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On the sums of arbitrary different biquadrates in two different ways

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Abstract: The quartic Diophantine equation $A^4 + hB^4 = C^4 + hD^4$, where h is a fixed arbitrary positive integer, has been studied by some mathematicians. In a recent paper, we studied this equation by using elliptic curve theory, and worked out some solutions of this equation for certain values of h, in particular for the values which has not already been found a solution in the range where $A, B, C, D \leq 100000$ by computer search. Finally we presented two conjecture such that if one of them is correct then we may solve this equation for every rational number h. In the present paper, we use the solutions of aforementioned Diophantine equation, as well as a simple idea to show that how some numbers can be written as the sums of two, three, four, five, or more different biquadrates in two different ways. In particular we give examples for the sums of 2, 3, $\cdots$, and 10, biquadrates expressed in two different ways.
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1 Introduction

The quartic Diophantine equation (DE)

$$A^4 + hB^4 = C^4 + hD^4, \tag{1}$$

where h is an arbitrary positive integer, has been studied by some authors. (see [3], [4], [1], [2], [5])
Currently, by computer search (see [6]), the small solutions of this DE are known for all positive

integers $h < 5000$, and $A, B, C, D < 100000$, except for

$h = 1198, 1787, 1987,$

$2459, 2572, 2711, 2797, 2971,$

$3086, 3193, 3307, 3319, 3334, 3347, 3571, 3622, 3623, 3628, 3644,$

$3646, 3742, 3814, 3818, 3851, 3868, 3907, 3943, 3980,$

$4003, 4006, 4007, 4051, 4054, 4099, 4231, 4252, 4358, 4406, 4414,$

$4418, 4478, 4519, 4574, 4583, 4630, 4643, 4684, 4870, 4955, 4999.$

In a recent paper [4], we studied this equation by using elliptic curve theory, and worked out some solutions of this equation for certain values of h, in particular for some above values which has not already been found a solution in the range where $A, B, C, D \leq 100000$ by computer search. Finally We presented two conjecture such that if one of them is correct then we may solve this equation for every rational number h.

In this paper by using the results of the previous paper as well as a simple idea, we easily show that how some numbers can be written as the sums of two, three, four, five, or more different biquadrates in two different ways.

2 The DE $A^4 + hB^4 = C^4 + hD^4$

Firstly, we prove the following theorem, which is one of the our main theorem in the pervious paper [4], then this theorem easily implies the above main result.

Theorem 2.1. *Consider the DE* (1)*, where h is a fixed arbitrary rational number. Then there exists a cubic elliptic curve of the form $E(h) : Y^2 = X^3 + FX^2 + GX + H$, where the coefficients F, G, and H, are all functions of h. If the elliptic curve $E(h)$ or its counterpart $E(h)_t$ resulting from $E(h)$ by switching h to ht^4 has positive rank, depending on the value of h and an appropriate rational number t, then the DE* (1) *has infinitely many integers solutions. By taking $h = \frac{v}{u}$, this also solves DE of the form $uA^4 + vB^4 = uC^4 + vD^4$ for appropriate integer values of u and v.*

Proof 2.1. Let: $A = m - q$, $B = m + p$, $C = m + q$, and $D = m - p$, where all variables are rational numbers. By substituting these variables in the DE (1) we get

$$-8m^3q - 8mq^3 + 8hm^3p + 8hmp^3 = 0. \tag{2}$$

Then after some simplifications and clearing the case $m = 0$ we obtain

$$m^2(hp - q) = -hp^3 + q^3. \tag{3}$$

We may assume that $hp - q = 1$ and $m^2 = -hp^3 + q^3$.
By plugging $q = hp - 1$ into the equation (3) and some simplifications we obtain the equation

$$m^2 = (h^3 - h)p^3 - (3h^2)p^2 + (3h)p - 1. \tag{4}$$

By multiplying both sides of this equation in $(h^3 - h)^2$ and letting

$$X = (h^3 - h)p \qquad Y = (h^3 - h)m, \tag{5}$$

we get the final elliptic curve

$$E(h) : Y^2 = X^3 - (3h^2)X^2 + (3h(h^3 - h))X - (h^3 - h)^2. \tag{6}$$

Note that by letting $X = Z + h^2$ in (6), we get the simple elliptic curve

$$E'(h) : Y^2 = Z^3 - (3h^2)Z - (h^4 + h^2). \tag{7}$$

If for a given h, the above elliptic curve $E(h)$ or its counterpart $E(h)_t$ resulting from $E(h)$ by switching h to ht^4 has positive rank, then by calculating m, p, q, A, B, C, D, from the relations (5), $q = hp - 1$, $A = m - q$, $B = m + p$, $C = m + q$, $D = m - p$, after some simplifications and canceling the denominators of A, B, C, D, we obtain infinitely many integer solutions for the DE (1). Now the proof of the theorem is completed.
It is interesting to see that for above solution, $A + C = B + D$, too.

Although, we were able to find an appropriate t such that $E(h)_t$ has positive rank in the case of rank zero $E(h)$ for many values of h, the proof for arbitrary h seems to be difficult at this point. For this reason, we stated it as a conjecture in [4].

conjecture 2.2. Let h be an arbitrary fixed rational number. Then there exists at least a rational number t such that the rank of the elliptic curve

$$E'(h)_t : Y^2 = Z^3 - (3h^2t^8)Z - (h^4t^{16} + h^2t^8), \tag{8}$$

is positive.

Remark 1. Note that by putting $h = \frac{v}{u}$, we may solve the DE in the form
$uA^4 + vB^4 = uC^4 + vD^4$, for appropriate integer values of u, and v.

Proof of the main result Let us by using the above results as well as a simple idea, show the sums of two, three, four, five, or more different biquadrates in two different ways.
Now, in the equation $A^4 + hB^4 = C^4 + hD^4$, let us take
$h = \pm h_1^4 \pm h_2^4 \pm + h_3^4 \pm \cdots \pm h_{n-1}^4$,
where h_i are choosen such that the rank of the elliptic curve (6) or (7) to be positive. Then we get

$$A^4 + (\pm h_1^4 \pm h_2^4 \pm + h_3^4 \pm \cdots \pm h_{n-1}^4)B^4 = C^4 + (\pm h_1^4 \pm h_2^4 \pm + h_3^4 \pm \cdots \pm h_{n-1}^4)D^4. \quad (9)$$

Now by multiplying h_i^4, to the numbers B^4, D^4, and by writing the positive terms in the one side and the negative terms in the other side, we get n positive terms of fourth powers in both sides, and then may obtain infinitely many nontrivial solutions for the DE $\sum_{i=1}^n a_i^4 = \sum_{i=1}^n b_i^4$. Now the proof of the our main result is completed.

Remark 2. Surprisingly, we may solve the general DE $\sum_{i=1}^n a_i x_i^4 = \sum_{j=1}^n a_j y_j^4$, by taking $h = \pm a_1 h_1^4 \pm \cdots \pm a_m h_m^4$, in the equation $A^4 + hB^4 = C^4 + hD^4$. Again here h_i are choosen such that the rank of the elliptic curve (6) or (7) to be positive.

Now we are going to work out many examples.

Example 1. $A^4 + B^4 = C^4 + D^4$
i.e., sums of 2 biquadrates in two different ways.

$h = 16$, here $h = 1$ replaced by $h = 2^4$.

$E(16) : Y^2 = X^3 - 768X^2 + 195840X - 16646400.$

$Rank = 1.$

Generator: $P = (X, Y) = (340, 680).$

Points: $2P = (313, -275)$, $3P = (\frac{995860}{729}, \frac{-727724440}{19683})$, $4P = (\frac{123577441}{302500}, \frac{305200800239}{166375000})$.

$(p, m, q) = (\frac{313}{4080}, \frac{-55}{816}, \frac{58}{255})$, $(p', m', q') = (\frac{2929}{8748}, \frac{-1070183}{118098}, \frac{9529}{2187})$,
$(p'', m'', q'') = (\frac{123577441}{1234200000}, \frac{305200800239}{678810000000}, \frac{46439941}{77137500})$.

Solutions:

$1203^4 + 76^4 = 653^4 + 1176^4$,

$1584749^4 + 2061283^4 = 555617^4 + 2219449^4$,

$103470680561^4 + 746336785578^4 = 713872281039^4 + 474466415378^4$.

Example 2. $X_1^4 + X_2^4 + X_3^4 = Y_1^4 + Y_2^4 + Y_3^4$
i.e., sums of 3 biquadrates in two different ways.

$h = \frac{39}{16} = \frac{5^4 - 1^4}{4^4}$.

$E(\frac{39}{16}) : Y^2 = X^3 - \frac{4563}{256}X^2 + \frac{5772195}{65536}X - \frac{2433942225}{16777216}$.

$Rank = 2$.

Generators: $P_1 = (X, Y) = (\frac{3289}{256}, \frac{3289}{256})$, and $P_2 = (X', Y') = (\frac{6565}{256}, \frac{43615}{512})$.

Points: $2P_1 = (\frac{4069}{256}, \frac{-14183}{512})$, $3P_1 = (\frac{9572761}{57600}, \frac{1752134549}{864000})$, P_2.

$(p, m, q) = (\frac{5008}{3795}, \frac{-8728}{3795}, \frac{2804}{1265})$, $(p', m', q') = (\frac{605392}{43875}, \frac{110806928}{658125}, \frac{36712}{1125})$,
$(p'', m'', q'') = (\frac{1616}{759}, \frac{488}{69}, \frac{1060}{253})$.

Solutions:

$8570^4 + 2325^4 + 1717^4 = 158^4 + 8585^4 + 465^4$,

$11166301^4 + 18732470^4 + 3178939^4 = 165354431^4 + 15894695^4 + 3746494^4$,

$1094^4 + 4365^4 + 469^4 = 4274^4 + 2345^4 + 873^4$.

Example 3. $X_1^4 + X_2^4 + X_3^4 + X_4^4 = Y_1^4 + Y_2^4 + Y_3^4 + Y_4^4$
i.e., sums of 4 biquadrates in two different ways.

$h = 23 = \frac{5^4 - 1^4 - 4^4}{2^4}$.

$E(23) : Y^2 = X^3 - 1587X^2 + 837936X - 147476736$.

Rank=1.

Generator: $P = (X, Y) = (880, 6512)$.

Points: P, $2P = (\frac{3424933}{5476}, \frac{275924489}{405224})$.

$(p, m, q) = (\frac{5}{69}, \frac{37}{69}, \frac{46}{69})$, $(p', m', q') = (\frac{3424933}{66500544}, \frac{275924489}{4921040256}, \frac{533605}{2891328})$.

Solutions:

$9^4 + 105^4 + 16^4 + 64^4 = 83^4 + 80^4 + 21^4 + 84^4$,

$1264542442^4 + 2646847655^4 + 22479447^4 + 89917788^4 =$
$2368240398^4 + 112397235^4 + 529369531^4 + 2117478124^4$.

Example 4. $X_1^4 + X_2^4 + X_3^4 + X_4^4 + X_5^4 = Y_1^4 + Y_2^4 + Y_3^4 + Y_4^4 + Y_5^4$
i.e., sums of 5 biquadrates in two different ways.

$h = \frac{3}{17} = \frac{1^4+2^4+3^4+11^4}{17^4}$.

$E(\frac{3}{17}) : Y^2 = X^3 - \frac{27}{289}X^2 - \frac{7560}{83521}X - \frac{705600}{24137569}$.

Rank=1.

Generator: $P = (X, Y) = (\frac{400}{289}, \frac{440}{289})$.

Points: P, $2P = (\frac{65437}{139876}, \frac{313237}{3077272})$.

$(p, m, q) = (\frac{-170}{21}, \frac{-187}{21}, \frac{-51}{21})$, $(p', m', q') = (\frac{-1112429}{406560}, \frac{-5325029}{8944320}, \frac{-200957}{135520})$.

Solutions:

$2312^4 + 357^4 + 714^4 + 1071^4 + 3927^4 = 4046^4 + 17^4 + 34^4 + 51^4 + 187^4$,

$134948261^4 + 29798467^4 + 59596934^4 + 89395401^4 + 327783137^4 =$
$315999247^4 + 19148409^4 + 38296818^4 + 57445227^4 + 210632499^4$.

Example 5. $X_1^4 + X_2^4 + X_3^4 + X_4^4 + X_5^4 + X_6^4 = Y_1^4 + Y_2^4 + Y_3^4 + Y_4^4 + Y_5^4 + Y_6^4$
i.e., sums of 6 biquadrates in two different ways.

$h = \frac{66}{25} = \frac{1^4+2^4+3^4+4^4+6^4}{5^4}$.

$E(\frac{66}{25}) : Y^2 = X^3 - \frac{13068}{625}X^2 + \frac{48756708}{390625}X - \frac{60637092516}{244140625}$.

Rank=1.

Generator: $P = (X, Y) = (\frac{1020552759889}{78568090000}, \frac{5339057694122399}{880905425080000})$.

Point: P.

$(p,m,q)=(\frac{47868328325}{58077532128},\frac{250424844940075}{651165290219136},\frac{1034770525}{879962608})$.

Solution:

$103059413079145^4+31484981684799^4+62969963369598^4+94454945054397^4+125939926739196^4+188909890108794^4=203229351055175^4+11450994089593^4+22901988179186^4+34352982268779^4+45803976358372^4+68705964537558^4$.

Example 6. $X_1^4+X_2^4+\cdots+X_7^4=Y_1^4+Y_2^4+\cdots+Y_7^4$
i.e., sums of 7 biquadrates in two different ways.

$h=\frac{77}{3}=\frac{6^4-1^4-2^4-3^4+4^4-5^4}{3^4}$.

$E(\frac{77}{3}):Y^2=X^3-\frac{5929}{3}X^2+\frac{35099680}{27}X-\frac{207790105600}{729}$.

Rank=1.

Generator: $P=(X,Y)=(\frac{92500}{81},\frac{7695260}{729})$.

Points: P, $2P=(\frac{6892959356452}{8759275281},\frac{975806887820176684}{819789332824071})$.

$(p,m,q)=(\frac{125}{1848},\frac{10399}{16632},\frac{53}{72})$, $(p',m',q')=(\frac{1723239839113}{36970630037880},\frac{243951721955044171}{346011823587522708\,0},\frac{282825681793}{1440414157320})$.

Solutions:

$2766^4+34572^4+23048^4+28810^4+4637^4+9274^4+13911^4=$
$33963^4+27822^4+18548^4+23185^4+5762^4+11524^4+17286^4$,

$X_1=653165044877947269$, $X_2=1215694385212406862$, $X_3=810462923474937908$,
$X_4=1013078654343672385$, $X_5=413359910863096694$, $X_6=82671982172619388$,
$X_7=124007973258929082$, $Y_1=1385020210743079782$, $Y_2=248015946517858164$,
$Y_3=165343964345238776$, $Y_4=206679955431548470$, $Y_5=2026157308687344477$,
$Y_6=405231461737468954$, $Y_7=607847192606203431$.

Example 7. $X_1^4+X_2^4+\cdots+X_8^4=Y_1^4+Y_2^4+\cdots+Y_8^4$
i.e., sums of 8 biquadrates in two different ways.

$h=10=\frac{7^4+1^4+2^4-3^4-4^4-5^4-6^4}{2^4}$.

$E(10):Y^2=X^3-300X^2+29700X-980100$.

$Rank = 1$.

Generator: $P = (X, Y) = (165, 495)$.

Points: P, $2P = (\frac{505}{4}, \frac{-85}{8})$, $3P = (\frac{172029}{961}, \frac{-20192733}{29791})$.

$(p, m, q) = (\frac{1}{6}, \frac{1}{2}, \frac{2}{3})$, $(p', m', q') = (\frac{101}{792}, \frac{-17}{1584}, \frac{109}{396})$, $(p'', m'', q'') = (\frac{5213}{28830}, \frac{-203967}{297910}, \frac{2330}{2883})$.

Solutions:
$3^4 + 4^4 + 4^4 + 5^4 + 14^4 = 7^4 + 7^4 + 8^4 + 10^4 + 12^4$,

$906^4 + 1295^4 + 185^4 + 370^4 + 657^4 + 876^4 + 1095^4 + 1314^4 =$
$838^4 + 1533^4 + 219^4 + 438^4 + 555^4 + 740^4 + 925^4 + 1110^4$,

$1334201^4 + 1576043^4 + 225149^4 + 450298^4 + 1160256^4 + 1547008^4 + 1933760^4 + 2320512^4 =$
$110399^4 + 2707264^4 + 386752^4 + 773504^4 + 675447^4 + 900596^4 + 1125745^4 + 1350894^4$.

Example 8. $X_1^4 + X_2^4 + \cdots + X_9^4 = Y_1^4 + Y_2^4 + \cdots + Y_9^4$
i.e., sums of 9 biquadrates in two different ways.

$h = \frac{21}{8} = \frac{8^4+1^4-2^4-3^4-4^4+5^4-6^4-7^4}{4^4}$.

$E(\frac{21}{8}) : Y^2 = X^3 - \frac{1323}{64}X^2 + \frac{498771}{4096}X - \frac{62678889}{262144}$.

$Rank = 1$.

Generator: $P = (X, Y) = (\frac{163241}{11552}, \frac{46525193}{3511808})$.

Point: P.

$(p, m, q) = (\frac{6928}{7581}, \frac{123409}{144039}, \frac{505}{361})$.

Solution:

$312344^4 + 2040328^4 + 255041^4 + 1275205^4 + 16446^4 + 24669^4 + 32892^4 + 49338^4 + 57561^4 =$
$1299616^4 + 65784^4 + 8223^4 + 41115^4 + 510082^4 + 765123^4 + 1020164^4 + 1530246^4 + 1785287^4$.

By choosing the other points on the elliptic curve such as $2P, 3P, \cdots$, (or changing the value of h, and getting new elliptic curve) we obtain infinitely many solutions for the above Diophantine equation as well.

$h = \frac{-3}{2} = \frac{8^4+1^4+2^4+3^4-4^4-5^4-6^4-7^4}{4^4}$.

$E(\frac{-3}{2}) : Y^2 = X^3 - \frac{27}{4}X^2 + \frac{135}{16}X - \frac{225}{64}$.

$Rank = 1$.

Generator: $P = (X, Y) = (\frac{85}{16}, \frac{55}{64})$.

Point: P.

$(p, m, q) = (\frac{-17}{6}, \frac{-11}{24}, \frac{13}{4})$.

Solution:

$356^4 + 632^4 + 79^4 + 158^4 + 237^4 + 228^4 + 285^4 + 342^4 + 399^4 =$
$268^4 + 456^4 + 57^4 + 114^4 + 171^4 + 316^4 + 395^4 + 474^4 + 553^4$.

Example 9. $X_1^4 + X_2^4 + \cdots + X_{10}^4 = Y_1^4 + Y_2^4 + \cdots + Y_{10}^4$
i.e., sums of 10 biquadrates in two different ways.

$h = -63 = \frac{14^4+1^4+2^4+3^4+4^4+5^4-6^4-11^4-13^4}{3^4}$.

$E(-63) : Y^2 = X^3 - 11907X^2 + 47246976X - 62492000256$.

$Rank = 1$.

Generator: $P = (X, Y) = (4960, 30752)$.

Points: $P, 2P = (\frac{4096948}{961}, \frac{74223316}{29791})$.

$(p, m, q) = (\frac{-5}{252}, \frac{-31}{252}, \frac{1}{4}), (p', m', q') = (\frac{-1024237}{60058656}, \frac{-18555829}{1861818336}, \frac{70925}{953312})$.

Solutions:

$141^4 + 252^4 + 18^4 + 36^4 + 54^4 + 72^4 + 90^4 + 78^4 + 143^4 + 169^4 =$
$48^4 + 182^4 + 13^4 + 26^4 + 39^4 + 52^4 + 65^4 + 108^4 + 198^4 + 234^4$,

$235608531^4 + 3521502324^4 + 25153588^4 + 50307176^4 + 75460764^4 + 100614352^4 + 125767940^4 +$
$39586554^4 + 72575349^4 + 85770867^4 =$
$179941044^4 + 92368626^4 + 6597759^4 + 13195518^4 + 19793277^4 + 26391036^4 + 32988795^4 +$
$150921528^4 + 276689468^4 + 326996644^4$.

Remark 3. By choosing the other points on the above elliptic curves such as $3P$, $4P$, $\cdots$, (or changing the value of h, and getting new elliptic curves) we obtain infinitely many solutions for each case of the above Diophantine equations.

The Sage software has been used for calculating the rank of the elliptic curves, (see [7]).

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