) \in \equiv_C^\omega$. Likewise, by (2.6), (2.7) and (2.8), $(4.2) \in \equiv_C^\omega$. Then, Theorems 3.7(ii) and 4.4 with $i = 1$ and $\diamond = \uplus_{\sqsupset}$ complete the argument. □

Lemma 4.10. *Let C' be a finitary Σ-logic and C'' a 1-extension of C' (cf. Definition 2.4). Suppose C' has DT with respect to $\sqsupset$, while (2.8) is satisfied in C''. Then, C'' is an extension of C'.*

Proof. By induction on any $n \in \omega$, we prove that C'' is an n-extension of C'. For consider any $X \in \wp_n(\mathrm{Fm}^\omega_\Sigma)$, in which case $n \neq 0$, and any $\psi \in C'(X)$. Then, in case $X = \varnothing$, we have $X \in \wp_1(\mathrm{Fm}^\omega_\Sigma)$, and so $\psi \in C'(X) \subseteq C''(X)$, for C'' is a 1-extension of C'. Otherwise, take any $\phi \in X$, in which case $Y \triangleq (X \setminus \{\phi\}) \in \wp_{n-1}(\mathrm{Fm}^\omega_\Sigma)$, and so, by DT with respect to $\sqsupset$, that C' has, and the induction hypothesis, we have $(\phi \sqsupset \psi) \in C'(Y) \subseteq C''(Y)$. Therefore, by (2.8)$[x_0/\phi, x_1/\psi]$ satisfied in C'', in view of its structurality, we eventually get $\psi \in C''(Y \cup \{\phi\}) = C''(X)$. Hence, since $\omega = (\bigcup \omega)$, we eventually conclude that C'' is an ω-extension of C', and so an extension of C', for this is finitary. ☐

Theorem 4.11. *Let M be a [finite] class of [finite hereditarily] simple [$\sqsupset$-implicative] Σ-matrices, $\mathsf{K} \triangleq \pi_0[\mathsf{M}]$ and C the logic of M. Then, C is self-extensional if[f], for each $\mathfrak{A} \in \mathsf{K}$ and all distinct $a, b \in A$, there are some $\mathcal{B} \in \mathsf{M}$ and some $h \in \hom(\mathfrak{A}, \mathfrak{B})$ such that $\chi^{\mathcal{B}}(h(a)) \neq \chi^{\mathcal{B}}(h(b))$.*

Proof. The "if" part is by Theorem 4.1(v)⇒(i) with $\mathsf{C} = \mathsf{K}$. [Conversely, assume C is self-extensional. Consider any $\mathfrak{A} \in \mathsf{K}$ and any $\bar{a} \in (A^2 \setminus \Delta_A)$. Then, by Theorem 4.9, $\mathfrak{A} \in \mathrm{IV}(C)$ is an $\sqsupset$-implicative intrinsic semi-lattice, in which case, by the commutativity identity for $\uplus_\sqsupset$, $a_{1-i} \neq (a_i \uplus^{\mathfrak{A}}_{\sqsupset} a_{1-i})$, for some $i \in 2$. Let $n \triangleq |A| \in (\omega \setminus 1)$. Take any bijection $\bar{c} : n \to A$. Let $g \in \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{A})$ extend $[x_j/c_j; x_k/c_0]_{j \in n; k \in (\omega \setminus n)}$, in which case $A = (\mathrm{img}\, \bar{c}) \subseteq (\mathrm{img}\, g) \subseteq A$, and so there is some $\bar{\varphi} \in (\mathrm{Fm}^\omega_\Sigma)^2$ such that $g(\bar{\varphi}) = \bar{a}$. Then, by (2.15), $S \triangleq g^{-1}[\mathrm{Fg}^{\mathfrak{A}}_C(\varnothing)] \in \mathrm{Fi}_C(\mathfrak{Fm}^\omega_\Sigma)$. Let us prove, by contradiction, that $\varphi_{1-i} \notin T \triangleq C(S \cup \{\varphi_i\})$. For suppose $\varphi_{1-i} \in T$, in which case, by DT, $(\varphi_i \sqsupset \varphi_{1-i}) \in C(S)$, and so $(\varphi_i \sqsupset \varphi_{1-i}) = \sigma(\varphi_i \sqsupset \varphi_{1-i}) \in S$, for $\sigma[S] = S \subseteq S$, where σ is the diagonal Σ-substitution. Then, $(a_i \sqsupset^{\mathfrak{A}} a_{1-i}) \in \mathrm{Fg}^{\mathfrak{A}}_C(\varnothing)$. Clearly, by (2.6), $F \triangleq \{a_i \sqsupset^{\mathfrak{A}} a_i\} \subseteq \mathrm{Fg}^{\mathfrak{A}}_C(\varnothing)$. Conversely, consider any $\phi \in C(\varnothing)$ and any $e \in \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{A})$, in which case, by the structurality of C, $\sigma'(\phi) \in C(\varnothing)$, where σ' is the Σ-substitution extending $[x_l/x_{l+1}]_{l \in \omega}$, and so, by (2.6) and Remark 4.8, $e(\phi) = e'(\sigma'(\phi)) = e'(x_0 \sqsupset x_0) = (a_i \sqsupset^{\mathfrak{A}} a_i) \in F$, where $e' \in \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{A})$ extends $[x_0/a_i; x_{m+1}/e(x_m)]_{m \in \omega}$ (in particular, $\mathcal{D} \triangleq \langle \mathfrak{A}, F \rangle \in \mathrm{Mod}_1(C)$; cf. Definition 2.7). And what is more, by (4.2), (2.8) is true in $\mathcal{D}$, in which case, by Lemma 4.10, $F \in \mathrm{Fi}_C(\mathfrak{A})$, and so $\mathrm{Fg}^{\mathfrak{A}}_C(\varnothing) \subseteq F$ (in particular, $\mathrm{Fg}^{\mathfrak{A}}_C(\varnothing) = F$). In this way, $(a_i \sqsupset^{\mathfrak{A}} a_{1-i}) = (a_i \sqsupset^{\mathfrak{A}} a_i)$, in which case, by (4.2), $(a_i \uplus^{\mathfrak{A}}_{\sqsupset} a_{1-i}) = ((a_i \sqsupset^{\mathfrak{A}} a_i) \sqsupset^{\mathfrak{A}} a_{1-i}) = a_{1-i}$, and so this contradiction shows that $\varphi_{1-i} \notin T$. Hence, there are some $\mathcal{B} \in \mathsf{M}$ and some $f \in \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{B})$ such that $(S \cup \{\varphi_i\}) \subseteq f^{-1}[D^{\mathcal{B}}] \not\ni \varphi_{1-i}$, in which case $\mathcal{E} \triangleq (\mathcal{B} \restriction (\mathrm{img}\, f))$, being a submatrix of $\mathcal{B}$, is simple, for $\mathcal{B}$ is hereditarily so, as well as, by Remark 2.8(ii), is $\sqsupset$-implicative, for $\mathcal{B}$ is so, while $U \triangleq f^{-1}[D^{\mathcal{E}}] = f^{-1}[D^{\mathcal{B}}] \in (\mathrm{img}\, C)$, whereas f is a surjective strict homomorphism from $\mathcal{F} \triangleq \langle \mathfrak{Fm}^\omega_\Sigma, U \rangle$ onto $\mathcal{E}$, and so, by Remark 2.6(iii), $\partial(\mathcal{F}) = f^{-1}[\partial(\mathcal{E})] = f^{-1}[\Delta_E] = (\ker f)$. Consider any $\bar{\psi} \in \theta \triangleq (\ker g)$, in which case, by (2.6), for all $\ell \in 2$, $g(\psi_\ell \sqsupset \psi_{1-\ell}) = (g(\psi_\ell) \sqsupset^{\mathfrak{A}} g(\psi_{1-\ell})) = (g(\psi_\ell) \sqsupset^{\mathfrak{A}} g(\psi_\ell)) \in \mathrm{Fg}^{\mathfrak{A}}_C(\varnothing)$, and so $(\psi_\ell \sqsupset \psi_{1-\ell}) \in S \subseteq U$. Hence, by (2.8), $(\psi_\ell \in U) \Rightarrow (\psi_{1-\ell} \in U)$, in which case $\bar{\psi} \in \theta^{\mathcal{F}}$, and so $\theta^{\mathcal{F}} \supseteq \theta \in \mathrm{Con}(\mathfrak{Fm}^\omega_\Sigma)$. Therefore, $\theta \in \mathrm{Con}(\mathcal{F})$, in which case $\theta \subseteq \partial(\mathcal{F}) = (\ker f)$, and so, by the Homomorphism Theorem, $h \triangleq (g^{-1} \circ f) \in \hom(\mathfrak{A}, \mathfrak{B})$, while $h(a_i) = f(\varphi_i) \in D^{\mathcal{B}} \not\ni f(\varphi_{1-i}) = h(a_{1-i})$]. ☐

4.3. **Self-extensionality of uniform finitely-valued logics versus truth discriminators.** A *truth discriminator for/of* a Σ-matrix $\mathcal{A}$ is any $\bar{h} : \mathrm{img}[\theta^{\mathcal{A}} \setminus \Delta_A] \to \hom(\mathfrak{A}, \mathfrak{A})$ such that, for every $\{a, b\} \in (\mathrm{dom}\, \bar{h})$, $\langle a, b \rangle \notin \ker(h_{\{a,b\}} \circ \chi^{\mathcal{A}})$. Then, since $\Delta_A \in \hom(\mathfrak{A}, \mathfrak{A})$, as the "unitary" common particular case of Theorems 4.7 and 4.11, we have:

Corollary 4.12. *Let $\mathcal{A}$ be a [finite hereditarily] simple [either implicative or both conjunctive and disjunctive] Σ-matrix and C the logic of $\mathcal{A}$. Then, C is self-extensional if[f] $\mathcal{A}$ has a truth discriminator.*

The effective procedure of verifying the self-extensionality of the logic of an n-valued, where $n \in (\omega \setminus 1)$, hereditarily simple either implicative or both conjunctive and disjunctive Σ-matrix resulted from Corollary 4.12 has the computational complexity n^{n+2} that is quite acceptable for (3|4)-valued logics. And what is more, it provides a quite useful heuristic tool of doing it, manual applications of which (suppressing the factor n^{n+2} at all) are presented below.

Corollary 4.13. *Let $n \in (\omega \setminus 3)$, $\mathcal{A}$ an n-valued hereditarily simple either implicative or both conjunctive and disjunctive Σ-matrix and C the logic of $\mathcal{A}$. Suppose every non-singular endomorphism of $\mathfrak{A}$ is diagonal (cf. Subsection 2.1). Then, the logic of $\mathcal{A}$ is not self-extensional.*

Proof. By contradiction. For suppose C is self-extensional. Then, as $n \in (\omega \setminus 3)$, $n \nleq 2$, for $3 \nleq 2$, in which case $\chi^{\mathcal{A}}$ is not injective, and so there are some distinct $a, b \in A$ such that $\chi^{\mathcal{A}}(a) = \chi^{\mathcal{A}}(b)$. On the other hand, by Corollary 4.12, there is some $e \in \hom(\mathfrak{A}, \mathfrak{A})$ such that $\chi^{\mathcal{A}}(e(a)) \neq \chi^{\mathcal{A}}(e(b))$, in which case $e(a) \neq e(b)$, and so e is not singular (in particular, e is diagonal). Hence, $\chi^{\mathcal{A}}(a) = \chi^{\mathcal{A}}(e(a)) \neq \chi^{\mathcal{A}}(e(b)) = \chi^{\mathcal{A}}(b)$. This contradiction completes the argument. ☐

Example 4.2 with $\Sigma = \{\bot, \top\}$, $\chi^{\mathcal{A}} = \Delta_2$ and $\mathfrak{A} = (\mathfrak{D}_{2,01} \restriction \Sigma)$ shows that the stipulation "$n \in (\omega \setminus 3)$" cannot be omitted in the formulation of Corollary 4.13.

4.3.1. *Self-extensionality versus equational implications and equality determinants.* According to [14], given any $m, n \in \omega$, a *[finitary] (Σ-)equational $\vdash^m_n$-{sequent }definition for/of* a Σ-matrix $\mathcal{A}$ is any $\mho \in \wp_{[\omega]}(\mathrm{Eq}^{m+n}_{\Sigma})$ such that, for all $\bar{a} \in A^m$ and all $\bar{b} \in A^n$, it holds that $(((\mathrm{img}\, a) \subseteq D^{\mathcal{A}}) \Rightarrow (((\mathrm{img}\, b) \cap D^{\mathcal{A}}) \neq \varnothing)) \Leftrightarrow (\mathfrak{A} \models (\bigwedge \mho)[x_i/a_i; x_{m+j}/b_j]_{i \in m; j \in n})$. Equational $\vdash^{0/1}_1$-definitions are also referred to as *equational "truth [predicate] definitions"/implications*, /(cf. [15]), respectively. Some kinds of equational sequent definitions are actually equivalent for implicative matrices, by:

Remark 4.14. Given a(n $\sqsupset$-implicative) Σ-matrix $\mathcal{A}$, (i) holds (as well as (ii–iv) do so), where:

(i) given a [finitary] equational $\vdash^2_2$-definition $\mho$ for $\mathcal{A}$, $\mho[x_{(2 \cdot i)+j}/x_i]_{i,j \in 2}$ is a [finitary] equational implication for $\mathcal{A}$ (cf. Theorems 10 and 12(ii)⇒(iii) of [14]);
(ii) given any [finitary] equational implication $\mho$ for $\mathcal{A}$, $\mho[x_0/(x_0 \sqsupset x_0), x_1/x_0]$ is a [finitary] equational truth definition for $\mathcal{A}$;
(iii) given any [finitary] equational truth definition $\mho$ for $\mathcal{A}$, $\mho[x_0/(x_0 \sqsupset (x_1 \sqsupset (x_2 \uplus_{\sqsupset} x_3)))]$ is a [finitary] equational $\vdash^2_2$-definition for $\mathcal{A}$;
(iv) in case $\mathcal{A}$ is truth-singular, $\{x_0 \approx (x_0 \sqsupset x_0)\}$ is a finitary equational truth definition for it. ☐

In this way, taking Theorems 10, 12(i)⇔(ii) and 13 of the work [14] as well as Remark 4.14 into account, an either implicative or both conjunctive and disjunctive consistent truth-non-empty finite Σ-matrix $\mathcal{M}$ with equality determinant has a finitary equational implication iff the multi-conclusion two-side sequent calculus $\widetilde{\mathcal{S}}^{(k,l)}_{\mathcal{M},\mathcal{T}}$ (cf. [13] as well as the paragraph -2 on p. 294 of [14] for more detail) is algebraizable (in the sense of [12, 11]). In this connection, by Lemma 9 and

Theorem 10 of [14] as well as Corollary 4.13, we immediately get the following universal negative result:

Corollary 4.15. *Let $n \in (\omega \setminus 3)$, $\mathcal{A}$ an n-valued consistent truth-non-empty either implicative or both conjunctive and disjunctive Σ-matrix with equality determinant and C the logic of $\mathcal{A}$. Suppose $\mathcal{A}$ has an equational implication. Then, C is not self-extensional.*

The converse does not, generally speaking, hold (cf. Example 5.34). In view of Theorem 10 and Lemma 8 of [14], Example 4.2 and the self-extensionality of inferentially inconsistent (in particular, one-valued) logics, the stipulation "$n \in (\omega \setminus 3)$" and the reservation "$n$-valued consistent truth-non-empty" cannot be omitted in the formulation of Corollary 4.15.

Example 4.16 (Łukasiewicz' finitely-valued logics; cf. [6])**.** Let $n \in (\omega \setminus 3)$, $\Sigma \triangleq (\Sigma_+ \cup \{\sim, \supset\})$ with binary $\supset$ (implication) and unary $\sim$ (negation) and $\mathcal{A}$ the Σ-matrix with $(\mathfrak{A} \restriction \Sigma_+) \triangleq \mathfrak{D}_n$ (cf. Subparagraph 2.2.1.2.1), $D^{\mathcal{A}} \triangleq \{1\}$, $\sim^{\mathfrak{A}} \triangleq (1-a)$ and $(a \supset^{\mathfrak{A}} b) \triangleq \min(1, 1-a+b)$, for all $a, b \in A$, in which case $\mathcal{A}$ is both consistent, truth-non-empty, $\wedge$-conjunctive and $\underline{\vee}$-disjunctive as well as has both an equational implication, by Example 7 of [14], and a equality determinant, by Example 3 of [13]. Hence, by Corollary 4.15, the logic of $\mathcal{A}$ is not self-extensional. $\square$

Example 4.17. In view of Example 2 of [13], Remark 1 as well as Theorem 10 and Lemma 9 of [14] and Corollaries 4.13 and 4.15, arbitrary three-valued expansions of both the *logic of paradox LP* [8] and Kleene's three-valued logic K_3 [4] are not self-extensional, because the matrix defining the former has the equational implication $(x_0 \wedge (x_1 \vee \sim x_1)) \approx (x_0 \wedge x_1)$, discovered in [10], while the matrix defining the latter has the same underlying algebra as that defining the former. Likewise, in view of "both Lemma 4.1 of [9] and Remark 4.14(i,iii)"/"Proposition 5.7 of [15]" as well as Corollary 4.15, arbitrary three-valued expansions of P^1/HZ [19]/[3] are not self-extensional, for their defining implicative/ matrices have equational "truth definition"/implication, respecttively. $\square$

Other generic applications of our universal elaboration are discussed in the next section.

5. Applications and examples

All along throughout this section, $\sim$ is supposed to be a primary unary connective of Σ viewed as negation. Let $\Sigma^{\{\supset\}}_{\sim(+)[01]} \triangleq (\{\sim\}(\cup\Sigma_+)[\cup\{\bot, \top\}]\{\cup\{\supset\}\})$ (cf. Subparagraph 2.2.1.2.1) {where $\supset$ is binary and viewed as implication}.

5.1. **Logics of three-valued super-classical matrices.**

Σ-matrices with $\sim$-reduct having a (canonical) $\sim$-classical submatrix {and so being both consistent and truth-non-empty} (and carrier $3 \div 2$; cf. Subparagraph 2.2.1.2.1) are said to be *([3-]canonical⟨ly⟩) $\sim$-super-classical*, In general, given any three-valued $\sim$-super-classical Σ-matrix $\mathcal{A}$ with $\sim$-classical submatrix $\mathcal{B}$ of its $\sim$-reduct, the bijection $e \triangleq (\chi^{\mathcal{B}} \cup ((A \setminus B) \times \{\frac{1}{2}\}) : A \to (3 \div 2)$ is an isomorphism from $\mathcal{A}$ onto the canonical $\sim$-super-classical Σ-matrix $\mathsf{C}_{[3]}(\mathcal{A}) \triangleq \langle e[\mathfrak{A}], e[D^{\mathcal{A}}]\rangle$, called the *[3-]canonization of $\mathcal{A}$*.

Throughout the rest of this subsection, unless otherwise specified, C is supposed to be the logic of an arbitrary but fixed canonical $\sim$-super-classical Σ-matrix $\mathcal{A}$ (that exhausts all Σ-logics of three valued $\sim$-super-classical Σ-matrices, in view of (2.15)), in which case this is false-singular iff it is not truth-singular iff $\Bbbk^{\mathcal{A}} \triangleq \chi^{\mathcal{A}}(\frac{1}{2}) = 1$, and so is false-/truth-singular, whenever it is $\sim$-paraconsistent/"both weakly $\underline{\vee}$-disjunctive and $(\underline{\vee}, \sim)$-paracomplete", respectively, in which case C is not

$\sim$-classical, in view of Remark 2.8(i)(c). And what is more, any proper submatrix $\mathcal{B}$ of $\mathcal{A}$ is either $\sim$-classical or one-valued, in which case $\mathcal{B}$ is simple, and so $\mathcal{A}$ is simple iff it is hereditarily so. Clearly, $\mathcal{A}$ is [weakly]/weakly $\diamond$-conjunctive/-disjunctive iff C is so. It appears that such is the case for both $\diamond$-disjunctivity and -implicativity, in view of the following preliminary results normally used below tacitly:

Lemma 5.1. *Let $\mathcal{B}$ be a Σ-matrix and C' the logic of $\mathcal{B}$. Suppose $\mathcal{B}$ is [not] false-singular [as well as both no-more-than-three-valued and $\sim$-super-classical]. Then, the following are equivalent:*

(i) *C' is $\veebar$-disjunctive;*
(ii) *$\mathcal{B}$ is $\veebar$-disjunctive;*
(iii) (2.2)*,* (2.4) *and* (2.5) *[as well as* (2.8) *for the material implication $\sqsupset = \sqsupset^{\sim}_{\veebar}$ (cf. Remark 2.8*(i)(b)*)] are satisfied in C' {viz., true in $\mathcal{B}$}.*

Proof. First, (ii)⇒(i) is immediate. Next, assume (i) holds. Then, (2.2), (2.4) and (2.5) are immediate. [And what is more, once $\mathcal{B}$ is not false-singular, it is both no-more-than-three-valued (and so truth-singular) and $\sim$-super-classical, in which case it is not $\sim$-paraconsistent, and so is C'. Then, by (i), (2.11) and Lemma 3.1, (2.8) with $\sqsupset = \sqsupset^{\sim}_{\veebar}$ is satisfied in C'.] Thus, (iii) holds. Finally, assume (iii) holds. Consider any $a, b \in B$. Then, by (2.2) with $i = 0$ and (2.4), C' is weakly $\veebar$-disjunctive, and so is $\mathcal{B}$, in which case $(a \veebar^{\mathfrak{B}} b) \in D^{\mathcal{B}}$, whenever either a or b is in $D^{\mathcal{B}}$. Now, assume $(\{a, b\} \cap D^{\mathcal{B}}) = \varnothing$. Then, in case $a = b$ (in particular, $\mathcal{B}$ is false-singular), by (2.5), we get $D^{\mathcal{B}} \not\ni (a \veebar^{\mathfrak{B}} a) = (a \veebar^{\mathfrak{B}} b)$. [Otherwise, $\mathcal{B}$ is not false-singular, in which case it is no-more-than-three-valued (in particular, truth-singular) and $\sim$-super-classical, while (2.8) with $\sqsupset = \sqsupset^{\sim}_{\veebar}$ is true in $\mathcal{B}$, and so, for some $c \in (B \setminus D^{\mathcal{B}}) = \{a, b\}$, it holds that $\sim^{\mathfrak{B}} c \in D^{\mathcal{B}}$, while $\sim^{\mathfrak{B}}\sim^{\mathfrak{B}} c = c$. Let d be the unique element of $\{a, b\} \setminus \{c\}$, in which case $\{a, b\} = \{c, d\}$. Then, since $\sim^{\mathfrak{B}} c \in D^{\mathcal{B}}$, we conclude that $(c \veebar^{\mathfrak{B}} d) = (\sim^{\mathfrak{B}}\sim^{\mathfrak{B}} c \veebar^{\mathfrak{B}} d) \notin D^{\mathcal{B}}$, for, otherwise, by (2.8) with $\sqsupset = \sqsupset^{\sim}_{\veebar}$, we would get $d \in D^{\mathcal{B}}$. Hence, by (2.4), we eventually get $(a \veebar^{\mathfrak{B}} b) \notin D^{\mathcal{B}}$.] Thus, (ii) holds. ☐

Corollary 5.2. *[Providing $\mathcal{A}$ is false-singular (in particular, $\sim$-paraconsistent)] $\mathcal{A}$ is $\sqsupset$-implicative iff C is [weakly] so.*

Proof. The "if" part is by Theorem 3.7(ii) and Lemma[s 3.8 and] 5.1 as well as (2.7), (2.8) and (2.10). The converse is immediate. ☐

Next, $\mathcal{A}$ is said to be *[extra-]classically-hereditary*, provided $[A \setminus]2$ forms a subalgebra of $\mathfrak{A}$. Likewise, $\mathcal{A}$ is said to be *classically-valued*, provided, for all $\varsigma \in \Sigma$, $(\operatorname{img} \varsigma^{\mathfrak{A}}) \subseteq 2$, in which case it is [not extra-]classically-hereditary.

5.1.1. *Non-classical logics.* Generally speaking, C, though being three-valued, need not be non-$\sim$-classical, in view of:

Example 5.3. Let $\Sigma \triangleq \Sigma_{\sim}$ and $(\mathcal{B}/\mathcal{D})|\mathcal{E}$ the canonical "$\sim$-negative false-/truth-singular $\sim$-super-classical"|$\sim$-classical Σ-matrix, respectively. Then, $\chi^{\mathcal{B}/\mathcal{D}} \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{B}/\mathcal{D}, \mathcal{E})$. Therefore, by (2.15), $\mathcal{B}/\mathcal{D}$ define the same $\sim$-classical Σ-logic of $\mathcal{E}$. ☐

On the other hand, $\sim$-classical Σ-logics are self-extensional, in view of Example 4.2. This makes studying the classicism of C especially acute.

Theorem 5.4. *Let $\mathcal{B}$ be a three-valued $\sim$-super-classical Σ-matrix and C' the logic of $\mathcal{B}$. [Suppose $\mathcal{B}$ is either false-singular or $\diamond$-conjunctive/-disjunctive.] Then,* (iv)⇐(iii)⇔(ii)⇒(i)[⇒(ii)]*, where:*

(i) *C' is $\sim$-classical;*
(ii) *$\mathcal{B}$ is a strictly surjectively homomorphic counter-image of a $\sim$-classical Σ-matrix;*

(iii) $\mathcal{B}$ *is not simple;*
(iv) $\mathcal{B}$ *is* $\sim$*-negative.*

Proof. Let $\mathcal{D}$ be a $\sim$-classical submatrix of $\mathcal{B}\restriction\{\sim\}$, in which case it, being $\sim$-negative, is both consistent and truth-non-empty, and so is $\mathcal{B}\restriction\{\sim\}$, in view of Remark 2.8(ii)(b) (in particular, $\mathcal{B}$ is so). First, assume (iii) holds, in which case $\Delta_B \neq \eth(\mathcal{B}) \subseteq \theta^{\mathcal{B}}$, and so $\eth(\mathcal{B}) = \theta^{\mathcal{B}}$, for $\mathrm{img}[\theta^{\mathcal{B}} \setminus \Delta_B] = (\{(\chi^{\mathcal{B}})^{-1}[\{i\}] \mid i \in 2\} \cap \wp_{3\setminus 2}(B))$ is a singleton, because $\mathcal{B}$ is three-valued as well as both consistent and truth-non-empty. Then, $\theta^{\mathcal{B}} \in \mathrm{Con}(\mathfrak{B})$, in which case $h \triangleq \chi^{\mathcal{B}}$ is a strict surjective homomorphism from $\mathcal{B}$ onto the classically-canonical (in particular, two-valued) Σ-matrix $\mathcal{E} \triangleq \langle h[\mathcal{B}], \{1\}\rangle$, and so $(h\restriction D) : D \to 2$, being surjective, for $\mathcal{D}$ is both consistent and truth-non-empty, is a strict surjective homomorphism from the $\sim$-negative Σ-matrix $\mathcal{D}$ onto $\mathcal{E}\restriction\{\sim\}$. Then, by Remark 2.8(ii)(a), $\mathcal{E}\restriction\{\sim\}$ is $\sim$-negative, and so is $\mathcal{E}$, in which case this, being two-valued, is $\sim$-classical, and so (ii) holds (in particular, (iii)⇒(ii) does so). The converse is by Remark 2.6(ii), for $|A| = 3 \nleqslant 2$.

Next, (ii)⇒(i) is by (2.15). [Conversely, assume (i) holds. We prove that (iii) holds, by contradiction. For suppose $\mathcal{A}$ is simple. Let $\mathcal{F}$ be any $\sim$-classical Σ-matrix defining C', in which case it is truth-singular, and so is $\mathcal{B} \in \mathbf{HSPS}\mathcal{F}$, in view of Lemma 3.3. Then, $\mathcal{B}$ is $\diamond$-conjunctive/-disjunctive (in particular, C' is so), and so is $\mathcal{F}$/“, in view of Lemma 5.1, for it is false-singular”, in which case this is $\diamond^{\sim/}$-disjunctive“, in view of Remark 2.8(i)(a), for it is $\sim$-negative”/, and so is C' (in particular, $\mathcal{B}$ is so, in view of Lemma 5.1). In this way, Theorem 3.5 yields (ii), and so (iii).]

Finally, (ii)⇒(iv) is by Remark 2.8(ii)(a). □

5.1.1.1. Examples.

5.1.1.1.1. Kleene-style logics. Let $\Sigma \triangleq \Sigma_{\sim,+[,01]}$ and $\mathcal{A}$ truth-/false-singular with $\sim^{\mathfrak{A}}\frac{1}{2} \triangleq \frac{1}{2}$ and $(\mathfrak{A}\restriction\Sigma_{+[,01]}) \triangleq \mathfrak{D}_{3[,01]}$. Then, $\mathcal{A}$ is both $\wedge$-conjunctive, $\vee$-disjunctive and non-$\sim$-negative (in particular, C is not $\sim$-classical; cf. Theorem 5.4(i)⇒(iv)) as well as both classically-hereditary and [not] extra-classically-hereditary, while $\mathfrak{A}$ is a distributive $(\wedge, \vee)$-lattice with zero 0 and unit 1, whereas C is [the *bounded version*|*expansion* $K_{3,01}/LP_{01}$ *of*] “Kleene's three-valued logic”/“the *logic of paradox*” K_3/LP [4]/[8], respectively.

5.1.1.1.2. Gödel-style logics. Let $\Sigma \triangleq \Sigma^{/\supset}_{\sim,+,01}$ and $\mathcal{A}$ [not] truth-singular with $(\mathfrak{A}\restriction\Sigma_{+,01}) \triangleq \mathfrak{D}_{3,01}$ and $\sim^{\mathfrak{A}}e \triangleq (0[+1])$ (viz., $\sim^{\mathfrak{A}}$ is the [dual] pseudo-complement operation)/“ as well as $\supset^{\mathfrak{A}}$ being the [dual] relative pseudo-complement operation”. Then, $\mathcal{A}$ is both $\wedge$-conjunctive, $\vee$-disjunctive and non-$\sim$-negative (in particular, C is not $\sim$-classical; cf. Theorem 5.4(i)⇒(iv)) as well as classically-hereditary but not extra-classically-hereditary, while C is [the *($\sim$-)paraconsistent counterpart* $PG_3^{*/}$ *of*] “the implication-less fragment G_3^* of”/ Gödel's three-valued logic G_3 [2].

5.1.1.1.3. Hałkowska-Zajac' logic. Let $\Sigma \triangleq \Sigma_{\sim,+}$ and $\mathcal{A}$ false-singular with $\sim^{\mathfrak{A}}\frac{1}{2} \triangleq \frac{1}{2}$ and $\mathfrak{A}$ being the distributive $(\wedge, \vee)$-lattice with zero $\frac{1}{2}$ and unit 1. Then, $\mathcal{A}$ is $\sim$-paraconsistent (in particular, C is not $\sim$-classical; cf. Remark 2.8(i)(c)) as well as both classically- and extra-classically-hereditary but weakly neither $\wedge$-conjunctive nor $\vee$-disjunctive, C being the logic HZ [3]. Nevertheless, since the identity $\sim\sim x_0 \approx x_0$ is true in $\mathfrak{A}$, $\mathfrak{A}$ is a distributive $(\vee^{\sim}, \wedge^{\sim})$-lattice (cf. Remark 2.8(i)(a) for definition of these secondary binary connectives) with zero 0 and unit $\frac{1}{2}$. Then, $\mathcal{A}$ is both $\vee^{\sim}$-conjunctive and $\wedge^{\sim}$-disjunctive.

5.1.1.1.4. Sette-style logics. Let $\Sigma \triangleq \Sigma^{\supset}_{\sim}$ and $\mathcal{A}$ classically-valued, non-$\sim$-negative, $\supset$-implicative (in particular, C is not $\sim$-classical; cf. Theorem 5.4(i)⇒(iv)) and

[not] false-singular. Then, C is [the *intuitionistic/((*$\uplus_{\supset}, \sim$*)-)paracomplete counterpart* I^1 *of*] P^1 [19].

5.1.1.2. Self-extensionality versus discriminating endomorphisms. A *(truth-)discriminating operator/endomorphism on/of* $\mathcal{A}$ is any $h \in (A^A / \operatorname{hom}(\mathfrak{A}, \mathfrak{A}))$ such that $\chi^{\mathcal{A}}(h(\frac{1}{2})) \neq \chi^{\mathcal{A}}(h(\Bbbk^{\mathcal{A}}))$, in which case $h(\frac{1}{2}) \neq h(\Bbbk^{\mathcal{A}})$, and so h is neither diagonal nor singular, the set of all them being denoted by $(\partial/\eth)(\mathcal{A})$, respectively. Then, since $\operatorname{img}[\theta^{\mathcal{A}} \setminus \Delta_A] = \{\{\frac{1}{2}, \Bbbk^{\mathcal{A}}\}\}$, by Example 4.2, Corollary 4.12 and Theorem 5.4(iii)⇒(i), we have:

Corollary 5.5. *[Providing* $\mathcal{A}$ *is either implicative or both conjunctive and disjunctive]* C *is self-extensional if[f] either it is* $\sim$*-classical or* $\eth(\mathcal{A}) \neq \varnothing$*.*

Though there are $3^3 = 27$ unary operations on A, only few of them may be discriminating operators/endomorphisms on/of $\mathfrak{A}$. More precisely, let $\Delta_2^+ \triangleq \Delta_2 \in 2^2$, $\Delta_2^- \triangleq (A^2 \setminus \Delta_2) \in 2^2$, $h_{+|-,a} \triangleq (\Delta_2^{+|-} \cup \{\langle \frac{1}{2}, a\rangle\}) \in A^A$, where $a \in A$, $\mathcal{H} \triangleq (\bigcup_{a \in A}\{h_{+,a}, h_{-,a}\})$ and $\mathcal{H}^{\mathcal{A}} \triangleq (\{h_{-,a} \mid a \in A, \chi^{\mathcal{A}}(a) = \Bbbk^{\mathcal{A}}\} \cup \{h_{+,1-\Bbbk^{\mathcal{A}}}\})$. Clearly,

$$(\mathcal{H} \cap \partial(\mathcal{A})) = \mathcal{H}^{\mathcal{A}}. \tag{5.1}$$

Conversely, we have:

Lemma 5.6. *Let* $\mathfrak{D}$ *is a subalgebra of* $\mathfrak{A}$ *and* $h \in \operatorname{hom}(\mathfrak{D}, \mathfrak{A})$*. Suppose* h *is non-singular (cf. Subsection 2.1). Then,* $2 \subseteq D$*, while* $h[2] = 2$*, in which case* $(h{\upharpoonright}2) : 2 \to 2$ *is bijective, and so belongs to* $\{\Delta_2^+, \Delta_2^-\}$*.*

Proof. First, note that the carrier of any subalgebra of $\mathfrak{A}{\upharpoonright}\{\sim\}$ (in particular, $(\operatorname{img} h)D$) belongs to $\{A, 2, \{\frac{1}{2}\}\}$. In particular, $D = (\operatorname{dom} h)$ is non-one=element, for $\operatorname{img} h$ is so, in which case $D \neq \{\frac{1}{2}\}$, and so $2 \subseteq D$. And what is more, for each $a \in A$, we have $(\sim^{\mathfrak{A}} a = a) \Rightarrow (a = \frac{1}{2})$. In particular, providing $\sim^{\mathfrak{A}} \frac{1}{2} = \frac{1}{2}$, we have $\sim^{\mathfrak{A}} h(\frac{1}{2}) = h(\frac{1}{2})$, in which case we get $h(\frac{1}{2}) = \frac{1}{2}$. Then, as 2 forms a subalgebra of $\mathfrak{A}{\upharpoonright}\{\sim\}$, $h[2]$ forms a no-more-than-two-element subalgebra of $\mathfrak{A}{\upharpoonright}\{\sim\}$, in which case $h[2] \in \{2, \{\frac{1}{2}\}\}$, and so $h[2] = 2$, for, otherwise, $h[2] = \{\frac{1}{2}\}$ would form a subalgebra of $\mathfrak{A}{\upharpoonright}\{\sim\}$, in which case we would have $\sim^{\mathfrak{A}} \frac{1}{2} = \frac{1}{2}$, and so would get $h(\frac{1}{2}) = \frac{1}{2}$ (in particular, $(\operatorname{img} h) = \{\frac{1}{2}\}$ would be a singleton, i.e., h would be singular). ☐

Then, since $\eth(\mathcal{A}) = (\partial(\mathcal{A}) \cap \operatorname{hom}(\mathfrak{A}, \mathfrak{A}))$, by (5.1) and Lemma 5.6, we have:

Corollary 5.7. $\eth(\mathcal{A}) \subseteq \mathcal{H}$*. In particular,* $\eth(\mathcal{A}) = (\mathcal{H}^{\mathcal{A}} \cap \operatorname{hom}(\mathfrak{A}, \mathfrak{A}))$*.*

Combining Corollaries 5.5 and 5.7, we eventually get:

Theorem 5.8. *[Providing* $\mathcal{A}$ *is either implicative or both conjunctive and disjunctive]* C *is self-extensional if[f] either it is* $\sim$*-classical or* $(\mathcal{H}^{\mathcal{A}} \cap \operatorname{hom}(\mathfrak{A}, \mathfrak{A})) \neq \varnothing$*.*

This yields a quite effective purely-algebraic criterion of the self-extensionality of C with either implicative or both conjunctive and disjunctive $\mathcal{A}$ that can inevitably be enhanced a bit more under separate studying the alternatives involved excluding *a priori* some elements of $\mathcal{H}^{\mathcal{A}}$ from $\eth(\mathcal{A})$ (i.e., from $\operatorname{hom}(\mathfrak{A}, \mathfrak{A})$; cf. Corollary 5.7), because, under the stipulation of C's being both self-extensional and non-$\sim$-classical, the alternatives under considerations are disjoint, as it is shown in the next subparagraph.

5.1.1.2.1. Self-extensionality versus equational truth-definitions.

Lemma 5.9. *Let* $\mho$ *be an equational truth definition for* $\mathcal{A}$*. Suppose* $\mathcal{A}$ *is either false-singular or* $\sqsupset$*-implicative, while* C *is not* $\sim$*-classical. Then, any non-singular endomorphism* h *of* $\mathfrak{A}$ *is diagonal. In particular, providing* $\mathcal{A}$ *is either implicative or both conjunctive and disjunctive,* C *is not self-extensional.*

Proof. Then, for any $a \in A$, we have $(a \in D^{\mathcal{A}}) \Leftrightarrow (\mathfrak{A} \models (\bigwedge \mho)[x_0/a]) \Rightarrow (\mathfrak{A} \models (\bigwedge \mho)[x_0/h(a)]) \Leftrightarrow (h(a) \in D^{\mathcal{A}})$, in which case $h \in \hom(\mathcal{A}, \mathcal{A})$ (in particular, $h(1) \neq 0$, for $1 \in D^{\mathcal{A}} \not\ni 0$), and so, by Lemma 5.6(i) with $\mathcal{D} = \mathcal{A} = \mathcal{B}$, $h{\restriction}2$ is diagonal. Therefore, if $h(\frac{1}{2})$ was equal to $\Bbbk^{\mathcal{A}}$, then h would be equal to $\chi^{\mathcal{A}}$, in which case $\theta^{\mathcal{A}} = (\ker h)$ would be a congruence of $\mathfrak{A}$, and so, by Theorem 5.4, C would be $\sim$-classical. Hence, in case $\mathcal{A}$ is false-singular, $h(\frac{1}{2}) = \frac{1}{2}$, for $\frac{1}{2} \in D^{\mathcal{A}} \not\ni 0$. Otherwise, $\mathcal{A}$ is $\sqsupset$-implicative, in which case $(\frac{1}{2} \sqsupset^{\mathfrak{A}} 0) = 1$ and $(1 \sqsupset^{\mathfrak{A}} 0) \neq 1$, and so $h(\frac{1}{2}) = \frac{1}{2}$, for otherwise, we would have $h(\frac{1}{2}) = 1$, in which case we would get $1 \neq 1$. Thus, in any case, $h(\frac{1}{2}) = \frac{1}{2}$, and so h is diagonal. In this way, Corollary 4.13 and Theorem 5.4(iii)⇒(i) complete the argument. ☐

This "equational truth definition" analogue of Corollary 4.15 provides another and much more transparent insight into the non-self-extensionality of the instances discussed in Example 4.17 and summarized below. In this connection, we first have:

Corollary 5.10. *Suppose $\mathcal{A}$ is both $\sqsupset$-implicative and either weakly $\overline{\wedge}$-conjunctive (in particular, $\wr$-negative with $\overline{\wedge} = \uplus^{\wr}_{\sqsupset}$; cf. Remark 2.8*(i)(a)*) or truth-singular. Then, $\mathcal{A}$ has a finitary equational truth-definition. In particular, C is not self-extensional, unless it is $\sim$-classical.*

Proof. The case, when $\mathcal{A}$ is truth-singular, is due to Remark 4.14(iv). Otherwise, $\mathcal{A}$ is weakly $\overline{\wedge}$-conjunctive, while $\{\frac{1}{2}\}$ does [not] form a subalgebra of $\mathfrak{A}$ [that is, there is some $\varphi \in \mathrm{Fm}^1_\Sigma$ such that $\varphi^{\mathfrak{A}}(a) \in 2$], so $\{(x_0 \sqsupset \phi) \approx \phi\}$ with $\phi \triangleq (\psi[\overline{\wedge}(\psi[x_0/\varphi])])$ and $\psi \triangleq (x_0 \mathbin{\overline{\wedge}} {\sim} x_0)$ is a finitary equational truth definition for $\mathcal{A}$. In this way, Lemma 5.9 completes the argument. ☐

This is why the contexts of the next two subparagraphs are disjoint, whenever C is self-extensional but not $\sim$-classical. Before coming to discussing them, we provide practically immediate applications of the above results of this subparagraph to some of the logics specified in Paragraph 5.1.1.1.

Remark 5.11. Suppose $\mathcal{A}$ is both $\sim$-paraconsistent (and so false-singular), conjunctive and $\underline{\vee}$-disjunctive as well as both classically- and extra-classically-hereditary. Then, $\{x_0 \approx (x_0 \mathbin{\underline{\vee}} {\sim} x_0)\}$ is an equational truth definition for $\mathcal{A}$, so, by Remark 2.8(i)(c) and Lemma 5.9, C is not self-extensional. ☐

This subsumes disjunctive conjunctive $\sim$-paraconsistent LP and HZ, providing a more transparent insight into the non-self-extensionality of them than that given by Example 4.17.

Remark 5.12. Suppose $\mathcal{A}$ is both classically-valued and $\diamond$-conjunctive/-disjunctive /(in particular, $\sqsupset$-implicative with $\diamond = \uplus_{\sqsupset}$). Then, it is $\wr$-negative, where $\wr x_0 \triangleq {\sim}(x_0 \diamond x_0)$, in which case, by Remark 2.8(i)(a), $\mathcal{A}$ is both $\overline{\wedge}$-conjunctive and $\underline{\vee}$-disjunctive, where $\overline{\wedge} \triangleq \diamond^{/\wr}$ and $\underline{\vee} \triangleq \diamond^{\wr/}$, and so, by Remark 2.8(i)(b), $\mathcal{A}$ is $\sqsupset^{\wr}_{\underline{\vee}}$-implicative. On the other hand, as $\frac{1}{2} \notin 2$, any idempotent binary operation on A, being term-wise definable in $\mathfrak{A}$, is so by either x_0 or x_1, in which case it is not symmetric, for A is not a singleton, and so $\mathfrak{A}$ is not a semi-lattice (in particular, is not a [distributive] lattice). And what is more, $\{((x_0 \sqsupset^{\wr}_{\underline{\vee}} x_0) \sqsupset^{\wr}_{\underline{\vee}} x_0) \approx (x_0 \sqsupset^{\wr}_{\underline{\vee}} x_0)\}$ is a finitary equational truth definition for $\mathcal{A}$, so, providing $\mathcal{A}$ is not $\sim$-negative (in which case it is $\sim$-paraconsistent|$(\underline{\vee}, \sim)$-paracomplete, whenever it is false-|truth-singular), by Remark 2.8(i)(c) and Lemma 5.9, C is not self-extensional. ☐

This subsumes both P^1 and I^1.

5.1.1.2.2. Conjunctive logics.

Lemma 5.13. *Let $\mathcal{B}$ be a consistent/truth-non-empty weakly $\diamond$-conjunctive/-disjunctive Σ-matrix. Suppose $\mathfrak{B}$ is a $\diamond$-semi-lattice with bound. Then, $\beta^{\mathfrak{B}}_{\diamond} \notin / \in D^{\mathcal{B}}$.*

Proof. By the weak $\diamond$-conjunctivity/-disjunctivity of $\mathcal{B}$, we do have $\beta^{\mathfrak{B}}_{\diamond} = (\beta^{\mathfrak{B}}_{\diamond} \diamond^{\mathfrak{B}} a) \notin / \in D^{\mathcal{B}}$, where $a \in ((B \setminus D^{\mathcal{B}})/D^{\mathcal{B}}) \neq \varnothing$. ☐

Lemma 5.14. *Suppose C is both self-extensional and $\bar{\wedge}$-conjunctive. Then, $\mathfrak{A}$ is a $\bar{\wedge}$-semi-lattice with bound such that the following hold:*

(i) $(0 \mathbin{\bar{\wedge}^{\mathfrak{A}}} 1) = \beta^{\mathfrak{A}}_{\bar{\wedge}}$;
(ii) $\frac{1}{2} \leq^{\mathfrak{A}}_{\bar{\wedge}} 1$;
(iii) *for every finite set I, all $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$ and any truth-non-empty subdirect product $\mathcal{D}$ of it, the following hold:*
  (a) *for each $j \in 2$, $(I \times \{j\}) \in D$;*
  (b) *providing $I \neq \varnothing$ (in particular, $\mathcal{D}$ is consistent), for each $\Sigma' \subseteq \Sigma$, $\{\langle a, I \times \{a\}\rangle \mid a = \varphi^{\mathfrak{A}}(0,1), \varphi \in \mathrm{Fm}^2_{\Sigma'}\}$ is an embedding of the submatrix of $\mathcal{A}{\restriction}\Sigma'$ generated by 2 into $\mathcal{D}{\restriction}\Sigma'$.*
(iv) [providing $\eth(\mathcal{A}) \neq \varnothing$, (g)$\Rightarrow$](a)$\Rightarrow$(b)$\Rightarrow$(c)$\Leftrightarrow$(d)$\Leftrightarrow$(e)$\Leftrightarrow$(f)$\Rightarrow$(g)$\Rightarrow$(h)[$\Rightarrow$(f)], where:
  (a) $h_{+,1-\Bbbk^{\mathcal{A}}} \in \hom(\mathfrak{A}, \mathfrak{A})$;
  (b) $\mathcal{A}$ is classically-hereditary;
  (c) $\beta^{\mathfrak{A}}_{\bar{\wedge}} = 0$;
  (d) $0 \leq^{\mathfrak{A}}_{\bar{\wedge}} \frac{1}{2}$;
  (e) $0 \leq^{\mathfrak{A}}_{\bar{\wedge}} 1$;
  (f) $\sim^{\mathfrak{A}} \frac{1}{2} \neq \frac{1}{2}$;
  (g) $h_{-,a} \in \hom(\mathfrak{A}, \mathfrak{A})$, for no $a \in A$;
  (h) $h_{-,\frac{1}{2}} \notin \hom(\mathfrak{A}, \mathfrak{A})$.

Proof. In that case, by Theorem 4.6(i)$\Rightarrow$(iv), $\mathfrak{A}$, being finite, is a $\bar{\wedge}$-semi-lattice with bound, so, by Lemma 5.13, $\beta^{\mathfrak{A}}_{\bar{\wedge}} \notin D^{\mathcal{A}}$. Let $\xi_{0[+1]} \triangleq [\sim]x_0$ as well as both $\phi_k \triangleq \xi_k(x_0 \mathbin{\bar{\wedge}} {\sim}x_0)$ and $\psi_k \triangleq \phi_k({\sim}x_0)$, where $k \in 2$.

(i) In case $\beta^{\mathfrak{A}}_{\bar{\wedge}} = 0$, we have $0 = \beta^{\mathfrak{A}}_{\bar{\wedge}} \leq^{\mathfrak{A}} 1$, and so get $(0 \mathbin{\bar{\wedge}^{\mathfrak{A}}} 1) = 0 = \beta^{\mathfrak{A}}_{\bar{\wedge}}$. Otherwise, as $1 \in D^{\mathcal{A}}$, we have $D^{\mathcal{A}} \not\ni \beta^{\mathfrak{A}}_{\bar{\wedge}} = \frac{1}{2}$, in which case $\mathcal{A}$ is truth-singular, and so is non-$\sim$-paraconsistent, that is, C is so. Then, by (2.11) and the conjunctivity of C, we have $x_1 \in C(\phi_0)$, in which case, by Theorem 4.6(i)$\Rightarrow$(iv), we get $\beta^{\mathfrak{A}}_{\bar{\wedge}} \leq^{\mathfrak{A}} (0 \mathbin{\bar{\wedge}^{\mathfrak{A}}} 1) = \phi^{\mathfrak{A}}_0(0) \leq^{\mathfrak{A}}_{\bar{\wedge}} \beta^{\mathfrak{A}}_{\bar{\wedge}}$, and so eventually get $(0 \mathbin{\bar{\wedge}^{\mathfrak{A}}} 1) = \beta^{\mathfrak{A}}_{\bar{\wedge}}$.

(ii) Consider the following complementary cases:
- $\mathcal{A}$ is is false-singular,
  in which case, by (i), for each $k \in 2$, $\phi^{\mathfrak{A}}_0(k) = \phi^{\mathfrak{A}}_0(0) = \beta^{\mathfrak{A}}_{\bar{\wedge}} = 0$, and so $(\phi|\psi)^{\mathfrak{A}}_1(k) = 1 \in D^{\mathcal{A}}$. Consider the following complementary subcases:
  - $\sim^{\mathfrak{A}} \frac{1}{2} = \frac{1}{2}$,
    in which case $\phi^{\mathfrak{A}}_1(\frac{1}{2}) = \frac{1}{2} \in D^{\mathcal{A}}$, for $\mathcal{A}$ is false-singular, and so ϕ_1 is true in $\mathcal{A}$ (in particular, $\phi_1 \in C(x_1)$). Then, by Theorem 4.6(i)$\Rightarrow$(iv), $\frac{1}{2} \leq^{\mathfrak{A}}_{\bar{\wedge}} \phi^{\mathfrak{A}}_1(0) = 1$.
  - $\sim^{\mathfrak{A}} \frac{1}{2} \neq \frac{1}{2}$,
    that is, $\sim^{\mathfrak{A}} \frac{1}{2} \in 2$, in which case $\psi^{\mathfrak{A}}_1(\frac{1}{2}) = \phi^{\mathfrak{A}}_1(\sim^{\mathfrak{A}} \frac{1}{2}) = 1 \in D^{\mathcal{A}}$, and so ψ_1 is true in $\mathcal{A}$ (in particular, $\psi_1 \in C(x_1)$). Then, by Theorem 4.6(i)$\Rightarrow$(iv), $\frac{1}{2} \leq^{\mathfrak{A}}_{\bar{\wedge}} \psi^{\mathfrak{A}}_1(0) = 1$.
- $\mathcal{A}$ is truth-singular,
  in which case it is non-$\sim$-paraconsistent, that is, C is so, and so, by

(2.11) and the $\overline{\wedge}$-conjunctivity of C, $x_1 \in C(\phi_0)$. Consider the following complementary subcases:

- $\frac{1}{2}$ is equal to either $\beta^{\mathfrak{A}}_{\overline{\wedge}}$ or $\sim^{\mathfrak{A}}\frac{1}{2}$,
  in which case we have $\frac{1}{2} = \phi^{\mathfrak{A}}_0(\frac{1}{2})$, and so, by Theorem 4.6(i)⇒(iv), get $\frac{1}{2} \leq^{\mathfrak{A}}_{\overline{\wedge}} 1$, for $x_1 \in C(\phi_0)$.
- $\beta^{\mathfrak{A}}_{\overline{\wedge}} \neq \frac{1}{2} \neq \sim^{\mathfrak{A}}\frac{1}{2}$,
  in which case, as $1 \in D^{\mathcal{A}}$, by (i), for each $k \in 2$, $\phi^{\mathfrak{A}}_0(k) = (0 \overline{\wedge}^{\mathfrak{A}} 1) = \beta^{\mathfrak{A}}_{\overline{\wedge}} = 0$, and so $(\phi|\psi)^{\mathfrak{A}}_1(k) = 1 \in D^{\mathcal{A}}$ (in particular, $\psi^{\mathfrak{A}}_1(\frac{1}{2}) = \phi^{\mathfrak{A}}_1(\sim^{\mathfrak{A}}\frac{1}{2}) = 1 \in D^{\mathcal{A}}$). Then, ψ_1 is true in $\mathcal{A}$, in which case $\psi_1 \in C(x_1)$, and so, by Theorem 4.6(i)⇒(iv), $\frac{1}{2} \leq^{\mathfrak{A}}_{\overline{\wedge}} \psi^{\mathfrak{A}}_1(0) = 1$.

(iii) Consider the following complementary cases:

- $\mathcal{A}$ is truth-singular,
  in which case $D^{\mathcal{A}} = \{1\}$, and so, for any $b \in D^{\mathcal{D}} \in \wp_{\infty\setminus 1}(D)$ and each $i \in I$, $\pi_i(b) = 1$ (in particular, $D \ni b = (I \times \{1\})$).
- $\mathcal{A}$ is false-singular,
  in which case $\beta^{\mathfrak{A}}_{\overline{\wedge}} = 0 \in C_i$, for each $i \in I$, as $\mathcal{C}_i \in \mathbf{S}_*(\mathcal{A})$, and so $\mathfrak{C}_i$, being a subalgebra of $\mathfrak{A}$, is a $\overline{\wedge}$-semi-lattice with bound 0, because $\mathfrak{A}$ is so. Then, $\mathfrak{D}$, being finite, as both A and I are so, is a $\overline{\wedge}$-semi-lattice with bound, in which case, by Lemma 2.2, for each $i \in I$, $\pi_i(\beta^{\mathfrak{D}}_{\overline{\wedge}}) = \beta^{\mathfrak{C}_i}_{\overline{\wedge}} = 0$, since $(\pi_i \restriction D) \in \hom(\mathfrak{D}, \mathfrak{C}_i)$ is surjective, and so $D \ni \beta^{\mathfrak{D}}_{\overline{\wedge}} = (I \times \{0\})$.

Thus, anyway, $(I \times \{1 - \Bbbk^{\mathcal{A}}\}) \in D$, in which case $D \ni \sim^{\mathfrak{D}}(I \times \{1 - \Bbbk^{\mathcal{A}}\}) = (I \times \{\Bbbk^{\mathcal{A}}\})$, and so the fact that $2 = \{\Bbbk^{\mathcal{A}}, 1 - \Bbbk^{\mathcal{A}}\}$ completes the argument.

(iv) First, (d/h) is a particular case of (c/g), while (d/e)⇒(e/c) is by (ii/i), whereas (b)⇒(e) is by the $\overline{\wedge}$-conjunctivity of $\mathcal{A}$ and the fact that $1 \in D^{\mathcal{A}} \not\ni 0$. Next, (a)⇒(b) is by the fact that $\mathrm{img}(h_{+,1-\Bbbk^{\mathcal{A}}}) = 2$. Further, assume (f) holds, in which case $l \triangleq \sim^{\mathfrak{A}}\frac{1}{2} \in 2$, and so $\xi^{\mathfrak{A}}_{1-l}(\frac{1}{2}) = 1 \in D^{\mathcal{A}}$. We prove (e) by contradiction. For suppose (e) does not hold, in which case $\beta^{\mathfrak{A}}_{\overline{\wedge}} \neq 0$, and so, by Lemma 5.13, $\beta^{\mathfrak{A}}_{\overline{\wedge}} = \frac{1}{2}$, for $1 \in D^{\mathcal{A}}$ (in particular, $\phi^{\mathfrak{A}}_0(\frac{1}{2}) = \frac{1}{2}$). Likewise, by (i), for each $k \in 2$, $\phi^{\mathfrak{A}}_0(k) = (0 \overline{\wedge}^{\mathfrak{A}} 1) = \beta^{\mathfrak{A}}_{\overline{\wedge}} = \frac{1}{2}$, in which case ϕ_{1-l} is true in $\mathcal{A}$, and so $\phi_{1-l} \in C(x_1)$. Then, by Theorem 4.6(i)⇒(iv), $0 \leq^{\mathfrak{A}}_{\overline{\wedge}} \phi^{\mathfrak{A}}_{1-l}(0) = 1$. Thus, (e) holds. [Conversely, assume (f) does not hold, in which case $\sim^{\mathfrak{A}}a = (1-a)$, for all $a \in A$. Take any $h \in \eth(\mathfrak{A}) \neq \varnothing$, in which case it is neither diagonal nor singular, and so, by Lemma 5.6, $(h \restriction 2) \in \{\Delta^+_2, \Delta^-_2\}$. Then, we have $h(\frac{1}{2}) = h(\sim^{\mathfrak{A}}\frac{1}{2}) = \sim^{\mathfrak{A}}h(\frac{1}{2}) = (1 - h(\frac{1}{2}))$, in which case we get $h(\frac{1}{2}) = \frac{1}{2}$, and so $h = h_{-,\frac{1}{2}}$, for, otherwise, h would be diagonal. Thus, (h)⇒(f) holds.] Now, assume (e) holds (that is, (c) does so), in which case, for each $k \in 2$, $\phi^{\mathfrak{A}}_0(k) = (0 \overline{\wedge}^{\mathfrak{A}} 1) = 0$, and so $\phi^{\mathfrak{A}}_1(k) = 1 \in D^{\mathcal{A}}$. We prove (f) by contradiction. For suppose $\sim^{\mathfrak{A}}\frac{1}{2} = \frac{1}{2}$, in which case $\phi^{\mathfrak{A}}_0(\frac{1}{2}) = \frac{1}{2}$, and so $\phi^{\mathfrak{A}}_1(\frac{1}{2}) = \frac{1}{2}$. Consider the following complementary cases:

- $\mathcal{A}$ is false-singular,
  in which case $\phi^{\mathfrak{A}}_1(\frac{1}{2}) = \frac{1}{2} \in D^{\mathcal{A}}$, and so ϕ_1 is true in $\mathcal{A}$ (in particular, $\phi_1 \in C(x_1)$). Then, by Theorem 4.6(i)⇒(iv), $1 \leq^{\mathfrak{A}}_{\overline{\wedge}} \phi^{\mathfrak{A}}_1(\frac{1}{2}) = \frac{1}{2}$, in which case, by (ii), $\frac{1}{2} = 1$, and so $\frac{1}{2} \in 2$.
- $\mathcal{A}$ is truth-singular,
  in which case it is not $\sim$-paraconsistent, and so, by (2.11) and the $\overline{\wedge}$-conjunctivity of C, $x_1 \in C(\phi_0)$. Then, by Theorem 4.6(i)⇒(iv), $\frac{1}{2} = \phi^{\mathfrak{A}}_0(\frac{1}{2}) \leq^{\mathfrak{A}}_{\overline{\wedge}} 0$, in which case, by (c), $\frac{1}{2} = 0$, and so $\frac{1}{2} \in 2$.

Thus, as $\frac{1}{2} \notin 2$, (f) does hold. Furthermore, if any $h : A \to A$ with $(h \restriction 2) = \Delta^-_2$ was an endomorphism of $\mathfrak{A}$, then, by (e), we would have $1 = h(0) = h(0 \overline{\wedge}^{\mathfrak{A}}$

$1) = (h(0) \mathbin{\overline{\wedge}}^{\mathfrak{A}} h(1)) = (1 \mathbin{\overline{\wedge}}^{\mathfrak{A}} 0) = (0 \mathbin{\overline{\wedge}}^{\mathfrak{A}} 1) = 0$, and so (g) holds. [Finally, (g)⇒(a) is by (5.1) and Lemma 5.6, for $\mathfrak{d}(\mathcal{A}) = (\partial(\mathcal{A}) \cap \hom(\mathfrak{A}, \mathfrak{A}))$.] □

Theorem 5.15. *Suppose C is $\overline{\wedge}$-conjunctive, non-$\sim$-classical and self-extensional. Then, $\mathfrak{d}(\mathcal{A}) \neq \varnothing$.*

Proof. Then, by Theorems 4.6(i)⇒(iv), 5.4 and Lemma 5.13, $\mathfrak{A}$, being finite, is a $\overline{\wedge}$-semi-lattice with bound $\beta^{\mathfrak{A}}_{\overline{\wedge}} \notin D^{\mathcal{A}}$, in which case, as $\frac{1}{2} \notin 2 \ni \Bbbk^{\mathcal{A}}$ (in particular, $\frac{1}{2} \neq \Bbbk^{\mathcal{A}}$), by the commutativity identity for $\overline{\wedge}$, there are some $\bar{a} \in (\{\frac{1}{2}, \Bbbk^{\mathcal{A}}\}^2 \setminus \Delta_A)$ and some $i \in 2$ such that $a_{1-i} \neq (a_i \mathbin{\overline{\wedge}}^{\mathfrak{A}} a_{1-i})$, and so $\mathcal{B} \triangleq \langle \mathfrak{A}, F \rangle$, where $a_i \in F \triangleq \{b' \in A \mid a_i \leq^{\mathfrak{A}}_{\overline{\wedge}} b'\} \not\ni a_{1-i}$, being both truth-non-empty and $\overline{\wedge}$-conjunctive, is a consistent model of C. In that case, $a_i \neq \beta^{\mathfrak{A}}_{\overline{\wedge}} \notin F$, so, by Lemma 5.14(i), $2 \nsubseteq D^{\mathcal{B}}$, for $\mathcal{B}$ is $\overline{\wedge}$-conjunctive. Likewise, by Lemma 5.14(ii), $(2 \cap D^{\mathcal{A}}) \neq \varnothing$, for $D^{\mathcal{B}} \neq \varnothing$. Therefore, since 2 forms a subalgebra of $\mathfrak{A}{\upharpoonright}\Sigma_{\sim}$, while $(\mathcal{A}{\upharpoonright}\Sigma_{\sim}){\upharpoonright}2$ is canonically $\sim$-classical, $(\mathcal{B}{\upharpoonright}\Sigma_{\sim}){\upharpoonright}2$ is a $\sim$-classical submatrix of $\mathcal{B}{\upharpoonright}\Sigma_{\sim}$, so $\mathcal{B}$ is $\sim$-super-classical. Let C' be the logic of $\mathcal{B}$. Consider the following complementary cases:

- C' is $\sim$-classical,
  in which case, as it is $\overline{\wedge}$-conjunctive, for its sublogic C is so, by Theorem 5.4(i)⇒(ii), $\mathcal{B}$ is a strictly surjectively homomorphic counter-image of a $\sim$-classical Σ-matrix $\mathcal{D}$. Then, by (2.15), $\mathcal{D}$ is a finite, simple, consistent and truth-non-empty model of C, for $\mathcal{B} \in \mathrm{Mod}(C)$, in which case, by Remarks 2.6(ii), 2.8(ii)(b), Lemmas 3.3, 5.14(iii)(b) and Theorem 5.4(iii)⇒(i), the submatrix $\mathcal{E}$ of $\mathcal{A}$ generated by 2 is embeddable into $\mathcal{D}$, and so is isomorphic to this, for $\mathcal{D}$ has no proper submatrix (in particular, $\mathcal{B}$ is a strictly homomorphic counter-image of $\mathcal{A}$).
- C' is not $\sim$-classical,
  in which case, by Theorem 5.4(iii)⇒(i), $\mathcal{B}$, defining C', is simple. Hence, by Lemma 3.3, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$, some subdirect product $\mathcal{G}$ of it and some $g \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{G}, \mathcal{B})$, in which case, by Remark 2.8(ii)(b), $\mathcal{G}$ is both consistent (in particular, $I \neq \varnothing$) and truth-non-empty, for $\mathcal{B}$ is so, and so, by Lemma 5.14(iii)(a), $a \triangleq (I \times \{1\}) \in G \ni b \triangleq (I \times \{0\})$. We prove, by contradiction, that $(I \times \{\frac{1}{2}\}) \in G$. For suppose $(I \times \{\frac{1}{2}\}) \notin G$, in which case $\mathcal{A}$ is classically-hereditary, for, otherwise, there would be some $\varphi \in \mathrm{Fm}^2_{\Sigma}$ such that $\varphi^{\mathfrak{A}}(0,1) = \frac{1}{2}$, and so $G \supseteq \{a, b\}$ would contain $\varphi^{\mathfrak{G}}(b, a) = (I \times \{\frac{1}{2}\})$. Consider the following complementary subcases:
  - $\mathcal{A}$ is truth-singular,
    in which case $\mathcal{B}$ is so, and so $D^{\mathcal{B}} = \{a_i\}$ (in particular, by Lemma 5.14(ii), $a_i \neq \frac{1}{2}$, for $1 \neq \frac{1}{2}$). Then, $\beta^{\mathfrak{A}}_{\overline{\wedge}} \neq a_i = \Bbbk^{\mathcal{A}} = 0$, in which case, as $1 \in D^{\mathcal{A}}$, $\beta^{\mathfrak{A}}_{\overline{\wedge}} = \frac{1}{2} \neq 0$, and so, by Lemma 5.14(iv)(b)⇒(c), $\mathcal{A}$ is not classically-hereditary.
  - $\mathcal{A}$ is false-singular,
    in which case, by Lemma 5.13, $\beta^{\mathfrak{A}}_{\overline{\wedge}} = 0$, and so, by Lemma 5.14(iv)(c)⇒(e/f), $(0 \leq^{\mathfrak{A}}_{\overline{\wedge}} 1)/({\sim}^{\mathfrak{A}}\frac{1}{2} \in 2)$, respectively. And what is more, by Lemma 5.14(ii), $\frac{1}{2} \leq^{\mathfrak{A}}_{\overline{\wedge}} 1$, in which case $1 = \delta\beta^{\mathfrak{A}}_{\overline{\wedge}}$, while $a_i \neq \frac{1}{2}$, for, otherwise, we would have $\frac{1}{2} = a_i \nleq^{\mathfrak{A}}_{\overline{\wedge}} a_{1-i} = \Bbbk^{\mathfrak{A}} = 1$, and so $a_i = \Bbbk^{\mathcal{A}} = 1$ (in particular, $D^{\mathcal{B}} = \{1\}$, for $1 = \delta\beta^{\mathfrak{A}}_{\overline{\wedge}}$). Furthermore, there is some $c \in G$ such that $g(c) = \frac{1}{2} \notin D^{\mathcal{B}}$, in which case $c \notin D^{\mathcal{G}}$, and so there is some $l \in I$ such that $\pi_l(c) = 0$, for $\mathcal{C}_l \in \mathbf{S}_*(\mathcal{A})$, while 0 is the only non-distinguished value of $\mathcal{A}$. Let $\mathcal{H}$ be the submatrix of $\mathcal{G}$ generated by $\{a, b, c\}$, in which case $(g{\upharpoonright}H) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{H}, \mathcal{B})$, for $g[\{a, b, c\}] = A$, and so, by (2.15), C', being defined by $\mathcal{B}$, is defined by $\mathcal{H}$. And what is more, since $\pi_l[\{a, b, c\}] = 2$ forms a subalgebra of

$\mathfrak{A}$, $\pi_l \restriction H$ is a surjective homomorphism from $\mathcal{H}$ onto $\mathcal{A}\restriction 2$. We prove that this is strict, by contradiction. For suppose there is some $d \in (G \setminus D^{\mathcal{G}})$ such that $\pi_l(d) \in (D^{\mathcal{A}} \cap 2) = \{1\}$, in which case $\pi_l(\sim^{\mathfrak{G}} d) = \sim^{\mathfrak{A}} \pi_l(d) = \sim^{\mathfrak{A}} 1 = 0 \notin D^{\mathcal{A}}$, and so $\sim^{\mathfrak{G}} d \notin D^{\mathcal{G}}$. Consider the following complementary (for $\sim^{\mathfrak{A}} \frac{1}{2} \in 2$) subsubcases:

- $\ast$ $\sim^{\mathfrak{A}} \frac{1}{2} = 1$,
  in which case $\mathcal{B}$ is $\sim$-negative, and so is $\mathcal{G}$, in view of Remark 2.8(ii)(a) (in particular, $\sim^{\mathfrak{G}} d \in D^{\mathcal{G}}$, for $d \in (G \setminus D^{\mathcal{G}})$).
- $\ast$ $\sim^{\mathfrak{A}} \frac{1}{2} = 0$,
  in which case $\sim^{\mathfrak{A}} \sim^{\mathfrak{A}} \frac{1}{2} = \sim^{\mathfrak{A}} 0 = 1 \in D^{\mathcal{B}}$. On the other hand, $g(d) \notin D^{\mathcal{B}} \not\ni g(\sim^{\mathfrak{G}} d) = \sim^{\mathfrak{A}} g(d)$, in which case $g(d) \notin 2$, and so $g(d) = \frac{1}{2}$ (in particular, $g(\sim^{\mathfrak{G}} \sim^{\mathfrak{G}} d) = \sim^{\mathfrak{A}} \sim^{\mathfrak{A}} g(d) \in D^{\mathcal{B}}$). However, since $d \notin D^{\mathcal{G}}$, there is some $m \in I$ such that $\pi_m(d) = 0$, in which case $\pi_m(\sim^{\mathfrak{G}} \sim^{\mathfrak{G}} d) = \sim^{\mathfrak{A}} \sim^{\mathfrak{A}} \pi_m(d) = \sim^{\mathfrak{A}} \sim^{\mathfrak{A}} 0 = 0 \notin D^{\mathcal{A}}$, and so $\sim^{\mathfrak{G}} \sim^{\mathfrak{G}} d \notin D^{\mathcal{G}}$ (in particular, $g(\sim^{\mathfrak{G}} \sim^{\mathfrak{G}} d) \notin D^{\mathcal{B}}$).

Thus, in any case, we come to a contradiction, in which case $(\pi_l \restriction H) \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{H}, \mathcal{A}\restriction 2)$, and so, by (2.15), C', being defined by $\mathcal{H}$, is defined by the $\sim$-classical Σ-matrix $\mathcal{A}\restriction 2$ (in particular, C' is $\sim$-classical).

Thus, anyway, we come to a contradiction, in which case $(I \times \{\frac{1}{2}\}) \in G$, and so, as $I \neq \varnothing$, while $a \in G \ni b$, $e \triangleq \{\langle a', I \times \{a'\}\rangle \mid a' \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{G}$. Therefore, by Remark 2.6(ii) and Theorem 5.4, $e' \triangleq (e \circ g)$ is am embedding of $\mathcal{A}$ into $\mathcal{B}$, in which case it is an isomorphism from $\mathcal{A}$ onto $\mathcal{B}$, because $|A| = 3 \nleq n$, for no $n \in 3 = |B|$, and so ${e'}^{-1} \in \hom(\mathcal{B}, \mathcal{A})$ is strict.

In this way, in any case, there is some strict $h \in \hom(\mathcal{B}, \mathcal{A}) \subseteq \hom(\mathfrak{A}, \mathfrak{A})$, in which case $h(a_i) \in D^{\mathcal{A}} \not\ni h(a_{1-i})$, for $a_i \in D^{\mathcal{B}} \not\ni a_{1-i}$, and so $h \in \eth(\mathcal{A})$, as required. □

Then, combining Corollary 5.5 and Theorem 5.15 with Lemmas 5.13 and 5.14(ii, iv), we immediately get the following two corollaries:

Corollary 5.16. *Suppose C is both $\overline{\wedge}$-conjunctive and non-$\sim$-classical, while $\mathcal{A}$ is false-/truth-singular. Then, C is self-extensional iff /either $h_{+,1-\Bbbk^{\mathcal{A}}}$ /“or $h_{-,\frac{1}{2}}$” is an endomorphism of $\mathfrak{A}$ [while $\mathfrak{A}$ is a $\overline{\wedge}$-semi-lattice with dual bound 1, whereas it is that with bound 0 and/iff $\sim^{\mathfrak{A}} \frac{1}{2} \neq \frac{1}{2}$ “as well as”/iff $\mathcal{A}$ is classically-hereditary].*

Corollary 5.17. *Suppose $\mathcal{A}$ is both $\overline{\wedge}$-conjunctive and $\underline{\vee}$-disjunctive, while C is not $\sim$-classical. Then, C is self-extensional iff $h_{+,1-\Bbbk^{\mathcal{A}}} \in \hom(\mathfrak{A}, \mathfrak{A})$ [while $\mathfrak{A}$ is a distributive $(\overline{\wedge}, \underline{\vee})$-lattice with zero 0 and unit 1, whereas $\sim^{\mathfrak{A}} \frac{1}{2} \neq \frac{1}{2}$ as well as $\mathcal{A}$ is classically hereditary].*

These immediately yield the self-extensionality of $[P]G_3^{(*)}$, for $h_{+,1-\Bbbk^{\mathcal{A}}}$ is an endomorphism of the underlying algebra of its conjunctive (disjunctive) defining matrix. And what is more, they immediately imply the non-self-extensionality of both P^1 and I^1, for the underlying algebras of their conjunctive (disjunctive) defining matrices are not semi-lattices at all (cf. Remark 5.12). Likewise, the non-self-extensionality of the conjunctive (disjunctive) HZ ensues from either the involutivity of the negation operation of the underlying algebra of its defining conjunctive (disjunctive) classically-hereditary matrix or the fact that the underlying algebra of this matrix, though being a distributive lattice, is not that with *both* zero 0 and unit 1. Finally, the above corollaries imply *immediately* the non-self-extensionality of $LP_{[01]}/K_{3[,01]}$, in view the involutivity of the negation operation of the underlying algebra of their conjunctive (disjunctive) classically-hereditary matrices, providing, as opposed to Example 4.17, a more [perhaps, the most] transparent and immediate *generic* insight into the non-self-extensionality of the latter independent from that of the former, and so into that of Łukasiewicz' finitely-valued logics [6], for these are expansions of K_3. On the other hand, Corollary 5.17 does not subsume Corollary 5.16, due to existence of self-extensional conjunctive but non-disjunctive non-$\sim$-classical Σ-logics of the kind under consideration, most representative instances of which are as follows:

Example 5.18. Let $\Sigma \triangleq \{\wedge, \sim\}$ and $\mathcal{A}$ the Σ-reduct of the [non-]truth-singular $\Sigma^{\supset}_{\sim,+,01}$-matrix specified in Subparagraph 5.1.1.1.2, in which case the former is both $\wedge$-conjunctive and non-$\sim$-negative, for the latter is so, and so $[P]G_3^{\wedge} \triangleq C$, being the Σ-fragment of the self-extensional [paraconsistent counterpart of] Gödel's three-valued logic $[P]G_3$ [2], is both $\wedge$-conjunctive and self-extensional as well as, by Theorem 5.4(i)$\Rightarrow$(iv), not $\sim$-classical. On the other hand, by induction on construction of any $\varphi \in \mathrm{Fm}^2_{\Sigma}$, we prove that either $\varphi^{\mathfrak{A}}(\frac{1}{2}, \frac{1}{2}) \neq \frac{1}{2}$ or there are some $a, b \in A$ such that $\max(a, b) \nleqslant \varphi^{\mathfrak{A}}(a, b)$. In case $\varphi = x_{0|1}$, taking $a \triangleq (0|1)$ and $b \triangleq (1|0)$, we get $\max(a, b) = 1 \nleqslant 0 = \varphi^{\mathfrak{A}}(a, b)$. Likewise, in case $\varphi = \sim\xi$, where $\xi \in \mathrm{Fm}^2_{\Sigma}$, as $(\mathrm{img} \sim^{\mathfrak{A}}) \subseteq 2 \not\ni \frac{1}{2}$, we have $\varphi^{\mathfrak{A}}(\frac{1}{2}, \frac{1}{2}) \neq \frac{1}{2}$. Finally, in case $\varphi = (\phi \wedge \psi)$, where $\phi, \psi \in \mathrm{Fm}^2_{\Sigma}$, if $\varphi^{\mathfrak{A}}(\frac{1}{2}, \frac{1}{2})$ is equal to $\frac{1}{2}$, then so is either $\phi^{\mathfrak{A}}(\frac{1}{2}, \frac{1}{2})$ or $\psi^{\mathfrak{A}}(\frac{1}{2}, \frac{1}{2})$, for $\mathcal{A}$ is classically-hereditary, while, if, for any $a, b \in A$, it holds that $\max(a, b) \leqslant \varphi^{\mathfrak{A}}(a, b) = \min(\phi^{\mathfrak{A}}(a, b), \psi^{\mathfrak{A}}(a, b))$, then both $\max(a, b) \leqslant \phi^{\mathfrak{A}}(a, b)$ and $\max(a, b) \leqslant \psi^{\mathfrak{A}}(a, b)$ hold, and so the induction hypothesis completes the argument. In particular, $\max \cap A^2$ is not term-wise definable in $\mathfrak{A}$. Therefore, by Lemma 5.1 and Corollary 5.17, $[P]G_3^{\wedge}$ is not disjunctive. ☐

5.1.1.2.3. Implicative logics. We start from marking the framework of the self-extensionality of C under its being non-$\sim$-classical and $\mathcal{A}$'s being implicative:

Corollary 5.19. *Suppose $\mathcal{A}$ is $\sqsupset$-implicative. Then, C is not self-extensional, unless it is either $\sim$-paraconsistent or $\sim$-classical. In particular, C is not self-extensional, whenever $\mathcal{A}$ is truth-singular (in particular, both $(\veebar, \sim)$-paracomplete and weakly $\veebar$-disjunctive).*

Proof. In case $\mathcal{A}$ is both false-singular and non-$\sim$-paraconsistent, it is $\sim$-negative. In this way, Remarks 2.8(i)(c), 4.14(iv), Lemma 5.9 and Corollary 5.10 complete the argument. ☐

Theorem 5.20. *Suppose $\mathcal{A}$ is $\sqsupset$-implicative, while C is not $\sim$-classical. Then, C is self-extensional iff $h_{-,\frac{1}{2}} \in \mathrm{hom}(\mathfrak{A}, \mathfrak{A})$ [while $\mathfrak{A}$ is an $\sqsupset$-implicative intrinsic semi-lattice with bound $\frac{1}{2} = \sim^{\mathfrak{A}} \frac{1}{2}$].*

Proof. Assume C is self-extensional. Then, by Theorem 4.9, $\mathfrak{A}$ is an $\sqsupset$-implicative intrinsic semi-lattice with bound $a \triangleq (\frac{1}{2} \sqsupset^{\mathfrak{A}} \frac{1}{2}) = (b \sqsupset^{\mathfrak{A}} b)$, for any $b \in A$, while, by Corollary 5.19, $\mathcal{A}$ is $\sim$-paraconsistent (in particular, false-singular), in which case $a \in D^{\mathcal{A}} = \{\frac{1}{2}, 1\}$, and so $a = \frac{1}{2}$ [in particular, $\sim^{\mathfrak{A}} a \in D^{\mathcal{A}}$, and so $\sim^{\mathfrak{A}} a = \frac{1}{2}$], for, otherwise, we would have $[\sim^{\mathfrak{A}}]a = 1$, in which case we would get $\sim^{\mathfrak{A}}[\sim^{\mathfrak{A}}]a = \sim^{\mathfrak{A}} 1 = 0 \notin D^{\mathcal{A}}$, and so $\mathcal{A}$ would be $\wr$-negative, where $\wr x_0 \triangleq (x_0 \sqsupset \sim[\sim](x_0 \sqsupset x_0))$ (in particular, by Corollary 5.10, C would not be self-extensional). In that case, for any $h \in \mathrm{hom}(\mathfrak{A}, \mathfrak{A})$, we have $h(\frac{1}{2}) = (h(\frac{1}{2}) \sqsupset^{\mathfrak{A}} h(\frac{1}{2})) = \frac{1}{2}$, and so Theorem 5.8 completes the argument. ☐

Corollaries 5.17/5.16 and 5.20, in particular, "provide one more insight into their context's being disjoint, in view of opposite requirements on the negation operation"/"taking Example 4.2 into account, immediately yield the following essential (mainly, due to elimination of the disjunctivity stipulation) enhancement of Theorem 5.8":

Corollary 5.21. *Suppose $\mathcal{A}$ is either implicative or conjunctive. Then, C is self-extensional iff either it is $\sim$-classical or $(\{h_{+,1-\Bbbk^{\mathcal{A}}}, h_{-,\frac{1}{2}}\} \cap \hom(\mathfrak{A},\mathfrak{A})) \neq \varnothing$.*

Finally, we present an instance of a $\sim$-paraconsistent implicative three-valued $\sim$-super-classical Σ-matrix, the logic of which is self-extensional:

Example 5.22. Let $\Sigma \triangleq \Sigma^{\supset}_{\sim}$ and $\mathcal{A}$ false-singular with $\sim^{\mathfrak{A}}\frac{1}{2} \triangleq \frac{1}{2}$ and, for all $a \in A$, $(a \supset^{\mathfrak{A}} a) \triangleq \frac{1}{2}$ as well as, for all $b \in (A \setminus \{a\})$, $(a \supset^{\mathfrak{A}} b) \triangleq b$. Then, $\mathcal{A}$ is both $\sim$-paraconsistent and $\supset$-implicative. And what is more, $h_{-,\frac{1}{2}} \in \hom(\mathfrak{A},\mathfrak{A})$. Hence, by Theorem 5.20, C is self-extensional. □

5.2. **Four-valued expansions of Belnap's four-valued logic.** A *[bounded] De Morgan lattice* [12] is any $\Sigma_{\sim,+[,01]}$-algebra, whose $\Sigma_{+[,01]}$-reduct is a [bounded] distributive lattice and that satisfies the following $\Sigma_{\sim,+}$-identities:

$$\sim\sim x_0 \approx x_0, \tag{5.2}$$

$$\sim(x_0 \vee x_1) \approx (\sim x_0 \wedge \sim x_1), \tag{5.3}$$

By $\mathfrak{DM}_{4[,01]}$ we denote the non-Boolean diamond [bounded] De Morgan lattice with $(\mathfrak{DM}_{4[,01]} \upharpoonright \Sigma_{+[,01]}) \triangleq \mathfrak{D}^2_{2[,01]}$ and $\sim^{\mathfrak{DM}_{4[,01]}}\langle i,j\rangle \triangleq \langle 1-j, 1-i\rangle$, for all $i,j \in 2$.

Here, it is supposed that $\Sigma \supseteq \Sigma_{\sim,+[,01]}$. Fix a Σ-matrix $\mathcal{A}$ with $(\mathfrak{A} \upharpoonright \Sigma_{\sim,+[,01]}) \triangleq \mathfrak{DM}_{4[,01]}$ and $D^{\mathcal{A}} \triangleq (2^2 \cap \pi_0^{-1}[\{1\}])$. Then, both $\mathcal{A}$ and $\boldsymbol{\partial}(\mathcal{A}) \triangleq \langle \mathfrak{A}, 2^2 \cap \pi_1^{-1}[\{1\}]\rangle$ are both $\wedge$-conjunctive and $\vee$-disjunctive, while $\{x_0, \sim x_0\}$ is an equality determinant for them (cf. Example 2 of [13]), so they as well as their submatrices are hereditarily simple, while:

$$(\theta^{\mathcal{A}} \cap \theta^{\boldsymbol{\partial}(\mathcal{A})}) = \Delta_A. \tag{5.4}$$

Let C be the logic of $\mathcal{A}$. Then, since $\mathcal{DM}_{4[,01]} \triangleq (\mathcal{A} \upharpoonright \Sigma_{\sim,+[,01]})$ defines [the bounded version/expansion of] Belnap's four-valued logic $B_{4[,01]}$ [1] (cf. [12, 17]), C is a four-valued expansion of $B_{4[,01]}$. Conversely, according to Corollary 4.9 of [17] any four-valued expansion of $B_{4[,01]}$ is defined by a unique expansion of $\mathcal{DM}_{4[,01]}$. Moreover, according to Theorem 4.20 of [17], C is $\sim$-subclassical iff Δ_2 forms a subalgebra of $\mathfrak{A}$, in which case $\mathcal{A} \upharpoonright 2$ is isomorphic to any $\sim$-classical model of C, and so defines a unique $\sim$-classical extension of C.

Given any $i \in 2$, put $DM_{3,i} \triangleq (2^2 \setminus \{\langle i, 1-i\rangle\})$. Then, providing this forms a subalgebra of $\mathfrak{A}$ (such is the case, when, e.g., $\Sigma = \Sigma_{\sim,+[,01]}$), we set $(\mathcal{A}/\mathcal{DM})_{3,i/[,01]} \triangleq ((\mathcal{A}/\mathcal{DM})_{/4[,01]} \upharpoonright DM_{3,i})$, the logic $(C/B)_{3,i/[,01]}$ of which is a both $\vee$-disjunctive and $\wedge$-conjunctive (for its defining matrix is so; cf. Remark 2.8(ii)) as well as inferentially consistent (for its defining matrix is both consistent and truth-non-empty) unitary three-valued both extension of $(C/B)_{4/[,01]}$, in view of (2.15), and a three-valued expansion of [the bounded version/expansion $LP_{01}|K_{3,01}$ of] "the *logic of paradox*"|"Kleene's three-valued logic" $LP|K_3$ [8]| [4], defined by $\mathcal{DM}_{3,i[,01]}$, whenever $i = (0|1)$.

Let $\mu : 2^2 \to 2^2, \langle i,j\rangle \mapsto \langle j,i\rangle$. Then,

$$D^{\boldsymbol{\partial}(\mathcal{A})} = \mu^{-1}[D^{\mathcal{A}}]. \tag{5.5}$$

Theorem 5.23 (cf. [16, 18])**.** *The following are equivalent:*

- (i) *C is self-extensional;*
- (ii) *$\mathfrak{A}$ is specular;*
- (iii) *$\boldsymbol{\partial}(\mathcal{A})$ is isomorphic to $\mathcal{A}$;*
- (iv) *C is defined by $\boldsymbol{\partial}(\mathcal{A})$;*
- (v) *$\boldsymbol{\partial}(\mathcal{A}) \in \mathrm{Mod}(C)$;*

Proof. First, assume (i) holds. Then, by Theorem 4.7, there is some $h \in \hom(\mathfrak{A}, \mathfrak{A})$ such that $\chi^{\mathcal{A}}(h(11)) \neq \chi^{\mathcal{A}}(h(10))$, in which case h is not singular, and so $B \triangleq (\operatorname{img} h)$ forms a non-one-element subalgebra of $\mathfrak{A}$. Hence $\Delta_2 \subseteq B$, in which case $\mathfrak{A}[\restriction B]$ is a $(\wedge, \vee)$-lattice with zero/unit $\langle 0/1, 0/1\rangle$, and so, by Lemma 2.2, $(h \restriction \Delta_2)$ is diagonal. Therefore, $h(10) \notin D^{\mathcal{A}}$, for $h(11) = (11) \in D^{\mathcal{A}}$. On the other hand, for all $a \in A$, it holds that $({\sim}^{\mathfrak{A}} a = a) \Leftrightarrow (a \notin \Delta_2)$. Therefore, $h(10) = (01)$. Moreover, if $h(01)$ was equal to 01 too, then we would have $(00) = h(00) = h((10) \wedge^{\mathfrak{A}} (01)) = ((01) \wedge^{\mathfrak{A}} (01)) = (01)$. Thus, $\hom(\mathfrak{A}, \mathfrak{A}) \ni h = \mu$, so (ii) holds.

Next, (ii)⇒(iii) is by (5.5) and the bijectivity of $\mu : A \to A$, while (iii)⇒(iv) is by (2.15), whereas (v) is a particular case of (iv).

Finally, (v)⇒(i) is by (5.4) and Theorem 4.1(vi)⇒(i) with $\mathsf{S} = \{\mathcal{A}, \boldsymbol{\partial}(\mathcal{A})\}$. □

This provides a new proof as well as a new *generic* insight to the already-known result of [16, 18] proved originally *ad hoc* therein, so justifying the first paragraph of Section 1.

Theorem 5.23 positively covers $B_{4[,01]}$. Moreover, in case $\Sigma = \Sigma_{\simeq,+[,01]} \triangleq (\Sigma_{\sim,+[,01]} \cup \{\neg\})$ with unary $\neg$ (classical — viz., Boolean — negation) and $\neg^{\mathfrak{A}}\langle i, j\rangle \triangleq \langle 1-i, 1-j\rangle$, for all $i, j \in 2$, being the complement operation, Theorem 5.23 equally covers the logic $CB_{4[,01]} \triangleq C$ of the $\neg$-negative (and so $\sqsupset^{\neg}_{\vee}$-implicative; cf. Remark 2.8(i)(b)) $\mathcal{DMB}_{4[,01]} \triangleq \mathcal{A}$ introduced in [12]. In view of Corollary 4.15, the self-extensionality of these three instances provides a new insight and a new proof to the non-algebraizability of the sequent calculi associated with the instances involved proved originally in [12] by a quite different (though equally generic) method based upon universal tools elaborated in [11]. This once more justifies the thesis of the first paragraph of Section 1. As for *some* of non-self-extensional instances, we first need the following consequence of Theorem 5.23(i)⇒(ii):

Corollary 5.24. *Suppose μ is an endomorphism of $\mathfrak{A}$ (i.e., C is self-extensional; cf. Theorem 5.23(i)⇒(ii)). Then, Δ_2 forms a subalgebra of $\mathfrak{A}$ (i.e., C is ${\sim}$-subclassical).*

Proof. By contradiction. For suppose there are some $f \in \Sigma$ of arity $n \in \omega$ and some $\bar{a} \in \Delta_2^n$ such that $f^{\mathfrak{A}}(\bar{a}) \notin \Delta_2$. Then, $f^{\mathfrak{A}}(\bar{a}) = f^{\mathfrak{A}}(\mu \circ \bar{a}) = \mu(f^{\mathfrak{A}}(\bar{a})) \neq f^{\mathfrak{A}}(\bar{a})$. This contradiction completes the argument. □

According to Corollary 5.2 of [17], any bilattice expansion of B_4 has no inferentially consistent proper (in particular, ${\sim}$-classical; cf. Remark 2.8(i)(c)) extension, and so, by Corollary 5.24, is not self-extensional.

Finally, consider (purely-)implicative expansions of $B_{4[,01]}$, in which case $\Sigma \supseteq (=)\Sigma^{\supset}_{\sim,+[,01]}$, where, for all $i, j, k, l \in 2$, $(\langle i, j\rangle \supset^{\mathfrak{A}} \langle k, l\rangle) \triangleq \langle \max(1-i, k), \max(1-i, l)\rangle$, and so $\mathcal{A}$ is $\supset$-implicative (in particular, C is so). Then, $\mu((01) \supset^{\mathfrak{A}} (01)) = \mu(11) = (11) \neq (10) = ((10) \supset^{\mathfrak{A}} (10)) = (\mu(01) \supset^{\mathfrak{A}} \mu(01))$, so, by Theorem 5.23(i)⇒(ii), any implicative expansion of B_4 is not self-extensional. On the other hand, according to Corollary 5.3 of [17], the *purely*-implicative expansion of $B_{4[,01]}$ is ${\sim}$-subclassical, in which case Corollary 5.24 is not applicable to proving its non-self-extensionality, as opposed to that of bilattice expansions, and so Theorem 5.23(i)⇒(ii) remains the main tool of disproving self-extensionality of expansions of B_4.

5.2.1. *No-more-than-three-valued extensions.*

Lemma 5.25. *Let $n \in (4 \setminus 1)$. Then, any n-valued model/extension of C is $\vee$-disjunctive.*

Proof. Let $\mathcal{B}$ be an n-valued model of C, in which case, by (2.15) and Remark 2.6(iv), $\mathcal{D} \triangleq (\mathcal{B}/\partial(\mathcal{B}))$, is an m-valued simple model of C, where $m \leqslant n \leqslant 3$, and so, by Corollary 3.13, $\mathfrak{D} \in \mathbf{V}(\mathfrak{A})$. Therefore, $\mathfrak{D}{\upharpoonright}\Sigma_+$, being an m-element lattice, for $\mathfrak{A}{\upharpoonright}\Sigma_+$ is a lattice, is a chain. Hence, $\mathcal{D}$, being $\wedge$-conjunctive, for C is so, is $\vee$-disjunctive, and so is $\mathcal{B}$, by Remark 2.8(ii), as required. □

Corollary 5.26. *Let $\mathcal{B}$ be a consistent truth-non-empty three-valued non-$\sim$-negative three-valued model of C and C' the logic of $\mathcal{B}$. Then, there is some $i \in 2$ such that $DM_{3,i}$ forms a subalgebra of $\mathfrak{A}$, while $\mathcal{B}$ is isomorphic to $\mathcal{A}_{3,i}$, and so $C' = C_{3,i}$.*

Proof. Then, by Lemma 5.25, $\mathcal{B}$ is $\vee$-disjunctive. Hence, by Theorem 3.5 and Remark 2.6(ii), there is some $h \in \hom_{\mathrm{S}}(\mathcal{B}, \mathcal{A})$, in which case $D \triangleq (\operatorname{img} h)$ forms a subalgebra of $\mathfrak{A}$, while h is a strict surjective homomorphism from $\mathcal{B}$ onto $\mathcal{D} \triangleq (\mathcal{A}{\upharpoonright}D)$. Therefore, if h was not injective, then $\mathcal{D}$ would be either one-valued, in which case it would be either inconsistent or truth-empty, and so would be $\mathcal{B}$, or two-valued, in which case D would be equal to Δ_2, and so, by Remark 2.8(ii), $\mathcal{B}$ would be $\sim$-negative, for $\mathcal{D}$ would be so. Thus, h is injective, in which case $|D| = 3$, and so $D = DM_{3,i}$, for some $i \in 2$. In this way, (2.15) completes the argument. □

Likewise, we have:

Corollary 5.27. *Let $\mathcal{B}$ be a consistent truth-non-empty two-valued model of C and C' the logic of $\mathcal{B}$. Then, Δ_2 forms a subalgebra of $\mathfrak{A}$, while $\mathcal{B}$ is isomorphic to $\mathcal{A}{\upharpoonright}\Delta_2$, in which case it is $\sim$-classical, and so is C'.*

Proof. Then, by Lemma 5.25, $\mathcal{B}$ is $\vee$-disjunctive. Hence, by Theorem 3.5 and Remark 2.6(ii), there is some strict $h \in \hom(\mathcal{B}, \mathcal{A})$, in which case $D \triangleq (\operatorname{img} h)$ forms a subalgebra of $\mathfrak{A}$, while h is a strict surjective homomorphism from $\mathcal{B}$ onto $\mathcal{D} \triangleq (\mathcal{A}{\upharpoonright}D)$. Therefore, if h was not injective, then $\mathcal{D}$ would be one-valued, in which case it would be either inconsistent or truth-empty, and so would be $\mathcal{B}$. Thus, h is injective, in which case $|D| = 2$, and so $D = \Delta_2$. In this way, Remark 2.8(ii) completes the argument. □

Lemma 5.28. *Let $\mathcal{B}$ be a finite $\sim$-negative model of C and C' the logic of $\mathcal{B}$. Then, Δ_2 forms a subalgebra of $\mathfrak{A}$, while $\mathcal{B}$ is a strict surjective homomorphic counter-image of $\mathcal{A}{\upharpoonright}\Delta_2$, an so C' is $\sim$-classical.*

Proof. Then, by the following claim, $\mathcal{B}$ is $\vee$-disjunctive:

Claim 5.29. *Any $\sim$-negative $\mathcal{B} \in \mathrm{Mod}(C)$ is $\vee$-disjunctive.*

Proof. Then, by Remark 2.8(i)(a), $\mathcal{B}$, being $\wedge$-conjunctive, for C is so, is $\wedge^{\sim}$-disjunctive. On the other hand, as (5.2) and (5.3) are true in $\mathfrak{A}$, so is $(x_0 \vee x_1) \approx (x_0 \wedge^{\sim} x_1)$, in which case, by Lemma 3.12, $(x_0 \vee x_1) \equiv^{\omega}_{C} (x_0 \wedge^{\sim} x_1)$, and so $((a \vee^{\mathfrak{B}} b) \in D^{\mathcal{B}}) \Leftrightarrow ((a(\wedge^{\sim})^{\mathfrak{B}} b) \in D^{\mathcal{B}})$, for all $a, b \in B$. Thus, $\mathcal{B}$, being $\wedge^{\sim}$-disjunctive, is equally $\vee$-disjunctive, as required. □

Therefore, by Theorem 3.5 and Remark 2.6(ii), there is some strict $h \in \hom(\mathcal{B}, \mathcal{A})$, in which case $D \triangleq (\operatorname{img} h)$ forms a subalgebra of $\mathfrak{A}$, while h is a strict surjective homomorphism from $\mathcal{B}$ onto $\mathcal{D} \triangleq (\mathcal{A}{\upharpoonright}D)$, and so, by Remark 2.8(ii), $\mathcal{D}$ is $\sim$-negative, for $\mathcal{B}$ is so. Hence, $D = \Delta_2$. Finally, (2.15) completes the argument. □

Then, Examples 4.2, 4.17, Corollary 5.26, Lemma 5.28 and the self-extensionality of inferentially inconsistent logics immediately yield:

Theorem 5.30. *Let C' be a three-valued extension of C. Then, the following are equivalent:*

(i) *C' is self-extensional;*
(ii) *C' is either inferentially inconsistent or $\sim$-classical;*
(iii) *for each $i \in 2$, if $DM_{3,i}$ forms a subalgebra of $\mathfrak{A}$, then $C' \neq C_{3,i}$.*

In general, since $\mathcal{DM}_4{\restriction}\{01\}$ is the only truth-empty submatrix of $\mathcal{DM}_4$, while $\{01\} \subseteq [\not\subseteq] DM_{3,1[-1]} \supseteq \Delta_2$, whereas any truth-empty Σ-matrix is both consistent and $\vee$-disjunctive, by (2.15), Corollaries 3.6, 5.26, 5.27, Theorem 3.5 and Lemma 5.28, we also have:

Theorem 5.31. *Let $\mathsf{M} \subseteq \mathrm{Mod}(C)$, C' the logic of M, $n \in (4 \setminus 1)$ and $\mathsf{M}^{(*)\{,\sim|\not\sim\}}_{[n]\langle,0/1\rangle}$ the class of all (truth-non-empty) [n-valued] $\{\sim$-negative$|$non-$\sim$-negative$\}$ $\langle$false-/truth-singular$\rangle$ consistent elements of M. Suppose, for each $\mathcal{B} \in \mathsf{M}$, $|B| \in 4$ (in particular, $\mathcal{A} \neq \mathcal{B} \in \mathbf{S}(\mathcal{A})$). Then, C' is defined by $\{\mathcal{A}{\restriction}\{01\} \mid (\mathsf{M} \setminus \mathsf{M}^*) \neq \varnothing = \mathsf{M}^{*,\not\sim}_{3,1}\} \cup \{\mathcal{A}{\restriction}\Delta_2 \mid (\bigcup_{i\in 2} \mathsf{M}^{*,\not\sim}_{3,i}) = \varnothing \neq (\mathsf{M}^{\sim} \cup \mathsf{M}^*_2)\} \cup \bigcup_{i\in 2}\{\mathcal{A}_{3,i} \mid \mathsf{M}^{*,\not\sim}_{3,i} \neq \varnothing\}$.*

By Theorem 5.30, any inferentially consistent non-$\sim$-classical *unitary* three-valued extension of C' is not self-extensional. Then, taking (2.13), Theorem 5.31, Remark 2.5 and Example 4.2 into account, for analyzing the "non-unitary" case it suffices to restrict our consideration by the following "double" one.

5.2.1.1. Double three-valued extension. Here, it is supposed that, for each $i \in 2$, $DM_{3,i}$ forms a subalgebra of $\mathfrak{A}$, in which case, by (2.15), the logic $(C/B)_{3/[,01]}$ of $\{(\mathcal{A}/\mathcal{DM})_{3,0/[,01]}, (\mathcal{A}/\mathcal{DM})_{3,1/[,01]}\}$ is a $\vee$-disjunctive proper extension of $C/B_{4[,01]}$ satisfying $\{x_0, \sim x_0\} \vdash (x_1 \vee \sim x_1)$, not being true in $\mathcal{A}/\mathcal{DM}_{4[,01]}$ under $[x_i/\langle 1-i, i\rangle]_{i\in 2}$, and so $\Delta_2 = (DM_{3,0} \cap DM_{3,1})$ forms a subalgebra of $\mathfrak{A}_{[3,0]}$, in which case $C_{[3]}$ is $\sim$-subclassical, in view of (2.15). Moreover, set $\boldsymbol{\partial}(\mathcal{A}_{3,i}) \triangleq (\boldsymbol{\partial}(\mathcal{A}){\restriction}DM_{3,i})$.

Theorem 5.32. *The following are equivalent:*

(i) *C_3 is self-extensional;*
(ii) *for each $i \in 2$, $(\mu{\restriction}DM_{3,i}) \in \mathrm{hom}(\mathfrak{A}_{3,i}, \mathfrak{A}_{3,1-i})$;*
(iii) *for some $i \in 2$, $(\mu{\restriction}DM_{3,i}) \in \mathrm{hom}(\mathfrak{A}_{3,i}, \mathfrak{A}_{3,1-i})$;*
(iv) *for each $i \in 2$, C_3 is defined by $\{\mathcal{A}_{3,i}, \boldsymbol{\partial}(\mathcal{A}_{3,i})\}$;*
(v) *for some $i \in 2$, C_3 is defined by $\{\mathcal{A}_{3,i}, \boldsymbol{\partial}(\mathcal{A}_{3,i})\}$;*
(vi) *for each $i \in 2$, $\boldsymbol{\partial}(\mathcal{A}_{3,i}) \in \mathrm{Mod}(C_3)$;*
(vii) *for some $i \in 2$, $\boldsymbol{\partial}(\mathcal{A}_{3,i}) \in \mathrm{Mod}(C_3)$;*
(viii) *$\mathfrak{A}_{3,0}$ and $\mathfrak{A}_{3,1}$ are isomorphic.*

Proof. First, assume (i) holds. Consider any $i \in 2$. Then, as $DM_{3,i} \ni a \triangleq \langle 1-i, i\rangle \neq b \triangleq \langle 1-i, 1-i\rangle \in \Delta_2 \subseteq DM_{3,i}$, by Theorem 4.7, there are some $j \in 2$, some $h \in \mathrm{hom}(\mathfrak{A}_{3,i}, \mathfrak{A}_{3,j})$ such that $\chi^{\mathcal{A}_{3,j}}(h(a)) \neq \chi^{\mathcal{A}_{3,j}}(h(b))$, in which case h is not singular, and so $B \triangleq (\mathrm{img}\, h)$ forms a non-one-element subalgebra of $\mathfrak{A}_{3,j}$. Therefore, $\Delta_2 \subseteq B$. Hence, $\mathfrak{A}_{3,i[-i+j]}[{\restriction}B]$ is a $(\wedge, \vee)$-lattice with zero/unit $\langle 0/1, 0/1\rangle$, in which case, by Lemma 2.2, $(h{\restriction}\Delta_2)$ is diagonal, and so $h(b) = b \in D^{\mathcal{A}_j}$. On the other hand, for all $c \in A$, it holds that $(\sim^{\mathfrak{A}} c = c) \Leftrightarrow (c \notin \Delta_2)$. Therefore, as $a \notin \Delta_2$, $h(a) \notin \Delta_2$, in which case $B \neq \Delta_2$, and so $B = DM_{3,j}$. Hence, if j was equal to i, we would have $h(a) = a$, in which case we would get $\chi^{\mathcal{A}_{3,j}}(h(a)) = \chi^{\mathcal{A}_{3,j}}(a) = (1-i) = \chi^{\mathcal{A}_{3,j}}(b) = \chi^{\mathcal{A}_{3,j}}(h(b))$, and so $j = (1-i)$, in which case $h(a) = \mu(a)$. Thus, $\mathrm{hom}(\mathfrak{A}_{3,i}, \mathfrak{A}_{3,1-i}) \ni h = (\mu{\restriction}DM_{3,i})$, and so (ii) holds.

Next, (iii/v/vii) is a particular case of (ii/iv/vi), respectively. Likewise, (vi/vii) is a particular case of (iv/v), while (ii/iii)⇒(iv/v) is by (2.15) and (5.5).

Further, assume (vii) holds. Then, as no false-/truth-singular Σ-matrix is isomorphic to any one not being so, while $\boldsymbol{\partial}(\mathcal{A}_{3,i})$ is false-/truth-singular iff $\mathcal{A}_{3,i}$ is not so, by Remarks 2.6(ii), 2.8(ii) and Theorem 3.5, we conclude that $\boldsymbol{\partial}(\mathcal{A}_{3,i})$ is isomorphic to $\mathcal{A}_{3,1-i}$, and so (2.15) yields (v). Now, assume (viii) holds. Let e be any isomorphism from $\mathfrak{A}_{3,0}$ onto $\mathfrak{A}_{3,1}$. Then, since these are both $(\wedge, \vee)$-lattices with zero/unit $\langle 0/1, 0/1\rangle$, by Lemma 2.2, $e{\restriction}\Delta_2$ is diagonal. Moreover, for all $c \in A$, it holds that $({\sim}^{\mathfrak{A}} c = c) \Leftrightarrow (c \notin \Delta_2)$. Therefore, $e(10) = (01)$, in which case $\hom(\mathfrak{A}_{3,0}, \mathfrak{A}_{3,1}) \ni e = (\mu{\restriction}DM_{3,0})$, and so (iii) with $i = 0$ holds.

Furthermore, (v)⇒(i) is by Theorem 4.1(vi)⇒(i) with $\mathsf{S} = \mathsf{M} = \{\mathcal{A}_{3,i}, \boldsymbol{\partial}(\mathcal{A}_{3,i})\}$ and (5.4).

□

First, by Theorems 5.23 and 5.32 we immediately have:

Corollary 5.33. *C_3 is self-extensional, whenever C is so.*

On the other hand, the converse does not hold, by:

Example 5.34 (cf. Example 11 of [14])**.** Let $\Sigma \triangleq (\Sigma_{\sim,+[,01]} \cup \{\amalg\})$ with binary $\amalg$ and $\amalg^{\mathfrak{A}} \triangleq ((\vee^{\mathfrak{A}}{\restriction}(DM^2_{3,0} \cup DM^2_{3,1})) \cup \{\langle\langle 01, 10\rangle, 11\rangle, \langle\langle 10, 01\rangle, 00\rangle\})$. Then, μ is not an endomorphism of $\mathfrak{A}$, while $(\mu{\restriction}DM_{3,0}) \in \hom(\mathfrak{A}_{3,0}, \mathfrak{A}_{3,1})$. Hence, by Theorems 5.23 and 5.32, C_3 is self-extensional, while C is not so, whereas $\mathcal{A}$ has no equational implication, in view of Theorem 10 and Lemma 9 of [14]. □

6. Conclusions

Aside from quite useful general results and their equally illustrative generic applications (sometimes, even multiple ones providing different insights, and so demonstrating the whole power of universal tools elaborated here) to infinite classes of particular logics, the universal negative result that the self-extensionality of uniform finitely-valued logics with equality determinant as well as either implication or both conjunction and disjunction excludes the algebraizability (in the sense of [12, 11]) of two-side sequent calculi (associated with such logics according to [13]), discovered here, looks especially remarkable (especially due to its providing a new insight into the non-"self-extensinality of"/"algebraizability of sequent calculi associated with" certain logics of such a kind proved originally *ad hoc*, and so justifying the thesis of the first paragraph of Section 1.

In view of Theorem 5.15, the most acute problem remaining still open is marking the framework of elimination of disjuctivity stipulation in Theorem 4.7.

References

1. N. D. Belnap, Jr, *A useful four-valued logic*, Modern uses of multiple-valued logic (J. M. Dunn and G. Epstein, eds.), D. Reidel Publishing Company, Dordrecht, 1977, pp. 8–37.
2. K. Gödel, *Zum intuitionistischen Aussagenkalkül*, Anzeiger der Akademie der Wissenschaften im Wien **69** (1932), 65–66.
3. K. Hałkowska and A. Zajac, *O pewnym, trójwartościowym systemie rachunku zdań*, Acta Universitatis Wratislaviensis. Prace Filozoficzne **57** (1988), 41–49.
4. S. C. Kleene, *Introduction to metamathematics*, D. Van Nostrand Company, New York, 1952.
5. J. Łoś and R. Suszko, *Remarks on sentential logics*, Indagationes Mathematicae **20** (1958), 177–183.
6. J. Łukasiewicz, *O logice trójwartościowej*, Ruch Filozoficzny **5** (1920), 170–171.
7. A. I. Mal'cev, *Algebraic systems*, Springer Verlag, New York, 1965.
8. G. Priest, *The logic of paradox*, Journal of Philosophical Logic **8** (1979), 219–241.
9. A. P. Pynko, *Algebraic study of Sette's maximal paraconsistent logic*, Studia Logica **54** (1995), no. 1, 89–128.
10. ______, *On Priest's logic of paradox*, Journal of Applied Non-Classical Logics **5** (1995), no. 2, 219–225.
11. ______, *Definitional equivalence and algebraizability of generalized logical systems*, Annals of Pure and Applied Logic **98** (1999), 1–68.
12. ______, *Functional completeness and axiomatizability within Belnap's four-valued logic and its expansions*, Journal of Applied Non-Classical Logics **9** (1999), no. 1/2, 61–105, Special Issue on Multi-Valued Logics.

13. ———, *Sequential calculi for many-valued logics with equality determinant*, Bulletin of the Section of Logic **33** (2004), no. 1, 23–32.
14. ———, *A relative interpolation theorem for infinitary universal Horn logic and its applications*, Archive for Mathematical Logic **45** (2006), 267–305.
15. ———, *Subquasivarieties of implicative locally-finite quasivarieties*, Mathematical Logic Quarterly **56** (2010), no. 6, 643–658.
16. ———, *Many-sorted General Algebraic Logic: Subprevarieties versus Extensions*, Kindle Direct Publishing, 2019.
17. ———, *Four-valued expansions of Dunn-Belnap's logic (I): Basic characterizations*, Bulletin of the Section of Logic **49** (2020), no. 4, 401–437.
18. ———, *Four-valued expansions of Belnap's logic: Inheriting basic peculiarities*, EasyChair Preprint no. 3118, EasyChair, 2020.
19. A. M. Sette, *On the propositional calculus P^1*, Mathematica Japonica **18** (1973), 173–180.

Department of Digital Automata Theory (100), V.M. Glushkov Institute of Cybernetics, Glushkov prosp. 40, Kiev, 03680, Ukraine
Email address: `pynko@i.ua`